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$FI-$Semi-Injective Modules

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Abstract This paper introduce and investigate the notion of $FI-M-$principally injective and $FI-$semi-injective (fully invariant-semi injective) modules. Clearly $FI-$semi-injective module does not satisfy the (C_1) condition, so we provide several sufficient conditions under which $FI-$semi-injective modules will be continuous. Apart from this we obtain more results related with uniform, weakly co-Hopfian and square free modules. We also prove that $FI-$semi-injective module satisfies summand intersection property(SIP), summand sum property(SSP). Furthermore, we characterize several rings in terms of $FI-$semi-injective modules.

Introduction: In recent years structure of principally injective rings, principally injective modules and their various properties have been extensively studied by many authors ([8], [10]). Recall that a ring R is principally injective, if every homomorphism from a principal right ideal to R is given by a left multiplication by an element of R. A module M is called principally injective if every homomorphism $f : aR \to M$, $a \in R$, extends to R; for example every injective modules are principally injective, and every right R-module is principally injective if and only if R is (von Neumann) regular ring. Sanh et.al.[11], extend this notion to module and generalize the idea of principal injectivity to $M-$principal injectivity for a given right $R-$module M. Recall that a right $R-$module N is called $M-$principally injective, if every homomorphism from an $M-$cyclic submodule $s(M)$ of M to N can be extended to a homomorphism from M to N. A module M is called semi injective if it is $M-$principally injective (see [9], [11], [14]). The modules we are interested in are those where this is required only for fully invariant $M-$cyclic submodule $s(M)$ of M, which extend the notion of semi-injective modules to $FI-$semi-injective modules. In a similar fashion motivated by the above defined notions, we introduce the concepts of $FI-M-$principally injective ($FI-$semi-injective) module as a proper generalizations of $M-$principally injective (semi-injective) module. Thus the class of $FI-$semi-injective modules is bigger than the class of semi injective modules, as we have the following implication:
Injective ⇒ Quasi-injective ⇒ Semi-injective ⇒ $FI-$semi-injective module.

In this paper, we show that the structure of $FI-M-$principally injective module is closed under the direct summand, finite direct sum and finite direct product. Further it is observed that the direct sum of $FI-$semi-injective module need not be $FI-$semi-injective, it holds only if modules in direct sum are reletively $FI-M_i-$principally injective. In Proposition 1.17, we show that $FI-$semi-injective module satisfies conditions (C_2) and (C_3) but not (C_1). This raised the following question: Do there exist $FI-$semi-injective modules which are continuous, or when will $FI-$semi-injective module satisfy (C_1) ? We provided several sufficient conditions under which $FI-$semi-injective modules to be continuous. Further, we relate $FI-$semi-injective module with uniform, weakly co-Hopfian and square free modules. We also prove that $FI-$semi-injective module satisfiy summand intersection property(SIP), summand sum property(SSP) and characterize several rings as $pp-$ring, semi simple Artinian ring and von-Neumann regular ring in terms of $FI-$semi-injective modules.

Throughout this paper, by a ring R we always mean an associative ring with identity and every R-module M is an unitary right R-module. Let M be a right R–module, the notation $N \subseteq M$ mean that N is a submodule of M. A submodule N of an R-module M is called essential submodule if $N \cap L \neq 0$, for each nonzero submodule L of M. A nonzero module M is called uniform if every nonzero submodule of M is essential in M, for example $M = \mathbb{Z}_{\mathbb{Z}}$ is a uniform module. A module N is called M–generated, if for some index set I, there exists an epimorphism $M^{(I)} \longrightarrow N$. If index set I is finite, then N is called finitely M-generated. In particular, a submodule N of M is called an M-cyclic submodule of M if $N = s(M)$ for some $s \in End(M_R)$ or if there exist an epimorphism from M to N, equivalently it is isomorphic to M/L for some submodule L of M. A submodule K of M is called fully invariant if $s(K) \subseteq K$ for all $s \in End(M_R)$. Clearly 0 and M are fully invariant submodule of M. Many distinguished submodules of a module are fully invariant for example, the socle, the Jacobson radical, the singular submodule, the torsion submodule etc.. Observe that the fully invariant submodule of R_R are exactly the two sided ideals of R and the $(S-R)$–submodules of $(S-R)$–bimodule M are fully invariant submodule of M, where $S = End(M_R)$. Furthermore, the fully invariant submodules of an injective module are quasi-injective and fully invariant submodules of quasi-injective modules are again quasi-injective. An R–module M is called duo if all of its submodules are fully invariant, for example, if M is a simple R–module, then M is a duo but $M \oplus M$ is not duo. A ring R is called a duo ring if R_R is a duo module. Clearly commutative rings and division rings are example of duo rings but any matrix ring of order 2 over such a ring is not a duo ring. A module M is called self generator if it generates all its submodule.

For usual definitions and standerd notations, we refer [1], [7] and [13].

1 $FI-M$–Principal Injectivity

Let M be a right R–module. A right R–module N is called fully invariant $-M$–principally injective (in short, $FI-M$–principally injective), if every homomorphism from a fully invariant M-cyclic submodule $s(M)$ of M to N can be extended to a homomorphism from M to N.

$$\begin{array}{ccccc} 0 & \longrightarrow & s(M) & \overset{i}{\longrightarrow} & M \\ & & f\downarrow & \swarrow g & \\ & & N & & \end{array}$$

Equivalently every homomorphism f from fully invariant M-cyclic submodule $s(M)$ of M to N, factors in the form $f = g \circ i$, for some homomorphism g from M to N and inclusion map i from $s(M)$ to M. N is called FI–principally injective if it is $FI-R$–principally injective. For example we can consider $\mathbb{Z}$ be the ring of integers and $N = \mathbb{Z}_4$ and $M = \mathbb{Z}_6$ are additive abelian groups, which forms a module over $\mathbb{Z}$, then we can easily verify that N is $FI-M$–principally injective and M is $FI-N$–principally injective but $\mathbb{Z}$ is not $FI-\mathbb{Z}$–principally injective module over $\mathbb{Z}$.

We begin with some basic properties of fully invariant M-cyclic submodules as,

Lemma 1.1. *Let M be a right R–module.*
(1) *For any fully invariant M-cyclic submodule N of M, there exists a maximal submodule K of M such that $K \cap N = 0$.*
(2) *Any sum and intersection of fully invariant M-cyclic submodules of M is again a fully invariant M-cyclic submodule of M, (in fact the fully invariant M-cyclic submodules form a complete modular sublattice of the lattice of submodules of M).*
(3) *Transitivity : If K is fully invariant N-cyclic submodule of N and N is fully invariant M-cyclic submodule of M, then K is fully invariant M-cyclic submodules of M.*
(4) *If $M = \oplus_{i\in I} M_i$ be a direct sum of fully invariant M–cyclic submodule $M_i (i \in I)$ and N be a fully invariant M-cyclic submodule of M, then $N = \bigoplus_{i\in I}(M_i \cap N) = \oplus_{i\in I}\pi_i(N)$, where π_i is the i-th projection of M.*

Proof: Proof is routine.

In the following results we have to prove that $FI - M-$principally injective modules are closed under finite direct sums, direct product and summands.

Proposition 1.2. *Two modules N_1 and N_2 are $FI - M-$principally injective if and only if $N_1 \bigoplus N_2$ is $FI - M-$principally injective.*

Proof: Let N be the fully invariant M-cyclic submodules of M and $i : N \longrightarrow M$ be the inclusion map, then for any homomorphism $f : N \longrightarrow N_1 \bigoplus N_2$. Since N_1 and N_2 are $FI - M-$principally injective, there exist g_1 from M to N_1 and g_2 from M to N_2 such that $g_1 \circ i = \pi_1 \circ f$ and $g_2 \circ i = \pi_2 \circ f$, where π_1 and π_2 be the natural epimorphism from $N_1 \bigoplus N_2$ to N_1 and N_2 respectively. Take $g = i_1 \circ g_1 + i_2 \circ g_2$ from M to $N_1 \bigoplus N_2$, where i_1 and i_2 be the inclusion map from N_1 and N_2 to $N_1 \bigoplus N_2$ respectively. Thus it is clear that g extend f from M to $N_1 \bigoplus N_2$. Converse is obvious.

Corollary 1.3. *$\oplus_{i \in I} N_i$ is $FI-M-$principally injective, if each component N_i is $FI-M-$principally injective for finite index set I.*

Corollary 1.4. *$\prod_{i \in I} N_i$ is $FI-M-$principally injective, if each component N_i is $FI-M-$principally injective for finite index set I.*

Proposition 1.5. *If fully invariant M-cyclic submodule N of M is $FI-M-$principally injective, then it is a direct summand of M.*

Proof: Assume that N is $FI - M-$principally injective and $f : N \longrightarrow M$ be a monomorphism. Then by $FI - M-$principally injectivity of N, there exist split homomorphism $g : M \longrightarrow N$ such that $g \circ f = I_N$. Thus we have $M = f(N) \oplus ker(g)$, hence $f(N)$ is a direct summand of M, since N is fully invariant so $f(N) \subseteq N$ is a direct summand of M.

Proposition 1.6. *Let K be a fully invariant M-cyclic submodule of M and N be $FI-M-$principally injective. Then N is both $(FI - K)-$ and $FI - M/K-$principally injective.*

Proof: Assume that L is fully invariant K-cyclic submodule of K, then by Lemma 1.1(iii), L is fully invariant M-cyclic submodule of M. Take $f : L \longrightarrow N$ is a homomorphism, by $FI - M-$principally injectivity of N, there exists a homomorphism $g : M \longrightarrow N$. Then the restriction map $g|_K : K \longrightarrow N$ extend f. Thus N is $FI - K-$principally injective.
Now for the second part assume that $f' : K'/K \longrightarrow N$ be a homomorphism, where K' is fully invariant M-cyclic submodule of M containing K. By $FI - M-$principally injectivity of N, $f' \circ \pi : K' \longrightarrow N$ can be extended to a homomorphism $g' : M \longrightarrow N$, where $\pi : K' \longrightarrow K'/K$ be a natural epimorphism. Now we define $g'' : M/K \longrightarrow N$ by $g''(m + K) = g'(m) \forall m \in M$, which is well defined. Then it is clear to see that g'' extend f' ie N is $FI - M/K-$principally injective.

Proposition 1.7. *If an $R-$module M is $FI - N-$principally injective, then every direct summand M_1 of M is $FI - N_1-$principally injective, where N_1 is fully invariant $N-$cyclic submodule of N.*

Proof: Let N_2 be fully invariant N_1-cyclic submodule of N_1, then by Lemma 1.1(iii), N_2 is fully invariant $N-$cyclic submodule of N. Take $i : N_2 \longrightarrow N_1$ and $i_1 : N_1 \longrightarrow N$ are injective map, then $i_1 \circ i : N_2 \longrightarrow N$ is injective. Let $f : N_2 \longrightarrow M_1$ be any homomorphism, then $i_2 \circ f : N_2 \longrightarrow M$ is a homomorphism, where $i_2 : M_1 \longrightarrow M$ be the injective homomorphism. Now by $FI - N-$principally injectivity of M, there exists a homomorphism $g : N \longrightarrow M$ such that $g \circ i_1 \circ i = i_2 \circ f \Rightarrow \pi \circ g \circ i_1 \circ i = \pi \circ i_2 \circ f$, where $\pi : M \longrightarrow M_1$ be a natural projection, which gives $h \circ i = I_{M_1} \circ f$, where $h = \pi \circ g \circ i_1 : N_1 \longrightarrow M_1$. Thus $h \circ i = f$ which shows that M_1 is $FI - N_1-$principally injective.

Corollary 1.8. *Let N_1 be a direct summand of N and M_1 be a direct summand of M. If N is $FI - M-$principally injective, then N_1 is $FI - M_1-$principally injective.*

Proposition 1.9. *Let M be a duo and N be right $R-$modules. Then N is $FI - M-$principally injective if and only if $Hom_R(M, N) \circ g = \{f \in Hom_R(M, N) | ker(g) \subseteq ker(f)\}$, for $g \in End(M_R)$.*

Proof: Assume that N is $FI - M$−principally injective and $f \in Hom_R(M,N)$ such that $ker(g) \subseteq ker(f)$. Then using the Factor Theorem [1], there exist a unique homomorphism $h : g(M) \longrightarrow N$ such that $f = h \circ g$. Now by $FI - M$−principally injectivity of N, there exist homomorphism $f' : M \longrightarrow N$ such that $f' \circ i = h$, where i is the inclusion map from $g(M)$ to M. Hence $f = f' \circ i \circ g = f' \circ g$, which implies that $f \in Hom_R(M,N) \circ g$, i.e. $\{f \in Hom_R(M,N)|ker(g) \subseteq ker(f)\} \subseteq Hom_R(M,N) \circ g$. Other part clearly holds, i.e. $Hom_R(M,N) \circ g \subseteq \{f \in Hom_R(M,N)|ker(g) \subseteq ker(f)\}$. Thus we have $Hom_R(M,N) \circ g = \{f \in Hom_R(M,N)|ker(g) \subseteq ker(f)\}$.
Conversely, let $Hom_R(M,N) \circ g = \{f \in Hom_R(M,N)|ker(g) \subseteq ker(f)\}$. Let $g(M)$ be fully invariant M−cyclic submodule of M and $h : g(M) \longrightarrow N$ be a homomorphism for $g \in End(M_R)$, then $h \circ g \in Hom_R(M,N)$ and $ker(g) \subseteq ker(h \circ g)$. So by assumption we have $h \circ g = f \circ g$ for some $f \in Hom_R(M,N)$, which shows that N is $FI - M$−principally injective.

Proposition 1.10. *Let M be a duo module and N be an right R−module. Then M is N−projective and every submodule of N is $FI - M$−principally injective if and only if N is $FI - M$−principally injective and every M−cyclic submodule of M is N−projective.*

Proof: Assume that, M is N−projective and every submodule $N' \subseteq N$ is $FI - M$−principally injective. Let $s(M)$ be an M−cyclic submodule of M for any $s \in End(M_R)$, as M is duo, so $s(M)$ be fully invariant M−cyclic submodule of M and $f : s(M) \longrightarrow N'$ be a homomorphism. Then by $FI - M$−principally injectivity of N', there exists a homomorphism $g : M \longrightarrow N'$, such that $g \circ i = f$, where $i : s(M) \longrightarrow M$ be the inclusion map. Now the N−projectivity of M implies that g can be lifted to a homomorphism $h : M \longrightarrow N$, such that $\pi \circ h = g$, where π is an epimorphism from N to N'. thus the composition map $h \circ i : s(M) \longrightarrow N$ lifted $f : s(M) \longrightarrow N'$, i.e. $s(M)$ is N−projective. Since N is submodule of itself so N is $FI - M$−principally injective.
Conversely, N is $FI - M$−principally injective and every fully invariant M−cyclic submodule $s(M)$ of M is N−projective. Let $f : s(M) \longrightarrow N'$, then by N−projectivity of $s(M)$, there exists homomorphism $g : s(M) \longrightarrow N$, such that $\pi \circ g = f$, where π is an epimorphism from N to N'. Now $FI - M$−principally injectivity of N implies that g can be extended to a homomorphism $h : M \longrightarrow N$, such that $h \circ i = g$. Thus it is clear that the composition map $\pi \circ h : M \longrightarrow N'$ is an extension of f, i.e. every submodule of N is $FI - M$−principally injective. Since M is fully invariant M−cyclic submodule of itself, so M is N−projective by assumption.

A ring R is called right pp−ring, if each of its principal right ideal is projective. In the following proposition we have extended the exercise of Wisbauer [13].

Proposition 1.11. *The following statements are equivalent for a projective module M:*
(i) Every homomorphic image of any $FI - M$−principally injective module is $FI - M$−principally injective;
(ii) Every homomorphic image of any M−injective module is $FI - M$−principally injective;
(iii) Every homomorphic image of any injective R−module is $FI - M$−principally injective.
(iv) Every fully invariant M−cyclic submodule of M is projective.

Proof: $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious.
$(iii) \Rightarrow (iv)$ let M_1 be fully invariant M−cyclic submodule of M. Now we claim that M_1 is projective. By [3] (Chapter 1, Proposition 5.1), M_1 is projective if and only if for any two R−module N and N_1, where N is injective and $\pi : N \longrightarrow N_1$ is an epimorphism, then any homomorphism $f : M_1 \longrightarrow N_1$ can be lifted to a homomorphism from $M_1 \longrightarrow N$. By assumption (iii) i.e. homomorphic image N_1 of an injective module N is $FI - M$−principally injective, therefore homomorphism f can be extended to a homomorphism $g : M \longrightarrow N$. Then clearly $g|_{M_1}$ lifts f, and hence M_1 is projective.
$(iv) \Rightarrow (i)$ Let M_1 be fully invariant M−cyclic submodule of M and N be $FI - M$−principally injective module. Take N_1 to be submodule of N and $\pi : N \longrightarrow N/N_1$ be the natural epimorphism. By assumption (iv) i.e. fully invariant M−cyclic submodule M_1 of M is projective, then any homomorphism $f : M_1 \longrightarrow N/N_1$ can be lifted to a homomorphism $g : M_1 \longrightarrow N$. Since N is $FI - M$−principally injective, therefore g can be extended to a homomorphism $h : M \longrightarrow N$. Clearly homomorphism $\pi \circ h : M \longrightarrow N/N_1$ extends f, i.e. N/N_1 is $FI - M$−principally injective module.

Corollary 1.12. *(Wisbauer, Exercises* $39.17(4)$*[13])The following statements are equivalent for a ring* R*:*
(i) *R is right pp-ring;*
(ii) *Every factor module of a principally injective module is principally injective;*
(iii) *Every factor module of an injective module is principally injective.*

An $R-$module M is called $FI-$semi injective, if it is $FI-M-$principally injective. A ring R is right $FI-$self-p-injective, if R_R is $FI-R-$principally injective. For example $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are $FI-$semi injective module over $\mathbb{Z}$. Every simple, semi simple, quasi-injective, semi-injective modules, $FI-$self-p-injective ring and their direct summands are all $FI-$semi injective module.

Corollary 1.13. *Any direct summand of $FI-$semi injective module is again $FI-$semi injective.*

Corollary 1.14. *Let M be a $FI-$semi injective module and $f, g \in S = End(M_R)$, then $f \in Sg$ if and only if $ker(g) \subseteq ker(f)$.*

Proof: Prove is obvious in the light of Proposition 1.9.

Two modules M_1 and M_2 are called relatively (or mutually)$FI-$principally injective, if M_1 is $FI-M_2-$principally injective and M_2 is $FI-M_1-$principally injective.

Proposition 1.15. *Two modules M_1 and M_2 are relatively $FI-$principally injective, if $M_1 \bigoplus M_2$ is $FI-$semi injective.*

Proof: It is enough to prove that M_1 is $FI-M_2-$principally injective. Let K be fully invariant M_2-cyclic submodule of M_2 with inclusion map $i : K \longrightarrow M_2$ and $\phi : K \longrightarrow M_1$ be a homomorphism. Define a homomorphism $g : K \longrightarrow M_1 \bigoplus M_2$ by $g(a) = (\phi(a), a)$ for all $a \in K$. Then by $FI-$semi injectivity of $M_1 \bigoplus M_2$ and Proposition 1.6, we get $M_1 \bigoplus M_2$ is $FI-M_2-$principally injective. Then g can be extended to f from M_2 to $M_1 \bigoplus M_2$, so $\pi_1 \circ f$ is a homomorphism extending ϕ, where π_1 is a natural epimorphism from $M_1 \bigoplus M_2$ to M_1 i.e. $g = f \circ i \Rightarrow \pi_1 \circ g = (\pi_1 \circ f) \circ i$, hence $\phi = \psi \circ i$, where $\varphi = \pi_1 \circ f$. Therefore M_1 is $FI-M_2-$principally injective.

Corollary 1.16. *If $\bigoplus_{i \in I} M_i$ is $FI-$semi injective for finite index set I, then M_i is $FI-M_j-$principally injective for all distinct $i, j \in I$.*

Proof: Applying induction on Proposition 1.15.

From [7], recall that the following conditions for an $R-$module M:
(C_1) Every submodule of M is essential in a direct summand of M.
(C_2) Every submodule isomorphic to a direct summand of M, is a direct summand of M.
(C_3) Direct sum of two direct summands, whose intersection is zero is a direct summand of M.

An $R-$module M is called extending (or CS) if and only if every closed submodule is a direct summand of M or it satisfies (C_1), continuous if it satisfies (C_1) with (C_2) and quasi-continuous if it satisfies (C_1) with (C_3).

Proposition 1.17. *Any $FI-$semi injective module satisfies the condition (C_2) and (C_3).*

Proof: For (C_2) let M_1 is a direct summand of M and $M_1 \simeq M_2$ then we have to show that $M_2 \subset^{\oplus} M$. Using Corollary 1.13, M_1 is $FI-M-$principally injective, since $M_1 \simeq M_2$, then M_2 is also $FI-M-$principally injective. Then by Proposition 1.5, M_2 is a direct summand of M. (C_3) is obtained from (Proposition 2.2, [7]).

Remark 1.18. Any $FI-$semi injective module does not satisfy the conditions (C_1). Therefore by above proposition we have every extending, $FI-$semi injective modules are continuous.

Corollary 1.19. *Any directly finite, extending, $FI-$semi injective module have the cancellation property.*

Proof: Proof is done with the help of above remark and (Corollary 3.25, [7]).

Thus we have the following implication as:
Injective $\Rightarrow$ Quasi-injective $\Rightarrow$ Semi-injective $\Rightarrow$ $FI-$semi-injective module $\not\Rightarrow$ Continuous.

Now we investigate the conditions under which $FI-$semi-injective modules are continuous:

Proposition 1.20. *An indecomposable $FI-$semi injective module is continuous if and only if M is uniform.*

Proof: Suppose M is continuous, it satisfies (C_1) and (C_2) conditions. Let $M = M_1 \oplus M_2$, by (C_1) condition every submodule is essential in a direct summand of M. Since M is indecomposable, implies that, either $M_1 = 0$ or $M_2 = 0$. Thus every submodule is essential in M, so M is uniform. Conversely, assume that $FI-$semi injective module is uniform. Since uniform module has (C_1) condition, then by Proposition 1.17, M satisfies the (C_2) condition. Therefore M is continuous.

Proposition 1.21. *If $M = \oplus_{i\in I} M_i$ is duo $FI-$semi injective module, where $M_i s$ are uniform, then M is continuous.*

Proof: Since M is $FI-$semi injective module, it satisfies the (C_2) condition. So it is enough to prove that M is uniform or M satisfies the (C_1) condition. By Lemma 1.1(iv), every submodule N of M can be written as $N = \bigoplus_{i\in I}(M_i \cap N)$, where $(M_i \cap N) \neq 0$. Since each M_i is uniform and $(M_i \cap N) \subset M_i$, so $(M_i \cap N)$ is essential in M_i i.e. M satisfies the (C_1) condition. Thus we see that N is essential in $\oplus_{i\in I} M_i = M$ i.e. M is uniform, so the proof follows by Proposition 1.20.

Proposition 1.22. *Let M be $FI-$semi injective module. If $S = End(M_R)$ is local, then for any non zero fully invariant $M-$cyclic submodules M_1 and M_2 of M, $M_1 \cap M_2 \neq 0$.*

Proof: Let $0 \neq s(M) = M_1$ and $0 \neq t(M) = M_2$ for $s, t \in S$ and $M_1 \cap M_2 = 0$, then we have the well defined map $f : (s+t)(M) \longrightarrow M$ as $(s+t)(m) \longmapsto s(m)$. By $FI-$semi injectivity, there exists $g \in S$ such that $g|_{(s+t)(m)} = f$ i.e. for any $m \in M$, $f(s+t)(m) = g(s+t)(m)$. It follows that $s = g(s+t)$. Then$(1-g) \circ s = g \circ t$. Since M_1 and M_2 are fully invariant, we have $(1-g) \circ s(M) \subset M_1$ and $g \circ t(M) \subset M_2$. It follows that $(1-g) \circ s = 0$ and $g \circ t = 0$ as $M_1 \cap M_2 = 0$. Since S is local, g or $(1-g)$ is invertible and this would imply that $s = 0$ or $t = 0$ a contradiction.

Proposition 1.23. *If M is duo $FI-$semi injective which is self generator module with local endomorphism ring. Then M is uniform, hence it is continuous.*

Proof: For any $0 \neq m \in M, mR$ contains a non zero $M-$cyclic submodule. Since M is self generator, using Proposition 1.22, M is uniform and we know that uniform $FI-$semi injective module is continuous by Proposition 1.20.

Proposition 1.24. *Let a module M is projective, semi perfect, duo and self generator. If M is $FI-$semi injective module, then it is continuous.*

Proof: By Proposition 1.17, $FI-$semi injective module satisfies (C_2) condition, so it is enough to prove that M satisfies (C_1) condition. Since M is projective and semi perfect, then by Theorem 4.44, [7] and 42.3, [13], M can be written as $M = \bigoplus_{i\in I} M_i$, where $M_i/Rad(M_i)$ are simple for all $i \in I$. Since each M_i is projective and semi perfect, so $Rad(M_i) \ll M_i$, and hence M_i is indecomposable. Also by Corollary 1.13, every direct summand of $FI-$semi injective module is $FI-$semi injective, then by (19.9, [13]), $End(M_i)$ are local ring for all $i \in I$. Since every direct summand of duo and self generator module is again a duo and self generator, then by proposition 1.23, each M_i is uniform. Now using Proposition 1.21, we get M is continuous.

Proposition 1.25. *(Theorem 3.5, [10]) If a semi perfect ring R is right duo right principally injective, then R_R is continuous.*

Proof: Prove is same as above proposition, taking $M_R = R_R$.

In the next theorem we provide a characterization of semi simple rings in term of $FI-$semi injective module.

Theorem 1.26. *Following statements are equivalent for a commutative ring R:*
(i) *The direct sum of any two $FI-$semi injective module is $FI-$semi injective;*
(ii) *Every $FI-$semi injective module is injective;*
(iii) *R is semi simple Artinian.*

Proof: $(i) \Rightarrow (ii)$ let M be $FI-$semi injective module and $E(M)$ be its injective hull. Suppose $N = M \oplus E(M)$ is $FI-$semi injective by assumption (i). Then N is $FI-N-$principally injective, hence M is $FI-N-$principally injective by Corollary 1.13. Consider the inclusion map $i : M \longrightarrow E(M)$ and $j : E(M) \longrightarrow M \oplus E(M)$, by Proposition 1.5, the map $j \circ i$ splits and hence M is direct summand of $E(M)$. Therefore M is injective.
$(ii) \Rightarrow (iii)$ assume every $FI-$semi injective module is injective. Since every simple module is $FI-$semi injective, it is injective and therefore R is $V-$ring and using the commutativity, thus R is a von-Neumann regular ring. Furthermore, every completely reducible $R-$module is $FI-$semi injective, it is injective. By [5], it follows that R is Noetherian ring if the countable direct sum of injective hulls of simple module is injective. Thus R being Noetherian and regular is semi simple Artinian.
$(iii) \Rightarrow (i)$ it is obvious.

Corollary 1.27. *Following statements are equivalent for a commutative ring R:*
(i) *The direct sum of any two semi injective module is semi injective;*
(ii) *Every semi injective module is injective;*
(iii) *R is semi simple Artinian.*

Note : Commutativity of the ring is used to prove only $(ii) \Rightarrow (iii)$ in the above Proposition and Corollary.

Proposition 1.28. *Let M be duo quasi projective, $FI-$semi injective module, then the following statements are equivalent for any $s \in End(M_R)$:*
(i) *Im(s) is $FI-M-$principally injective;*
(ii) *Im(s) is a direct summand of M;*
(iii) *Im(s) is $M-$projective.*

Proof: $(i) \Rightarrow (ii)$ follows from proposition 1.5. $(ii) \Rightarrow (iii)$ follows from the projectivity of M. $(iii) \Rightarrow (i)$ since the short exact sequence $0 \longrightarrow ker(s) \longrightarrow M \longrightarrow Im(s) \longrightarrow 0$ splits, so $Im(s)$ is isomorphic to direct summand of M. Therefore it is a direct summand by (C_2) condition. Hence it is $FI-M-$principally injective.

Remark 1.29. If every fully invariant $M-$cyclic submodule of M is a direct summand of M, then M is $FI-$semi injective module. Hence for any $R-$module M and $S = End(M_R)$, if S is von-Neumann regular, then M is $FI-$semi injective.

A right $R-$module M is called direct projective, if for any direct summand N of M, every epimorphism $f : M \longrightarrow N$ splits (i.e. $ker(f)$ is a direct summand of M). Now combining Proposition 1.5 and (37.7, [13]), we can state the following proposition;

Proposition 1.30. *Let $S = End(M_R)$ be the endomorphism ring of a module M;*
(i) *If S is von-Neumann regular, then every fully invariant $M-$cyclic submodule of M is $FI-M-$principally injective.*
(ii) *If M is direct projective and every fully invariant $M-$cyclic submodule of M is $FI-M-$principally injective, then S is von-Neumann regular.*

Proof: It is obvious, based on the definitions.

A module M is said to have the summand intersection (summand sum) property, if the intersection (sum) of two direct summand is again a direct summand. In short denoted by SIP (SSP)..

Proposition 1.31. *Every duo $FI-$semi injective module has the SIP and SSP.*

Proof: Let N and L are two direct summands of M. Take $M = N \oplus N_1 = L \oplus L_1$. Since every direct summands of M is an $M-$cyclic submodule of M, so by using Lemma 1.1(iv), L can be written as $L \cap (N \oplus N_1) = (L \cap N) \oplus (L \cap N_1)$. Hence M can be expressed as $M = (L \cap N) \oplus (L \cap N_1) \oplus L_1$, which shows that $N \cap L$ is a direct summand of M, i.e. M has the summand intersection property.
Now consider $N + L = N + (L \cap N) \oplus (L \cap N_1) = (N + (L \cap N)) \oplus (L \cap N_1) = N \oplus (L \cap N_1)$. By Proposition 1.17, $FI-$semi injective module satisfies (C_3) condition, so $N \oplus (L \cap N_1)$ is a direct summand of M. Thus $N + L$ is a direct summand of M, and hence M has the summand sum property.

A module M is called Hopfian (resp., co-Hopfian), if every surjective (resp., injective) endomorphism of M is an automorphism. For example every Noetherian module is Hopfian and every Artinian module is co-Hopfian. A module M is called weakly co-Hopfian if any injective endomorphism f of M is essential in M i.e. $f(M)$ is essential in M. A module is called directly finite, if it has no proper isomorphic direct summand.

Lemma 1.32. *(Proposition 1.25, [7]) An $R-$module M is directly finite if and only if $f \circ g = 1$ implies that $g \circ f = 1$ for any $f, g \in End(M_R)$.*

In the following Propositions we relate $FI-$semi injective module with weakly co-Hopfian, uniform and square free modules.

Proposition 1.33. *An $FI-$semi injective module M is co-Hopfian if and only if it is directly finite.*

Proof: Let f be any injective endomorphism of M and $I_M : M \longrightarrow M$ be the identity map on M. Then by $FI-$semi injectivity of M, there exists an endomorphism g of M such that $g \circ f = I_M$. Then by Lemma 1.32, we get $f \circ g = I_M$, which implies that f is surjective homomorphism. therefore M is co-Hopfian. Converse is obvious.

Corollary 1.34. *If M be $FI-$semi injective and Hopfian module, then it is co-Hopfian.*

Proof: It is well know that Hopfian and co-Hopfian module are directly finite. Then the proof can be obtained with the help of Proposition 1.33.

Corollary 1.35. *If M is an indecomposable $FI-$semi injective module, then it is co-Hopfian.*

Proof: Since every indecomposable module is directly finite, then the result follows from Proposition 1.33.

Proposition 1.36. *Every uniform module is weakly co-Hopfian. But the converse need not be true.*

Proof: Consider $f : M \longrightarrow M$ be any injective endomorphism, then $f(M)$ is $M-$cyclic submodule of M and hence essential in M because M is uniform. Therefore M is weakly co-Hopfian. For the converse we consider $Q^p = \{a/p^n | a \in \mathbb{Z}, n \geq 0\}$ and p is prime, then $M = \oplus_{p \in I} Q^p, I$ is the set of prime natural numbers. Then we can easily verify that M is weakly co-Hopfian but not uniform.

Note : Every co-Hopfian module is weakly co-Hopfian but converse need not be true, for example $\mathbb{Z}_{\mathbb{Z}}$ is weakly co-Hopfian but not co-Hopfian.

Proposition 1.37. *An $FI-$semi injective module is weakly co-Hopfian if and only if it is co-Hopfian.*

Proof: Consider $f : M \longrightarrow M$ be any injective endomorphism. Since M is $FI-$semi injective module, then the short exact sequence $0 \longrightarrow M \longrightarrow M$ splits, i.e. $f(M)$ is a direct summand of M. Also M is weakly co-Hopfian, then $f(M)$ is essential in M, which gives $f(M) = M$ i.e. f is an epimorphism, and hence M is co-Hopfian. Converse is clearly true.

Recall from [7], A module M is said to be square free, if it does not contain a direct sum of two isomorphic submodules.

Proposition 1.38. *Every square free module is uniform.*

Proof: We claim that any monomorphism $f : M \longrightarrow M$ is essential. Consider $f(M) \cap N = 0$ for some submodule N of M, since f is monomorphism so $f(N)$ is isomorphic to N and $f(N) \cap N = 0$. Since M is square free, which gives that $N = 0$. Hence $f(M)$ is essential submodule of M and so M is uniform.

Proposition 1.39. *Every square free module is weakly co-Hopfian.*

Proof: Proof follows from above Proposition 1.38 and Proposition 1.36.

Corollary 1.40. *If M is square free $FI-$semi injective module. Then M is co-Hopfian.*

Proof: Proof follows from above Propositions 1.39 and Proposition 1.37.

Proposition 1.41. *Let M be a $FI-$semi injective module and N be a fully invariant $M-$cyclic submodule which is essential in M. Then N is weakly co-Hopfian if and only if M is weakly co-Hopfian.*

Proof: Proof is similar to Proposition 2.6 [9].

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