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CHROMATICALLY UNIQUE 6-BRIDGE GRAPHS $\theta(r, r, r, s, t, u)$

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Abstract Let G and H be two graph and $P(G, x)$ and $P(H, x)$ are their chromatic polynomial, respectively. The two graphs G and H are said to be *chromatic equivalent* denoted by $G \sim H$ if $P(G, x) = P(H, x)$. A graph G is called *chromatically unique* graph if no other graph has the chromatic polynomial as the graph G. In this paper, the chromatic uniqueness of a new family of 6-bridge graph $\theta(r, r, r, s, t, u)$, where $2 \le r \le s \le t \le u$ is investigated.

1 Introduction

The graphs considered are finite and undirected graphs. For such a graph G, denote the chromatic polynomial of G. Let G and H be two graph and $P(G, x)$ and $P(H, x)$ are their chromatic polynomial, respectively. The two graphs G and H are said to be *chromatic equivalent* denoted by $G \sim H$ if $P(G, x) = P(H, x)$. A graph G is called *chromatically unique* graph if no other graph has the chromatic polynomial as the graph G. For each integer $k \ge 2$, let θ_k be the multigraph with two vertices and k edges. Any subdivision of θ_k is called the multi-bridge graph or k bridge graph. We denote $\theta(y_1, y_2, y_3, \dots, y_k)$, where $y_1, y_2, \dots, y_k \in \mathbb{N}$ and $y_1 \le y_2 \le \cdots \le y_k$ be a graph obtained by replacing the edges θ_k by paths of length $y_1, y_2, y_3, \dots, y_k$, respectively, and the girth of a graph is the length of its shortest cycle.

2 Chromaticity Of k-bridge graphs

The study on the chromaticity of k-bridge graph have been studied by many researchers. A 2-bridge graph is simply a cycle graph is χ-unique. A 3-bridge graph of the form $\theta(1, y_1, y_2)$ is called the theta graph. Chao and Whitehead [2] proved that every theta graph is χ-unique. Loerinc [17] extended the above result to all 3-bridge graphs are χ-unique. Chen et al. [22] and Xu et al. [3] solved the chromaticity of 4 bridge graph. The study on the chromaticity of 5-bridge graph has been done by the several researchers in [1, 9, 11, 12, 13, 18]. A very useful survey of the result about the chromatic uniqueness and chromatically equivalent graphs can be found in [4]. Chromaticity for k-bridge hypergraphs was studied by Bokhary et al. in [19, 20, 21]

Theorem 2.1. *(Xu et al. [22]) For $k \ge 2$, the graph $\theta_k(h)$ is χ unique.*

Theorem 2.2. *(Dong et al. [5]) If $2 \le y_1 \le y_2 \le \cdots \le y_k < y_1 + y_2$ where $k \ge 3$, then the graph $\theta(y_1, y_2, \dots, y_k)$ is χ-unique.*

Theorem 2.3. *(Dong et al. [5]) For any k, $y_1, y_2, ..., y_k \in N$,*

$$Q(\theta(y_1, y_2, ..., y_k), x) = x\prod_{i=1}^{k}(x^{y_i} - 1) - \prod_{i=1}^{k}(x^{y_i} - x) \tag{2.1}$$

Theorem 2.4. *(Dong et al. [5]) For any graph G and H,*
1. If $H \sim G$, then Q(H,x)= Q(G,x).
2. If Q(H,x)= Q(G,x) and v(H)= v(G), then $H \sim G$.

Theorem 2.5. *(Dong et al. [5]) Suppose that $\theta(y_1, y_2, \ldots, y_k) \sim \theta(y_1, y_2, ..., y_k)$ where $k \geq 3$, $2 \leq y_1 \leq y_2 \leq \cdots \leq y_k$ and $2 \leq x_1 \leq x_2 \leq \cdots \leq x_k$, then $y_i = x_i$ for all i= 1,2,3,...k.*

Theorem 2.6. *(Dong et al. [5]) Let $H \sim \theta(y_1, y_2, \ldots, y_k)$ where $k \geq 3$ and $y_i \geq 2$ for all i, then one of them is true:*
1. $H \cong \theta(y_1, y_2, \ldots, y_k)$
2. $H \in g_e(\theta(x_1, x_2, \ldots, x_k), C_{x_{i+1}}, \ldots, C_{x_{k+1}})$, where $3 \leq t \leq k-1$ and $x_i \geq 2$, for all i=1,2,3,..k.

Theorem 2.7. *(Dong et al. [5]) Let $k, t, x_1, x_2, \ldots, x_k \in N$ where $3 \leq t \leq k-1$ and $x_i \geq 2$ for all $i = 1, 2, 3, \ldots, k$. If $H \in g_e(\theta(x_1, x_2, ..., x_t), C_{x_{t+1}+1}, ..., C_{x_{k+1}})$, then*

$$Q(H, x) = x \prod_{i=1}^{k}(x^{x_i} - 1) - \prod_{i=1}^{t}(x^{x_i} - x) \prod_{i=t+1}^{k}(x^{x_i} - 1). \tag{2.2}$$

Theorem 2.8. *(Koh & Teo [15]) If $G \sim H$, then*
1. v(G)= v(H),
2. e(G)= e(H),
3. g(G)= g(H),
4. G and H have the same number of shortest cycle.
where v(G), v(H), e(G), e(H), g(G) and g(H) denote the number of vertices, the number of edges and the girth of G and H, respectively.

The chromaticity on several families of 6-bridge graph has been done by several authors which are given below.

Lemma 2.9. *[14] A 6-bridge graph $\theta(y_1, y_2, \ldots, y_6)$ is χ unique if the positive integer $y_1, y_2, \ldots, y_6$ assume exactly two distinct values.*

Lemma 2.10. *[6] The graph 6-bridge $\theta(3, 3, 3, s, s, t)$, where $r \leq s \leq t$, is χ-unique.*

Lemma 2.11. *[8] The 6-bridge graph $\theta(r, r, r, s, s, t)$, where $r \leq s \leq t$, is χ-unique.*

Lemma 2.12. *[7] The 6-bridge graph $\theta(3, 3, 3, s, t, u)$, where $3 \leq s \leq t$, is χ-unique.*

Lemma 2.13. *[9] The 6-bridge graph $\theta(r, r, s, s, t, t)$, where $r \leq s \leq t$, is χ-unique.*

Lemma 2.14. *[10] The 6-bridge graph $\theta(r, r, s, s, s, t)$, where $r \leq s \leq t$, is χ-unique.*

In this paper, we have extended this study to a new family of 6-bridge graph $\theta(r, r, r, s, t, u)$ where $2 \leq r \leq s \leq t \leq u$ and showed that this family of 6-bridge graph is chromatically unique.

3 Chromatically unique 6-bridge graph $\theta(r, r, r, s, t, u)$

In this section we present our main result on the chromaticity of 6- bridge graph.

Theorem 3.1. *The 6-bridge graph $\theta(r, r, r, s, t, u)$ where $r \leq s \leq t \leq u$ is chromatically unique.*

Proof. Let G be the 6-bridge graph of the form $\theta(r, r, r, s, t, u)$ and $2 \leq r \leq s \leq t \leq u$. By Theorem 2.2, G is χ unique if $u < 2r$. Suppose $r \geq 2$ and $H \sim G$, we shall solve $Q(G) = Q(H)$ to get all the solutions. Let the lowest remaining power and the highest remaining power be denoted by l.r.p and h.r.p, respectively. By Theorem 2.8, $g(G) = g(H) = 2r$ and H has the same number of shortest cycles as G. Thus, we have

$$3r + s + t + u = x_1 + x_2 + x_3 + x_4 + x_5 + x_6. \tag{3.1}$$

By Theorem 2.6 and 2.7, there are three cases to consider, that are
$H \in g_e(\theta(x_1, x_2, x_3), C_{x_4+1}, C_{x_5+1}, C_{x_6+1})$, where $2 \leq x_1 \leq x_2 \leq x_3$ and $2 \leq x_4, x_5, x_6$, or
$H \in g_e(\theta(x_1, x_2, x_3, x_4), C_{x_5+1}, C_{x_6+1})$, where $2 \leq x_1 \leq x_2 \leq x_3 \leq x_4$ and $2 \leq x_5, x_6$, or
$H \in g_e(\theta(x_1, x_2, x_3, x_4, x_5), C_{x_6+1})$, where $2 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$ and $2 \leq x_6$.

Case A

$H \in g_e(\theta(x_1,x_2,x_3), C_{x_4+1}, C_{x_5+1}, C_{x_6+1})$, where $2 \le x_1 \le x_2 \le x_3$ and $2 \le x_4, x_5, x_6$. As $G \cong \theta(r,r,r,s,t,u)$ and $H \in g_e(\theta(x_1,x_2,x_3), C_{x_4+1}, C_{x_5+1}, C_{x_6+1})$, then by Theorem 2.7, we have

$Q(G) = x(x^r-1)^3(x^s-1)(x^t-1)(x^u-1) - (x^r-x)^3(x^s-x)(x^t-x)(x^u-x)$.

$Q(H) = x(x^{x_1}-1)(x^{x_2}-1)(x^{x_3}-1)(x^{x_4}-1)(x^{x_5}-1)(x^{x_6}-1) - (x^{x_1}-x)(x^{x_2}-x)(x^{x_3}-x)(x^{x_4}-1)(x^{x_5}-1)(x^{x_6}-1)$.

By using Equation 3, $Q(G)=Q(H)$ yields

$Q_1(G) = x^{3r+s+1}+x^{3r+t+1}+x^{3r+u+1}+x^{s+t+1}+x^{s+u+1}+x^{t+u+1}+3x^{2r+s+t+1}+3x^{2r+s+u+1}+3x^{2r+t+u+1}+3x^{2r+1}+3x^{r+s+t+u+1}+3x^{r+s+1}+3x^{r+t+1}+3x^{r+u+1}+x^{3r+3}+x^{s+t+u+3}+x^{s+5}+x^{t+5}+x^{u+5}+3x^{2r+s+3}+3x^{2r+t+3}+3x^{2r+u+3}+3x^{r+s+t+3}+3x^{r+s+u+3}+3x^{s+t+u+3}+3x^{r+5}-(x^{3r+1}+x^{s+t+u+1}+x^{s+1}+x^{t+1}+x^{u+1}+3x^{2r+s+1}+3x^{2r+t+1}+3x^{2r+u+1}+3x^{r+s+t+1}+3x^{r+s+u+1}+3x^{r+t+u+1}+3x^{r+1}+x^{3r+s+2}+x^{3r+t+2}+x^{3r+u+2}+x^{s+t+4}+x^{s+u+4}+x^{s+t+4}+x^{t+u+4}+3x^{2r+s+t+2}+3x^{2r+s+u+2}+3x^{2r+t+u+2}+3x^{r+s+t+u+2}+3x^{r+s+4}+3x^{r+t+4}+3x^{r+u+4}+3x^{2r+4}+x^6)$.

$Q_1(H) = x^{x_1+x_2+x_3+x_4+x_5}+x^{x_1+x_2+x_3+x_4+x_6}+x^{x_1+x_2+x_3+x_4+1}+x^{x_1+x_2+x_3+x_5+x_6}+x^{x_1+x_2+x_3+x_5+1}+x^{x_1+x_2+x_3+x_6+1}+x^{x_1+x_2+x_3}+x^{x_1+x_4+x_5+x_6+1}+x^{x_1+x_4+x_5+2}+x^{x_1+x_4+x_6+2}+x^{x_1+x_5+x_6+2}+x^{x_1+x_4+1}+x^{x_1+x_5+1}+x^{x_1+x_6+1}+x^{x_1+2}+x^{x_2+x_4+x_5+x_6+1}+x^{x_2+x_4+x_5+2}+x^{x_2+x_4+x_6+2}+x^{x_2+x_4+1}+x^{x_2+x_5+x_6+2}+x^{x_2+x_5+1}+x^{x_2+x_6+1}+x^{x_2+2}+x^{x_3+x_4+x_5+x_6+1}+x^{x_3+x_4+x_5+2}+x^{x_3+x_4+x_6+2}+x^{x_3+x_4+1}+x^{x_3+x_5+x_6+2}+x^{x_3+x_5+1}+x^{x_3+x_6+1}+x^{x_3+2}+x^{x_4+x_5+x_6+3}+x^{x_4+x_5+1}+x^{x_4+x_6+1}+x^{x_4+3}+x^{x_5+x_6+1}+x^{x_5+3}+x^{x_6+3}-(x^{x_1+x_2+x_3+x_4+x_5+1}+x^{x_1+x_2+x_3+x_4+x_6+1}+x^{x_1+x_2+x_3+x_4}+x^{x_1+x_2+x_3+x_5+x_6+1}+x^{x_1+x_2+x_3+x_5}+x^{x_1+x_2+x_3+x_6}+x^{x_1+x_2+x_3+1}+x^{x_1+x_4+x_5+x_6+2}+x^{x_1+x_4+x_5+1}+x^{x_1+x_4+x_6+1}+x^{x_1+x_5+x_6+1}+x^{x_1+x_4+2}+x^{x_1+x_5+2}+x^{x_1+x_6+2}+x^{x_1+1}+x^{x_2+x_4+x_5+x_6+2}+x^{x_2+x_4+x_5+1}+x^{x_2+x_4+x_6+1}+x^{x_2+x_4+2}+x^{x_2+x_5+x_6+2}+x^{x_2+x_5+2}+x^{x_2+x_6+2}+x^{x_2+1}+x^{x_3+x_4+x_5+x_6+2}+x^{x_3+x_4+x_5+1}+x^{x_3+x_4+x_6+1}+x^{x_3+x_4+2}+x^{x_3+x_5+x_6+1}+x^{x_3+x_5+2}+x^{x_3+x_6+2}+x^{x_3+1}+x^{x_4+x_5+x_6+1}+x^{x_4+x_5+3}+x^{x_4+x_6+3}+x^{x_4+1}+x^{x_5+x_6+3}+x^{x_5+1}+x^{x_6+1}+x^3)$.

By comparing the l.r.p of $Q_1(G)$ and the l.r.p of $Q_1(H)$, we get $r=2$. Thus, $g(G)=g(H)=2r=4$. Since G has 3 cycles of length four, therefore H also has three cycles of length 4. Without loss of generality, we have four cases to consider,

1. $x_4=x_5=x_6=3$, or
2. $x_4=x_5=3, x_6 \neq 3$, or
3. $x_4=3, x_5\neq 3, x_6\neq 3$, or
4. $x_4\neq 3, x_5\neq 3, x_6\neq 3$.

Case1 :

$x_4=x_5=x_6=3$.

Note that, for $r=2$, the l.r.p in $Q_1(G)$ is $-3x^3$ and l.r.p in $Q_1(H)$ is $-x^3$. Thus, we have either $x_1=x_2=2$, or $x_1=x_3=2$, or $x_2=x_3=2$.

Case1.1 :

If $x_1=x_2=2$ then H has four cycles of length 4, a contradiction.

Case1.2 :

If $x_1=x_3=2$ then so is $x_2=2$. This implies that, H has six cycles of length 4, a contradiction.

Case1.3 :

If $x_2=x_3=2$ then $x_1=2$. This implies that, H has six cycles of length 4, a contradiction.

Case2 :

$x_4=x_5=3, x_6\neq 3$.

Since the girth of H is 4 therefore $x_6 \ge 4$. Given that H has three cycles of length 4, so $x_1+x_2=4$, implies that $x_1=x_2=2$. Thus, Equation 3 becomes $s+t+u=x_3+x_6+4$. By using this, we have

$Q_2(G) = 4x^{s+7}+4x^{t+7}+4x^{u+7}+x^{s+t+1}+x^{s+u+1}+x^{t+u+1}+6x^{s+t+5}+6x^{s+u+5}+6x^{t+u+5}+3x^{s+3}+3x^{t+3}+3x^{u+3}+x^{s+5}+x^{t+5}+x^{u+5}+x^9+2x^7+3x^5-(x^{s+t+u+1}+x^{s+1}+x^{t+1}+x^{u+1}+3x^{s+5}+3x^{t+5}+3x^{u+5}+3x^{s+t+3}+3x^{s+u+3}+3x^{t+u+3}+x^{s+8}+x^{t+8}+x^{u+8}+x^{s+t+4}+x^{s+u+4}+x^{t+u+4}+3x^{s+t+6}+3x^{s+u+6}+3x^{t+u+6}+3x^{s+6}+3x^{t+6}+3x^{u+6}+3x^8+x^6)$

$Q_2(H) = 3x^{x_3+8}+3x^{x_3+10}+3x^{x_3+4}+3x^{x_3+2}+3x^{x_6+9}+3x^{x_6+7}+3x^{x_6+3}+3x^{x_3+x_6+5}+x^{x_3+x_6+1}+2x^{10}+6x^6-(3x^{x_3+7}+3x^{x_3+5}+x^{x_3+11}+x^{x_3+1}+2x^{x_6+10}+6x^{x_6+6}+x^{x_6+1}+3x^{x_3+x_6+4}+x^{x_3+x_6+2}+3x^9+3x^7)$.

Considering the l.r.p in $Q_2(G)$ and the l.r.p in $Q_2(H)$, we have $s=t=u=4$. Thus,

$G \cong (2,2,2,4,4,4)$ and is χ-unique by Lemma 2.9.

Case3 :

$x_4 = 3, x_5 \neq 3, x_6 \neq 3.$

Since the girth of H is 4, therefore $x_5 \geq 4$ and $x_6 \geq 4$. It is given that H has three cycles of length 4, so $x_1 + x_2 = 4$, $x_1 + x_3 = 4$, implying that $x_1 = x_2 = x_3 = 2$. But, in this case H has four cycles of length 4, a contradiction.

Case4 :

$x_4 \neq 3, x_5 \neq 3, x_6 \neq 3$

Since the girth of H is 4, therefore $x_4, x_5, x_6 \geq 4$. It is given that H has three cycles of length 4, so $x_1 + x_2 = 4$, $x_1 + x_3 = 4$ and $x_2 + x_3 = 4$ implying that $x_1 = x_2 = x_3 = 2$. Thus, Equation 3 becomes $s + t + u = x_4 + x_5 + x_6$. By using this, we have

$Q_3(G) = 4x^{s+7} + 4x^{t+7} + 4x^{u+7} + x^{s+t+1} + x^{s+u+1} + x^{t+u+1} + 6x^{s+t+5} + 6x^{s+u+5} + 6x^{t+u+5} + 3x^{s+3} + 3x^{t+3} + 3x^{u+3} + x^{s+5} + x^{t+5} + x^{u+5} + x^9 + 2x^7 + 3x^5 - (x^{s+t+u+1} + x^{s+1} + x^{t+1} + x^{u+1} + 3x^{s+5} + 3x^{t+5} + 3x^{u+5} + 3x^{s+t+3} + 3x^{s+u+3} + 3x^{t+u+3} + x^{s+8} + x^{t+8} + x^{u+8} + x^{s+t+4} + x^{s+u+4} + x^{t+u+4} + 3x^{s+t+6} + 3x^{s+u+6} + 3x^{t+u+6} + 3x^{s+6} + 3x^{t+6} + 3x^{u+6} + 3x^8 + x^6).$

$Q_3(H) = x^{x_4+x_5+6} + x^{x_4+x_6+6} + x^{x_5+x_6+6} + x^{x_4+7} + x^{x_5+7} + x^{x_6+7} + 3x^{x_4+x_5+4} + 3x^{x_4+x_6+4} + 3x^{x_5+x_6+4} + 4x^{x_4+3} + 4x^{x_5+3} + 4x^{x_6+3} + x^{x_4+x_5+1} + x^{x_4+x_6+1} + x^{x_5+x_6+1} + x^6 + 3x^4 - (x^{x_4+x_5+7} + x^{x_4+x_6+7} + x^{x_5+x_6+7} + x^{x_4+6} + x^{x_5+6} + x^{x_6+6} + 4x^{x_4+x_5+3} + 4x^{x_4+x_6+3} + 4x^{x_5+x_6+3} + 3x^{x_4+4} + 3x^{x_5+4} + 3x^{x_6+4} + x^{x_4+1} + x^{x_5+1} + x^{x_6+1} + x^7 + x^3).$

By comparing the l.r.p in $Q_3(G)$ and $Q_3(H)$, we have $s = 2$, or $t = 2$, or $u = 2$.

If $s = 2$, then $G \cong \theta(2,2,2,2,t,u)$ and is χ-unique.

If $t = 2$, then $G \cong \theta(2,2,2,2,2,u)$ and is χ-unique by Lemma 2.9.

If $u = 2$, then $G \cong \theta(2,2,2,2,2,2)$ and is χ-unique by Theorem 2.1.

CaseH :

In this case, $H \in g_e(\theta(x_1,x_2,x_3,x_4), C_{x_5+1}, C_{x_6+1})$, where $2 \leq x_1 \leq x_2 \leq x_3 \leq x_4$ and $2 \leq x_5, x_6$. Since, $G \cong \theta(r,r,r,s,t,u)$ and $H \in g_e(\theta(x_1,x_2,x_3,x_4), C_{x_5+1}, C_{x_6+1})$, therefore by Theorem 2.2, we have

$Q_4(G)= x(x^r-1)^3(x^s-1)(x^t-1)(x^u-1) - (x^r-x)^3(x^s-x)(x^t-x)(x^u-x)$

$Q_4(H)= x(x^{x_1}-1)(x^{x_2}-1)(x^{x_3}-1)(x^{x_4}-1)(x^{x_5}-1)(x^{x_6}-1) - (x^{x_1}-x)(x^{x_2}-x)(x^{x_3}-x)(x^{x_4}-x)(x^{x_5}-1)(x^{x_6}-1)$

By using Equation 3, we have

$Q_5(G) = x^{3r+s+1} + x^{3r+t+1} + x^{3r+u+1} + x^{s+t+1} + x^{s+u+1} + x^{t+u+1} + 3x^{2r+s+t+1} + 3x^{2r+s+u+1} + 3x^{2r+t+u+1} + 3x^{2r+1} + 3x^{r+s+t+u+1} + 3x^{r+s+1} + 3x^{r+t+1} + 3x^{r+u+1} + x^{3r+3} + x^{s+t+u+3} + x^{s+5} + x^{t+5} + x^{u+5} + 3x^{2r+s+3} + 3x^{2r+t+3} + 3x^{2r+u+3} + 3x^{r+s+t+3} + 3x^{r+s+u+3} + 3x^{s+t+u+3} + 3x^{r+5} - (x^{3r+1} + x^{s+t+u+1} + x^{s+1} + x^{t+1} + x^{u+1} + 3x^{2r+s+1} + 3x^{2r+t+1} + 3x^{2r+u+1} + 3x^{r+s+t+1} + 3x^{r+s+u+1} + 3x^{r+t+u+1} + 3x^{r+1} + x^{3r+s+2} + x^{3r+t+2} + x^{3r+u+2} + x^{s+t+4} + x^{s+u+4} + x^{s+t+4} + x^{t+u+4} + 3x^{2r+s+t+2} + 3x^{2r+s+u+2} + 3x^{2r+t+u+2} + 3x^{r+s+t+u+2} + 3x^{r+s+4} + 3x^{r+t+4} + 3x^{r+u+4} + 3x^{2r+4} + x^6).$

$Q_5(H) = x^{x_1+x_2+x_3+x_4+x_5} + x^{x_1+x_2+x_3+x_4+x_6} + x^{x_1+x_2+x_3+x_4+1} + x^{x_1+x_2+x_5+x_6+1} + x^{x_1+x_2+x_5+2} + x^{x_1+x_2+x_6+2} + x^{x_1+x_2+1} + x^{x_1+x_3+x_5+x_6+1} + x^{x_1+x_3+x_5+2} + x^{x_1+x_3+x_6+2} + x^{x_1+x_3+1} + x^{x_1+x_4+x_5+x_6+1} + x^{x_1+x_4+x_5+2} + x^{x_1+x_4+x_6+2} + x^{x_1+x_4+1} + x^{x_1+x_5+x_6+3} + x^{x_1+x_5+1} + x^{x_1+x_6+1} + x^{x_1+3} + x^{x_2+x_3+x_5+x_6+1} + x^{x_2+x_3+x_5+2} + x^{x_2+x_3+x_6+2} + x^{x_2+x_3+1} + x^{x_2+x_4+x_5+x_6+1} + x^{x_2+x_4+x_5+2} + x^{x_2+x_4+x_6+2} + x^{x_2+x_4+1} + x^{x_2+x_5+x_6+3} + x^{x_2+x_5+1} + x^{x_2+x_6+1} + x^{x_2+3} + x^{x_3+x_4+x_5+x_6+1} + x^{x_3+x_4+x_5+2} + x^{x_3+x_4+x_6+2} + x^{x_3+x_4+1} + x^{x_3+x_5+x_6+2} + x^{x_3+x_5+1} + x^{x_3+x_6+1} + x^{x_3+3} + x^{x_4+x_5+x_6+3} + x^{x_4+x_5+1} + x^{x_4+x_6+1} + x^{x_4+3} + x^{x_5+x_6+1} + x^{x_5+4} + x^{x_6+4} - (x^{x_1+x_2+x_3+x_4+x_5+1} + x^{x_1+x_2+x_3+x_4+x_6+1} + x^{x_1+x_2+x_3+x_4} + x^{x_1+x_2+x_3+x_5+x_6+2} + x^{x_1+x_2+x_5+1} + x^{x_1+x_2+x_6+1} + x^{x_1+x_2+2} + x^{x_1+x_3+x_5+x_6+2} + x^{x_1+x_3+x_5+1} + x^{x_1+x_3+x_6+1} + + x^{x_1+x_3+2} + x^{x_1+x_4+x_5+x_6+2} + x^{x_1+x_4+x_5+1} + x^{x_1+x_4+x_6+1} + x^{x_1+x_5+x_6+1} + x^{x_1+x_4+2} + x^{x_1+x_5+3} + x^{x_1+x_6+3} + x^{x_1+1} + x^{x_2+x_3+x_5+x_6+2} + x^{x_2+x_3+x_5+1} + x^{x_2+x_3+x_6+1} + x^{x_2+x_3+2} + x^{x_2+x_4+x_5+x_6+2} + x^{x_2+x_4+x_5+1} + x^{x_2+x_4+x_6+1} + x^{x_2+x_4+2} + x^{x_2+x_5+x_6+1} + x^{x_2+x_5+3} + x^{x_2+x_6+3} + x^{x_2+1} + x^{x_3+x_4+x_5+x_6+2} + x^{x_3+x_4+x_5+1} + x^{x_3+x_4+x_6+1} + x^{x_3+x_4+2} + x^{x_3+x_5+x_6+1} + x^{x_3+x_5+3} + x^{x_3+x_6+3} + x^{x_3+1} + x^{x_4+x_5+x_6+1} + x^{x_4+x_5+3} + x^{x_4+x_6+3} + x^{x_4+1} + x^{x_5+x_6+4} + x^{x_5+1} + x^{x_6+1} + x^4).$

since $2 \leq r \leq s \leq t \leq u$, therefore by comparing the l.r.p in $Q_5(G)$ and the l.r.p in $Q_5(H)$, we have $r = 2$, or $r = 3$.

Case1 :

If $r = 2$, then $g(G) = g(H) = 2r = 4$. Since, G has three cycles of length 4, therefore H also has three cycles of length 4. Without loss of generality, we have the following three cases:

1. $x_5 = x_6 = 3$, or

2. $x_5 = 3, x_6 \neq 3$, or
3. $x_5 \neq 3, x_6 \neq 3$.

Case1.1 :
$x_5 = x_6 = 3$.
Since, H has three cycles of length 4 therefore $x_1 + x_2 = 4$ implying that $x_1 = x_2 = 2$.
Note that, for $r = 2$, the l.r.p in $Q_1(G)$ is $-3x^3$ and l.r.p in $Q_1(H)$ is $-x^3$. Thus, either $x_3 = 2$ or $x_4 = 2$.
If $x_3 = 2$ then H has five cycles of length 4, a contradiction.
If $x_4 = 2$, then $x_3 = 2$ implying that H has eight cycles of length 4, a contradiction.

Case1.2 :
$x_5 = 3, x_6 \neq 3$.
Since the girth of H is 4 therefore $x_6 \geq 4$. It is given that H has three cycles of length 4, so $x_1 + x_2 = 4$, $x_1 + x_3 = 4$ implying that $x_1 = x_2 = x_3 = 2$. But, then H has four cycles of length 4, a contradiction.

Case1.3 :
$x_5 \neq 3, x_6 \neq 3$.
Since the girth of H is 4 therefore $x_5,\ x_6 \geq 4$. It is given that H has three cycles of length 4, so $x_1 + x_2 = 4$, $x_1 + x_3 = 4$ and ($x_1 + x_4 = 4$ or $x_2 + x_3 = 4$) implying that $x_1 = x_2 = x_3 = 2$. We have two cases to consider.

Case1.3.1 :
If $x_1 + x_4 = 4$ then $x_1 = x_2 = 2$ implying that $x_1 = x_2 = x_3 = x_4 = 2$. But, then H has six cycles of length 4, a contradiction.

Case1.3.2 :
If $x_2 + x_3 = 4$ then $x_1 = x_2 = x_3 = 2$. In this case, Equation 3 becomes $s+t+u = x_4+x_5+x_6$ and we obtain the following after simplification,
$Q_6(G) = 4x^{s+7} + 4x^{t+7} + 4x^{u+7} + x^{s+t+1} + x^{s+u+1} + x^{t+u+1} + 6x^{s+t+5} + 6x^{s+u+5} + 6x^{t+u+5} + 3x^{s+3} + 3x^{t+3} + 3x^{u+3} + x^{s+5} + x^{t+5} + x^{u+5} + x^9 + 2x^7 - (x^{s+1} + x^{t+1} + x^{u+1} + 3x^{s+5} + 3x^{t+5} + 3x^{u+5} + 3x^{s+t+3} + 3x^{s+u+3} + 3x^{t+u+3} + x^{s+8} + x^{t+8} + x^{u+8} + x^{s+t+4} + x^{s+u+4} + x^{t+u+4} + 3x^{s+t+6} + 3x^{s+u+6} + 3x^{t+u+6} + 3x^{s+6} + 3x^{t+6} + 3x^{u+6} + 3x^8)$.
$Q_6(H) = x^{x_4+x_5+6} + x^{x_4+x_6+6} + 3x^{x_4+x_5+4} + 3x^{x_4+x_6+4} + x^{x_4+7} + 6x^{x_5+x_6+5} + 3x^{x_5+6} + 3x^{x_6+6} + 4x^{x_4+3} + 3x^{x_5+3} + 3x^{x_6+3} + x^{x_4+x_5+1} + x^{x_4+x_6+1} + x^{x_5+x_6+1} + x^{x_5+4} + x^{x_6+4} + 3x^5 - (x^{x_4+x_5+7} + x^{x_4+x_6+7} + x^{x_4+6} + 3x^{x_5+x_6+6} + 6x^{x_5+5} + 6x^{x_6+5} + 4x^{x_4+x_5+3} + 4x^{x_4+x_6+3} + 3x^{x_5+x_6+3} + 3x^{x_4+4} + x^{x_5+x_6+4} + x^{x_4+1} + x^{x_5+1} + x^{x_6+1} + 2x^6 + x^4)$.
By comparing the l.r.p in $Q_6(G)$ and the l.r.p in $Q_6(H)$, we have $s = 3$, or $t = 3$, or $u = 3$.
If $s = 3$, then
$Q_7(G) = 3x^{t+7} + 3x^{u+7} + x^{t+4} + x^{u+4} + x^{t+u+1} + 5x^{t+8} + 5x^{u+8} + 6x^{t+u+5} + 3x^{t+3} + 3x^{u+3} + 4x^{10} + x^9 + 2x^7 + 3x^6 - (x^{t+1} + x^{u+1} + 2x^{t+5} + 2x^{u+5} + 6x^{t+6} + 6x^{u+6} + 3x^{t+u+3} + x^{t+u+4} + 3x^{t+u+6} + 3x^{t+9} + 3x^{u+9} + x^{11} + 3x^9 + 5x^8)$
$Q_7(H) = x^{x_4+x_5+6} + x^{x_4+x_6+6} + 3x^{x_4+x_5+4} + 3x^{x_4+x_6+4} + x^{x_4+7} + 6x^{x_5+x_6+5} + 3x^{x_5+6} + 3x^{x_6+6} + 4x^{x_4+3} + 3x^{x_5+3} + 3x^{x_6+3} + x^{x_4+x_5+1} + x^{x_4+x_6+1} + x^{x_5+x_6+1} + x^{x_5+4} + +x^{x_6+4} + 3x^5 - (x^{x_4+x_5+7} + x^{x_4+x_6+7} + x^{x_4+6} + 3x^{x_5+x_6+6} + 6x^{x_5+5} + 6x^{x_5+5} + 4x^{x_4+x_5+3} + 4x^{x_4+x_6+3} + 3x^{x_5+x_6+3} + 3x^{x_4+4} + x^{x_5+x_6+4} + x^{x_4+1} + x^{x_5+1} + x^{x_6+1} + 2x^6)$.
By comparing the l.r.p in $Q_7(G)$ and the l.r.p in $Q_7(H)$, we have $x_4 = x_5 = x_6 = 4$. Replace these values in $Q_7(G)$ and $Q_7(H)$, we get
$Q_8(G) = 3x^{t+7} + 3x^{u+7} + x^{t+4} + x^{u+4} + x^{t+u+1} + 5x^{t+8} + 5x^{u+8} + 6x^{t+u+5} + 3x^{t+3} + 3x^{u+3} + 4x^{10} + x^9 + 2x^7 + 3x^6 - (x^{t+1} + x^{u+1} + 2x^{t+5} + 2x^{u+5} + 6x^{t+6} + 6x^{u+6} + 3x^{t+u+3} + x^{t+u+4} + 3x^{t+u+6} + 3x^{t+9} + 3x^{u+9} + x^{11} + 3x^9 + 5x^8)$
$Q_8(H) = 6x^{13} + 5x^{10} + 5x^{12} + 10x^7 - (2x^{15} + x^{14} + 9x^9 + 10x^{11} + x^8 + 2x^6)$.
But, $Q_8(G) \neq Q_8(H)$, a contradiction. Similarly, we can get contradiction for the case when $t = 3$ and $u = 3$.

Case2 :
If $r = 3$, then $g(G) = g(H) = 2r = 6$. Since, G has three cycles of length 6 therefore H also has three cycles of length 6. Without loss of generality, we have three cases to consider:
1. $x_5 = x_6 = 5$ or
2. $x_5 = 5, x_6 \neq 5$ or
3. $x_5 \neq 5, x_6 \neq 5$

Case2.1 :

$x_5 = x_6 = 5$
It follows that $x_1 + x_2 = 6$. Thus, we have either $x_1 = 2$, $x_2 = 4$ or $x_1 = x_2 = 3$
Case2.1.1 :
$x_1 = 2, x_2 = 4$
It follows from Equation 3 that $s + t + u = x_3 + x_4 + 7$. Since $3 \leq s \leq t \leq u$ and $4 \leq x_3 \leq x_4$ therefore by canceling the equal terms of $Q_8(H)$ and $Q_8(G)$, we can get a term $-x^3$ in $Q_8(H)$ but not in $Q_8(G)$, a contradiction.
Case2.1.2 :
$x_1 = x_2 = 3$
It follows from Equation 3 that $s + t + u = x_3 + x_4 + 7$ and we have
$Q_9(G) = x^{s+10} + x^{t+10} + x^{u+10} + x^{s+t+1} + x^{s+u+1} + x^{t+u+1} + 3x^{s+t+7} + 3x^{s+u+7} + 3x^{t+u+7} + x^{s+4} + x^{t+4} + x^{u+4} + x^{s+t+u} + x^{s+5} + x^{t+5} + x^{u+5} + +3x^{s+9} + 3x^{t+9} + 3x^{u+9} + 3x^{s+t+6} + 3x^{s+u+6} + 3x^{t+u+6} + x^{12} + x^8 + 2x^7 - (x^{s+t+u+1} + x^{s+1} + x^{t+1} + x^{u+1} + 6x^{s+7} + 6x^{t+7} + 6x^{u+7} + 4x^{s+t+4} + 4x^{s+u+4} + 4x^{t+u+4} + x^{s+11} + x^{t+11} + x^{u+11} + 3x^{s+t+8} + 3x^{s+u+8} + 3x^{t+u+8} + 4x^{10} + x^6)$.
$Q_9(H) = 3x^{x_3+x_4+7} + x^{x_3+x_4+1} + x^{x_3+13} + x^{x_3+3} + 4x^{x_3+10} + x^{x_4+13} + x^{x_4+3} + 4x^{x_4+10} + 2x^{x_3+14} + 2x^{x_3+4} + 2x^{x_4+4} + 2x^{x_4+14} + 2x^{x_3+6} + 2x^{x_4+6} + x^{17} + 2x^{16} + 2x^{13} + 6x^9 - (3x^{x_3+x_4+6} + 2x^{x_3+15} + 4x^{x_3+9} + 2x^{x_3+5} + x^{x_4+15} + 4x^{x_4+9} + 2x^{x_4+5} + 2x^{x_3+8} + 2x^{x_4+8} + x^{x_3+x_4+2} + x^{x_3+11} + x^{x_3+1} + x^{x_4+11} + x^{x_4+1} + x^{18} + 3x^{14} + 3x^{11} + 2x^{12} + x^8)$. By comparing the l.r.p in $Q_9(G)$ and the l.r.p in $Q_9(H)$, we have $x_3 = 5$ or $x_4 = 5$.
Case2.1.2.1 :
If $x_3 = 5$ then we obtain the following after simplification,
$Q_{10}(G) = x^{s+10} + x^{t+10} + x^{u+10} + x^{s+t+1} + x^{s+u+1} + x^{t+u+1} + 3x^{s+t+7} + 3x^{s+u+7} + 3x^{t+u+7} + x^{s+4} + x^{t+4} + x^{u+4} + x^{s+t+u} + x^{s+5} + x^{t+5} + x^{u+5} + +3x^{s+9} + 3x^{t+9} + 3x^{u+9} + 3x^{s+t+6} + 3x^{s+u+6} + 3x^{t+u+6} + x^{12} + x^8 + 2x^7 - (x^{s+t+u+1} + x^{s+1} + x^{t+1} + x^{u+1} + 6x^{s+7} + 6x^{t+7} + 6x^{u+7} + 4x^{s+t+4} + 4x^{s+u+4} + 4x^{t+u+4} + x^{s+11} + x^{t+11} + x^{u+11} + 3x^{s+t+8} + 3x^{s+u+8} + 3x^{t+u+8} + 4x^{10})$.
$Q_{10}(H) = 2x^{x_4+14} + x^{x_4+13} + 3x^{x_4+12} + 4x^{x_4+10} + 3x^{x_4+6} + 2x^{x_4+4} + x^{x_4+3} + 2x^{19} + x^{17} + x^{16} + 4x^{15} + 8x^9 - (2x^{x_4+15} + 2x^{x_4+8} + 4x^{x_4+11} + 4x^{x_4+9} + x^{x_4+7} + x^{x_4+1} + 2x^{x_4+5} 2x^{20} + 7x^{14} + 2x^{12} + x^{11})$.
But, $Q_{10}(G) \neq Q_{10}(H)$, a contradiction.
Case2.1.2.2 :
If $x_4 = 5$ then we have either $x_3 = 3$ or $x_3 = 4$ or $x_3 = 5$.
Case2.1.2.2(a) :
If $x_3 = 3$ then H has five cycles of length 6, a contradiction.
Case2.1.2.2(b) :
If $x_3 = 4$ then by simplifying $Q_{10}(H)$, we get $-x^5$ as l.r.p in $Q_{12}(H)$. Which implies that either $s = 4$ or $t = 4$ or $u = 4$.
• If $s = 4$ then Equation 3 becomes $t + u = 12$. Since $3 \leq 4 \leq t \leq u$, we have following possibilities:
(i) If $t = 4$ and $u = 8$ then $G \cong \theta(3, 3, 3, 4, 4, 8)$ and is χ-unique by Lemma 2.10.
(ii) If $t = 5, u = 7$ then
$Q_{11}(G) = 2x^{16} + 5x^{18} + 3x^8 + 4x^9 + 2x^7 - (6x^{11} + 2x^{12} + 2x^{14} + x^{15} + 3x^{20} + 2x^{10} + x^6)$.
$Q_{11}(H) = 5x^{16} + 4x^{15} + 3x^{18} + 3x^8 + 6x^9 + 3x^{10} + 2x^{11} + 2x^{17} + x^7 - (4x^{15} + 2x^9 + 2x^{20} + 2x^{10} + 4x^{11} + 4x^{12} + x^{16} + x^{18} + 3x^{14} + x^8)$.
$Q_{11}(G) \neq Q_{11}(H)$, a contradiction.
(iii) If $t = u = 6$ then again we get $Q_{11}(G) \neq Q_{11}(H)$, a contradiction.
• If $t = 4$ then Equation 3 becomes $s + u = 12$. Since $3 \leq s \leq 4 \leq u$, we have following possibilities:
(i) if $s = 3$ and $u = 9$ then $G \cong \theta(3, 3, 3, 3, 4, 9)$ and is χ-unique by Lemma 2.10.
(ii) If $s = 4$ and $u = 8$ then $G \cong \theta(3, 3, 3, 4, 4, 8)$ and is χ-unique by Lemma 2.10.
• If $u = 4$ then Equation 3 becomes $s + t = 12$. But $3 \leq s \leq t \leq 4$, a contradiction.
Case2.1.2.2(c) :
If $x_3 = 5$ then Equation 3 becomes $s + t + u = 17$ and we get,
$Q_{12}(G) = x^{s+10} + x^{t+10} + x^{u+10} + x^{s+t+1} + x^{s+u+1} + x^{t+u+1} + 3x^{s+t+7} + 3x^{s+u+7} + 3x^{t+u+7} + x^{s+4} + x^{t+4} + x^{u+4} + x^{s+t+u} + x^{s+5} + x^{t+5} + x^{u+5} + +3x^{s+9} + 3x^{t+9} + 3x^{u+9} + 3x^{s+t+6} + 3x^{s+u+6} + 3x^{t+u+6} + x^{12} + 2x^7 - (x^{s+t+u+1} + x^{s+1} + x^{t+1} + x^{u+1} + 6x^{s+7} + 6x^{t+7} + 6x^{u+7} + 4x^{s+t+4} + 4x^{s+u+4} + 4x^{t+u+4} + x^{s+11} + x^{t+11} + x^{u+11} + 3x^{s+t+8} + 3x^{s+u+8} + 3x^{t+u+8})$.
$Q_{12}(H) = 4x^{17} + 8x^{15} + 4x^{19} + 10x^9 + 2x^{11} + x^{18} - (3x^{16} + 4x^{20} + 11x^{14} + x^6 + 3x^{12} + 3x^{13})$.
By comparing the l.r.p in $Q_{12}(G)$ and the l.r.p in $Q_{12}(H)$, we get $s = 5$ or $t = 5$ or $u = 5$

(i) If $s = 5$ then $G \cong \theta(3, 3, 3, 5, t, u)$ and is χ-unique by Lemma 2.12.
(ii) If $t = 5$ then either $s = 3$ or $s = 4$ or $s = 5$.
(a) If $s = 3$ then $G \cong \theta(3, 3, 3, 3, 5, u)$ and is χ-unique by Lemma 2.12.
(b) If $s = 4$ then $G \cong \theta(3, 3, 3, 4, 5, u)$ and is χ-unique by Lemma 2.12.
(c) If $s = 5$ then $G \cong \theta(3, 3, 3, 5, 5, u)$ and is χ-unique by Lemma 2.10.
(iii) If $u = 5$ then we have the following possibilities, either $s = t = 3$ or $s = 3, t = 4$ or $s = t = 4$ or $s = t = 5$:
(a) If $s = t = 3$ then $G \cong \theta(3, 3, 3, 3, 3, 5)$ and is χ-unique by Lemma 2.9.
(b) If $s = 3, t = 4$ then $G \cong \theta(3, 3, 3, 3, 4, 5)$ and is χ-unique by Lemma 2.10.
(c) If $s = t = 4$ then $G \cong \theta(3, 3, 3, 4, 4, 5)$ and is χ-unique by Lemma 2.10.
(d) If $s = t = 5$ then $G \cong \theta(3, 3, 3, 5, 5, 5)$ and is χ-unique by Lemma 2.9.

Case2.2 :
$x_5 = 5, x_6 \neq 5$
In this case $x_1 + x_2 = 6$ and $x_1 + x_3 = 6$ implying that $x_2 = x_3$. Hence, we have either $x_1 = 2, x_2 = x_3 = 4$ or $x_1 = x_2 = x_3 = 3$.

Case2.2.1 : If $x_1 = 2, x_2 = x_3 = 4$ then Equation 3 becomes $s+t+u = x_2+x_3+6$. By replacing these values in $Q_9(G)$ and $Q_9(H)$ and canceling the equal terms, we obtain $Q_9(G) \neq Q_9(H)$, a contradiction.

Case2.2.2 : If $x_1 = x_2 = x_3 = 3$ then H has four cycles of length 6, a contradiction.

Case2.3 :
$x_5 \neq 5, x_6 \neq 5$
Since the girth of H is 6 therefore $x_5, x_6 \geq 6$. It is given that H has three cycles of length 6, so $x_1 + x_2 = 6$ $x_1 + x_3 = 6$ and ($x_1 + x_4 = 6$ or $x_2 + x_3 = 6$). Therefore, we have two cases to consider.

Case2.3.1 :
If $x_1 + x_4 = 6$ then by considering $x_1 + x_2 = 6$ we get $x_1 + x_3 = 6$ and then $x_2 = x_3 = x_4$. Hence, either $x_1 = 2$ or $x_2 = x_3 = x_4 = 4$ or $x_1 = x_2 = x_3 = x_4 = 3$.

Case2.3.1.1 :
If $x_1 = 2$ then $x_3 = x_4 = 4$ and by canceling the equal terms, we obtain $Q_9(G) \neq Q_9(H)$, a contradiction.

Case2.3.1.2 :
If $x_1 = x_2 = x_3 = x_4 = 3$ then H has six cycles of length 6, a contradiction.

Case2.3.2 :
If $x_2 + x_3 = 6$ then by considering $x_1 + x_2 = 6$ we get $x_1 + x_3 = 6$ and then $x_1 = x_2 = x_3$.
Thus Equation 3 becomes $9 + s + t + u = x_4 + x_5 + x_6$. Replace these values in $Q_5(G)$ and $Q_5(H)$ and compare the l.r.p of both, we get either $s = 3$ or $t = 3$ or $u = 3$
If $s = 3$ then $G \cong \theta(3, 3, 3, 3, t, u)$ and is χ-unique.
If $t = 3$ then $G \cong \theta(3, 3, 3, 3, 3, u)$ and is χ-unique by Lemma 2.9.
If $u = 3$ then $G \cong \theta(3, 3, 3, 3, 3, 3)$ and is χ-unique by Theorem 2.1.

Case C :
$H \in g_e(\theta(x_1, x_2, x_3, x_4, x_5), C_{x_6+1})$, where $2 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$ and $2 \leq x_6$ As $G \cong \theta(r, r, r, s, t, u)$ and $H \in g_e(\theta(x_1, x_2, x_3, x_4, x_5), C_{x_6+1})$, we have
$Q_{13}(G)= x(x^r - 1)^3(x^s - 1)(x^t - 1)(x^u - 1) - (x^r - x)^3(x^s - x)(x^t - x)(x^u - x)$
$Q_{13}(H)= x(x^{x_1} - 1)(x^{x_2} - 1)(x^{x_3} - 1)(x^{x_4} - 1)(x^{x_5} - 1)(x^{x_6} - 1) - (x^{x_1} - x)(x^{x_2} - x)(x^{x_3} - x)(x^{x_4} - x)(x^{x_5} - x)(x^{x_6} - 1)$.
By using Equation 3, we get
$Q_{14}(G) = x^{3r+s+1} + x^{3r+t+1} + x^{3r+u+1} + x^{s+t+1} + x^{s+u+1} + x^{t+u+1} + 3x^{2r+s+t+1} + 3x^{2r+s+u+1} + 3x^{2r+t+u+1} + 3x^{2r+1} + 3x^{r+s+t+u+1} + 3x^{r+s+1} + 3x^{r+t+1} + 3x^{r+u+1} + x^{3r+3} + x^{s+t+u+3} + x^{s+5} + x^{t+5} + x^{u+5} + 3x^{2r+s+3} + 3x^{2r+t+3} + 3x^{2r+u+3} + 3x^{r+s+t+3} + 3x^{r+s+u+3} + 3x^{s+t+u+3} + 3x^{r+5} - (x^{3r+1} + x^{s+t+u+1} + x^{s+1} + x^{t+1} + x^{u+1} + 3x^{2r+s+1} + 3x^{2r+t+1} + 3x^{2r+u+1} + 3x^{r+s+t+1} + 3x^{r+s+u+1} + 3x^{r+t+u+1} + 3x^{r+1} + x^{3r+s+2} + x^{3r+t+2} + x^{3r+u+2} + x^{s+t+4} + x^{s+u+4} + x^{s+t+4} + x^{t+u+4} + 3x^{2r+s+t+2} + 3x^{2r+s+u+2} + 3x^{2r+t+u+2} + 3x^{r+s+t+u+2} + 3x^{r+s+4} + 3x^{r+t+4} + 3x^{r+u+4} + 3x^{2r+4} + x^6)$.
$Q_{14}(H) = x^{x_1+x_2+x_3+x_4+x_5} + x^{x_1+x_2+x_3+x_6+1} + x^{x_1+x_2+x_3+2} + x^{x_1+x_2+x_4+x_6+1} + x^{x_1+x_2+x_4+2} + x^{x_1+x_2+x_5+x_6+1} + x^{x_1+x_2+x_5+2} + x^{x_1+x_2+x_6+3} + x^{x_1+x_2+1} + x^{x_1+x_3+x_4+x_6+1} + x^{x_1+x_3+x_4+2} + x^{x_1+x_3+x_5+x_6+1} + x^{x_1+x_3+x_5+2} + x^{x_1+x_3+x_6+3} + x^{x_1+x_3+1} + x^{x_1+x_4+x_5+x_6+1} + x^{x_1+x_4+x_5+2} + x^{x_1+x_4+x_6+3} + x^{x_1+x_4+1} + x^{x_1+x_5+x_6+3} + x^{x_1+x_5+1} + x^{x_1+x_6+1} + x^{x_1+4} + +x^{x_2+x_3+x_4+x_6+1} +$

$x^{x_2+x_3+x_4+2} + x^{x_2+x_3+x_5+x_6+1} + x^{x_2+x_3+x_5+2} + x^{x_2+x_3+x_6+3} + x^{x_2+x_3+1} + x^{x_2+x_4+x_5+x_6+1} + x^{x_2+x_4+x_5+2} + x^{x_2+x_4+x_6+3} + x^{x_2+x_4+1} + x^{x_2+x_5+x_6+3} + x^{x_2+x_5+1} + x^{x_2+x_6+1} + x^{x_2+4} + x^{x_3+x_4+x_5+x_6+1} + x^{x_3+x_4+x_5+2} + x^{x_3+x_4+x_6+3} + x^{x_3+x_4+1} + x^{x_3+x_5+x_6+3} + x^{x_3+x_5+1} + x^{x_3+x_6+1} + x^{x_3+4} + x^{x_4+x_5+x_6+3} + x^{x_4+x_5+1} + x^{x_4+x_6+1} + x^{x_4+4} + x^{x_5+x_6+1} + x^{x_5+4} + x^{x_6+5} - (x^{x_1+x_2+x_3+x_4+x_5+1} + x^{x_1+x_2+x_3+x_6+2} + x^{x_1+x_2+x_3+1} + x^{x_1+x_2+x_4+x_6+2} + x^{x_1+x_2+x_4+1} + x^{x_1+x_2+x_5+x_6+2} + x^{x_1+x_2+x_5+1} + x^{x_1+x_2+x_6+1} + x^{x_1+x_2+3} + +x^{x_1+x_3+x_4+x_6+1} + x^{x_1+x_3+x_4+1} + x^{x_1+x_3+x_5+x_6+2} + x^{x_1+x_3+x_5+1} + x^{x_1+x_3+x_6+1} + +x^{x_1+x_3+3} + x^{x_1+x_4+x_5+x_6+2} + x^{x_1+x_4+x_5+1} + x^{x_1+x_4+x_6+1} + x^{x_1+x_5+x_6+1} + x^{x_1+x_4+3} + x^{x_1+x_5+3} + x^{x_1+x_6+4} + x^{x_1+1} + +x^{x_2+x_3+x_4+x_6+2} + x^{x_2+x_3+x_4+1} + x^{x_2+x_3+x_5+x_6+2} + x^{x_2+x_3+x_5+1} + x^{x_2+x_3+x_6+1} + x^{x_2+x_3+3} + +x^{x_2+x_4+x_5+x_6+2} + x^{x_2+x_4+x_5+1} + x^{x_2+x_4+x_5+x_6+2} + x^{x_2+x_4+x_5+1} + x^{x_2+x_4+x_6+1} + x^{x_2+x_4+3} + x^{x_2+x_5+x_6+1} + x^{x_2+x_5+3} + x^{x_2+x_6+4} + x^{x_2+1} + x^{x_3+x_4+x_5+x_6+2} + x^{x_3+x_4+x_5+1} + x^{x_3+x_4+x_6+1} + x^{x_3+x_4+3} + x^{x_3+x_5+x_6+1} + x^{x_3+x_5+3} + x^{x_3+x_6+4} + x^{x_3+1} + x^{x_4+x_5+x_6+1} + x^{x_4+x_5+3} + x^{x_4+x_6+4} + x^{x_4+1} + x^{x_5+x_6+4} + x^{x_5+1} + x^{x_6+1} + x^5)$.

The l.r.p in $Q_{14}(G)$ is x^{r+1} and the l.r.p in $Q_{14}(H)$ is 5 implying that $r = 4$. Since $r \geq 2$, we have three cases to consider either 1) $r = 2$ or 2) $r = 3$ or 3) $r = 4$.

Case1 :

If $r = 2$ then $g(G) = g(H) = 2r = 4$. Since G has three cycles of length 4, therefore H has three cycles of length 4. Without loss of generality, we have two cases to consider, either (i) $x_6 = 3$ or (ii). $x_6 \neq 3$

Case1.1 :

If $x_6 = 3$ then either $x_1 + x_2 = 4$ or $x_1 + x_3 = 4$. Thus $x_1 = x_2 = x_3 = 2$. But, H has four cycles of length 4, a contradiction.

Case1.2 :

If $x_6 \neq 3$. Since the girth of H is 4 therefore $x_6 \geq 4$. It is given that H has three cycles of length 4, so $x_1 + x_2 = 4$, $x_1 + x_3 = 4$ and ($x_1 + x_4 = 4$ or $x_2 + x_3 = 4$). Therefore, we have two cases to consider.

Case1.2.1 :

If $x_1 + x_4 = 4$.

Since $x_1 + x_2 = 4$, $x_1 + x_3 = 4$, therefore $x_1 = x_2 = x_3 = x_4 = 2$. But H has six cycles of length 4, a contradiction.

Case1.2.2 :

if $x_2 + x_3 = 4$.

Since $x_1 + x_2 = 4$, $x_1 + x_3 = 4$, therefore $x_1 = x_2 = x_3 = 2$. In this case Equation 3 becomes $s + t + u = x_4 + x_5 + x_6$ and we get

$Q_{15}(G) = 4x^{s+7} + 4x^{t+7} + 4x^{u+7} + x^{s+t+1} + x^{s+u+1} + x^{t+u+1} + 6x^{s+t+5} + 6x^{s+u+5} + 6x^{t+u+5} + 3x^{s+3} + 3x^{t+3} + 3x^{u+3} + x^{s+5} + x^{t+5} + x^{u+5} + x^9 + 2x^7 + x^5 - (x^{s+1} + x^{t+1} + x^{u+1} + 3x^{s+5} + 3x^{t+5} + 3x^{u+5} + 3x^{s+t+3} + 3x^{s+u+3} + 3x^{t+u+3} + x^{s+8} + x^{t+8} + x^{u+8} + x^{s+t+4} + x^{s+u+4} + x^{t+u+4} + 3x^{s+t+6} + 3x^{s+u+6} + 3x^{t+u+6} + 3x^{s+6} + 3x^{t+6} + 3x^{u+6} + 3x^8 + x^6)$.

$Q_{15}(H) = x^{x_4+x_5+6} + 3x^{x_4+x_5+4} + 6x^{x_5+x_6+5} + 6x^{x_4+x_6+5} + 4x^{x_6+7} + 3x^{x_5+3} + 3x^{x_6+3} + 3x^{x_4+3} + x^{x_4+x_5+1} + x^{x_4+x_6+1} + x^{x_5+x_6+1} + x^{x_5+4} + x^{x_6+5} + 3x^6 + x^{x_4+4} + 4x^{x_4+x_5+x_6+3} - (x^{x_4+x_5+7} + 2x^{x_4+x_6+6} + 6x^{x_4+5} + 3x^{x_5+x_6+6} + 6x^{x_5+5} + 3x^{x_6+6} + 3x^{x_6+5} + 4x^{x_4+x_5+3} + 4x^{x_4+x_6+3} + 3x^{x_5+x_6+3} + x^{x_5+x_6+4} + x^{x_4+1} + x^{x_5+1} + x^{x_6+1} + 3x^{x_4+x_5+x_6+4} + x^{x_6+8} + 4x^7)$.

By comparing the l.r.p in $Q_{15}(G)$ and the l.r.p in $Q_{15}(H)$, we have either $s = 4$ or $t = 4$ or $u = 4$.

Case1.2.2.1 :

If $s = 4$ then l.r.p of $Q_{15}(H)$ is $3x^6$ implying that $x_4 = x_5 = x_6 = 5$. But in this case after simplification we get $Q_{15}(G) \neq Q_{15}(H)$, a contradiction.

Case1.2.2.2 :

If $t = 4$ then we get

$Q_{16}(G) = x^{s+7} + 6x^{s+9} + 3x^{s+3} + x^{u+7} + 6x^{u+9} + 3x^{u+3} + x^{s+u+1} + 5x^{s+u+5} + 4x^{s+u+7} + 4x^{11} + 5x^7 - (x^{s+1} + x^{s+5} + 2x^{s+8} + x^{s+10} + x^{s+6} + x^{u+1} + x^{u+5} + 2x^{u+8} + x^{u+10} + x^{u+6} + 3x^{s+u+3} + 3x^{s+u+6} + x^{12} + 3x^{10} + x^9 + 3x^8)$

$Q_{16}(H) = x^{x_4+x_5+6} + x^{x_4+x_5+1} + x^{x_4+x_6+1} + x^{x_5+x_6+1} + 3x^{x_4+x_5+4} + 6x^{x_4+x_6+5} + 6x^{x_5+x_6+5} + 4x^{x_4+x_5+x_6+3} + 4x^{x_6+7} + 3x^{x_4+6} + 3x^{x_5+6} + 3x^{x_6+3} + x^{x_4+4} + x^{x_5+4} + x^8 + 4x^6 - (x^{x_4+x_5+7} + x^{x_6+8} + 3x^{x_4+x_6+6} + 3x^{x_5+x_6+6} + 3x^{x_4+x_5+3} + 4x^{x_4+x_6+3} + 3x^{x_5+x_6+3} + 3x^{x_4+x_6+3} + x^{x_4+x_5+x_6+1} + 6x^{x_4+5} + 6x^{x_5+5} + 2x^{x_6+5} + x^{x_4+x_6+4} + x^{x_5+x_6+4} + x^{x_6+1} + x^{x_4+1} + x^{x_5+1} + 4x^7)$

By comparing the l.r.p in $Q_{16}(G)$ and the l.r.p in $Q_{16}(H)$, we have either $s = 2$ or $s = 3$ or $s = 4$, where $2 \leq s \leq 4 \leq u$.

If $s = 2$ then $G \cong \theta(2, 2, 2, 2, 4, u)$ and is χ-unique.

If $s = 4$ then $G \cong \theta(2, 2, 2, 4, 4, u)$ and is χ-unique by Lemma 2.11.
If $s = 3$ then compare the l.r.p in $Q_{16}(G)$ which is $-x^4$ to the l.r.p in $Q_{16}(H)$, we get either $x_4 = 3$ or $x_5 = 3$.
If $x_4 = 3$ then Equation 3 becomes $4 + u = x_5 + x_6$. Since $4 \leq u$ and $3 \leq x_5$, we obtain $Q_{16}(G) \neq Q_{16}(H)$, a contradiction.
If $x_5 = 3$ then Equation 3 becomes $4 + u = x_4 + x_6$. Since $4 \leq u$ and $2 \leq x_4 \leq 3$, we obtain $Q_{16}(G) \neq Q_{16}(H)$, a contradiction.

Case2 :

If $r = 3$, then $g(G) = g(H) = 2r = 6$. Since G has three cycles of length 6, therefore H has three cycles of length 6. Without loss of generality, we have two cases to consider either 1. $x_6 = 5$ or 2. $x_6 \neq 5$.

Case2.1 :

If $x_6 = 5$ then $x_1 + x_2 = 6$ and $x_1 + x_3 = 6$ implying that $x_2 = x_3$. Thus we have $x_1 = 2, x_2 = x_3 = 4$ or $x_1 = x_2 = x_3 = 3$.

Case2.1.1 :

if $x_1 = 2, x_2 = x_3 = 4$ then Equation 3 becomes $s + t + u = x_4 + x_5 + 6$. But, in this case we get after simplification that $Q_{14}(G) \neq Q_{14}(H)$, a contradiction.

Case2.1.2 :

If $x_1 = x_2 = x_3 = 3$ then H has four cycles of length 6, a contradiction.

Case2.2 :

If $x_6 \neq 5$. Since the girth of H is 6, therefore $x_6 \geq 6$. it is given that H has three cycles of length 6, so $x_1 + x_2 = 6$, $x_1 + x_3 = 6$ and ($x_1 + x_4 = 6$ or $x_2 + x_3 = 6$). Therefore, we have two cases to consider.

Case2.2.1 :

if $x_1 + x_4 = 6$.
since $x_1 + x_2 = 6$ therefore $x_1 + x_3 = 6$ and then $x_2 = x_3 = x_4$. Hence we have either $x_1 = 2$ and $x_2 = x_3 = x_4 = 4$ or $x_1 = x_2 = x_3 = x_4 = 3$.

Case2.2.1.1 :

If $x_1 = 2$ and $x_2 = x_3 = x_4 = 4$ then by canceling the equal terms, we obtain $Q_{14}(G) \neq Q_{14}(H)$, a contradiction.

Case2.2.1.2 :

If $x_1 = x_2 = x_3 = x_4 = 3$ then H has six cycles of length 6, a contradiction.

Case2.2.2 :

If $x_2 + x_3 = 6$. Since $x_1 + x_2 = 6$ therefore $x_1 + x_3 = 6$ and then $x_1 = x_2 = x_3$.
It follows from Equation 3 that $s + t + u = x_4 = x_5 = x_6$ and we have
$Q_{17}(G) = x^{s+10} + x^{t+10} + x^{u+10} + x^{s+t+1} + x^{s+u+1} + x^{t+u+1} + 3x^{s+t+7} + 3x^{s+u+7} + 3x^{t+u+7} + 3x^{s+4} + 3x^{t+4} + 3x^{u+4} + x^{s+t+u} + x^{s+5} + x^{t+5} + x^{u+5} + +3x^{s+9} + 3x^{t+9} + 3x^{u+9} + 3x^{s+t+6} + 3x^{s+u+6} + 3x^{t+u+6} + x^{12} + 3x^8 - (x^{s+1} + x^{t+1} + x^{u+1} + 6x^{s+7} + 6x^{t+7} + 6x^{u+7} + 4x^{s+t+4} + 4x^{s+u+4} + 4x^{t+u+4} + x^{s+11} + x^{t+11} + x^{u+11} + 3x^{s+t+8} + 3x^{s+u+8} + 3x^{t+u+8} + 4x^{10})$.
$Q_{17}(H) = x^{x_4+x_5+9} + 3x^{x_4+x_6+7} + 3x^{x_4+8} + 3x^{x_5+x_6+7} + 3x^{x_5+8} + 3x^{x_6+9} + 3x^{x_4+x_5+5} + 3x^{x_4+x_6+6} + 3x^{x_5+x_6+6} + 4x^{x_4+4} + 4x^{x_5+4} + 3x^{x_6+4} + x^{x_4+x_5+x_6+3} + x^{x_4+x_5+1} + x^{x_4+x_6+1} + x^{x_5+x_6+1} + x^{x_6+5} + x^{x_6+10} + x^{11} + 3x^7 + -(x^{x_4+x_5+10} + x^{x_4+x_6+7} + 3x^{x_5+x_6+8} + 3x^{x_4+6} + 3x^{x_5+6} + x^{x_6+11} + 3x^{x_4+x_5+4} + 2x^{x_4+x_6+8} + 3x^{x_4+7} + 3x^{x_5+7} + 6x^{x_6+7} + x^{x_4+x_5+3} + 4x^{x_4+x_6+4} + 4x^{x_5+x_6+4} + x^{x_4+1} + x^{x_5+1} + x^{x_6+1} + 3x^9 + x^5)$.
By comparing the l.r.p in $Q_{17}(G)$ and the l.r.p in $Q_{17}(H)$, we have either $s = 4$ or $t = 4$ or $u = 4$.

Case2.2.2.1 :

If $s = 4$ then after simplifying we have $Q_{17}(G) \neq Q_{17}(H)$, a contradiction.

Case2.2.2.2 :

If $t = 4$ then either $s = 3$ or $s = 4$.
If $s = 3$ then $G \cong \theta(3, 3, 3, 3, 4, u)$ and is χ-unique.
If $s = 4$ then $G \cong \theta(3, 3, 3, 4, 4, u)$ and is χ-unique by Lemma 2.10.

Case2.2.2.3 :

If $u = 4$ then $s = t = 3$ or $s = t = 4$ or $s = 3, t = 4$
If $s = t = 3$ then $G \cong \theta(3, 3, 3, 3, 3, 4)$ and is χ-unique by Lemma 2.9.
If $s = 3, t = 4$ then $G \cong \theta(3, 3, 3, 3, 4, 4)$ and is χ-unique by Lemma 2.9.
If $s = 4$ then $G \cong \theta(3, 3, 3, 4, 4, 4)$ and is χ-unique by Lemma 2.9.

Case3 :

If $r = 4$ then $g(G) = g(H) = 2r = 8$. Since G has three cycles of length 8, therefore H has three cycles of length 8. Without loss of generality, we have two cases to consider, either 1. $x_6 = 7$ or 2. $x_6 \neq 7$

Case3.1 :

If $x_6 = 7$.

Since $x_1 + x_2 = 8$ and $x_1 + x_3 = 8$, therefore $x_2 = x_3$. Thus, we have either $x_1 = 2$ and $x_2 = x_3 = 6$ or $x_1 = 3$ and $x_2 = x_3 = 5$ or $x_1 = x_2 = x_3 = 4$

Case3.1.1 :

If $x_1 = 2$ and $x_2 = x_3 = 6$. in this case Equation 3 becomes $s + t + u = x_4 + x_5 + 9$.

Since $4 \leq s \leq t \leq u$, after simplification we get the term $-x^3$ in Q_{14} but not in $Q_{14}(H)$, a contradiction.

Case3.1.2 :

If $x_1 = 3$ and $x_2 = x_3 = 5$.

Since $4 \leq s \leq t \leq u$, after simplification we get the term $-x^4$ in Q_{14} but not in $Q_{14}(H)$, a contradiction.

Case3.1.3 :

If $x_1 = x_2 = x_3 = 4$ then H has four cycles of length 8, a contradiction.

Case3.2 :

$x_6 \neq 7$.

Since the girth of H is 8, therefore $x_6 \geq 8$. It is given that H has three cycles of length 8, so $x_1 + x_2 = 8$, $x_1 + x_3 = 8$ and ($x_1 + x_4 = 8$ or $x_2 + x_3 = 8$). Therefore, we have two cases to consider.

Case3.2.1 :

If $x_1 + x_4 = 8$.

Since $x_1 + x_2 = 8$ therefore $x_1 + x_3 = 8$ and then $x_2 = x_3 = x_4$. Hence, we have either $x_1 = 2$ and $x_2 = x_3 = x_4 = 6$ or $x_1 = 3, x_2 = x_3 = 5$ or $x_1 = x_2 = x_3 = x_4 = 4$.

Case3.2.1.1 :

If $x_1 = 2$ and $x_3 = x_4 = 6$ then $Q_{14}(G) \neq Q_{14}(H)$ because l.r.p in Q_{14} is $-x^5$ and the l.r.p in $Q_{14}(H)$ is $-x^3$, a contradiction.

Case3.2.1.2 :

If $x_1 = 3$ and $x_2 = x_3 = 5$ then $Q_{14}(G) \neq Q_{14}(H)$ because l.r.p in Q_{14} is $-x^5$ and the l.r.p in $Q_{14}(H)$ is $-x^4$, a contradiction.

Case3.2.1.2 :

If $x_1 = x_2 = x_3 = x_4 = 4$ then H has six cycles of length 6, a contradiction.

Case3.2.2 :

$x_2 + x_3 = 8$.

Since $x_1 + x_2 = 8$ therefore we obtain $x_1 = x_2 = x_3$ and hence $x_1 = x_2 = x_3 = 4$. In this case Equation 3 becomes $s + t + u = x_4 + x_5 + x_6$ and we obtain

$Q_{18}(G) = x^{s+13} + x^{t+13} + x^{u+13} + 4x^{s+5} + 4x^{t+5} + 4x^{u+5} + 3x^{s+11} + 3x^{t+11} + 3x^{u+11} + x^{s+t+u+3} + x^{s+t+1} + x^{s+u+1} + x^{t+u+1} + 3x^{s+t+9} + 3x^{s+u+9} + 3x^{t+u+9} + 3x^{s+t+7} + 3x^{s+u+7} + 3x^{t+u+7} + 6x^9 + x^{15} - (x^{s+t+u+1} + x^{s+1} + x^{t+1} + x^{u+1} + 3x^{s+9} + 3x^{t+9} + 3x^{u+9} + 3x^{s+t+5} + 3x^{s+u+5} + 3x^{t+u+5} + 3x^{s+14} + 3x^{t+14} + 3x^{u+14} + +x^{s+t+4} + x^{s+u+4} + x^{t+u+4} + 3x^{s+t+10} + 3x^{s+u+10} + 3x^{t+u+10} + +3x^{s+8} + 3x^{t+8} + 3x^{u+8} + x^{13} + 3x^{12} + x^6)$.

$Q_{18}(H) = x^{x_4+x_5+12} + 2x^{x_4+x_6+9} + 3x^{x_4+5} + 3x^{x_5+x_6+9} + 3x^{x_5+10} + 3x^{x_6+11} + 3x^{x_4+x_5+6} + 3x^{x_4+x_6+7} + 3x^{x_5+5} + 3x^{x_5+x_6+7} + x^{x_4+4} + x^{x_5+4} + 4x^{x_6+5} + x^{x_4+x_5+x_6+3} + x^{x_4+x_5+1} + x^{x_4+x_6+1} + x^{x_5+x_6+1} + 3x^{x_4+10} + x^{x_6+13} + 3x^9 + x^{14} + 3x^8 + -(x^{x_4+x_5+13} + x^{x_5+x_6+4} + 3x^{x_5+x_6+10} + 3x^{x_4+9} + 3x^{x_5+9} + 3x^{x_6+9} + 3x^{x_4+x_5+5} + 3x^{x_4+x_6+5} + 3x^{x_4+7} + 3x^{x_5+7} + 3x^{x_6+8} + x^{x_4+x_5+3} + x^{x_4+x_6+4} + 2x^{x_4+x_6+10} + x^{x_6+14} + x^{x_4+x_5+x_6+1} + 3x^{x_5+x_6+5} + x^{x_4+1} + x^{x_5+1} + x^{x_6+1} + 3x^{11} + x^{13} + x^5)$.

By comparing the l.r.p in $Q_{18}(G)$ and the l.r.p in $Q_{18}(H)$, we have either $s = 4$ or $t = 4$ or $u = 4$.

If $s = 4$ then $G \cong \theta(4, 4, 4, 4, t, u)$ and is χ-unique.

If $t = 4$ then $G \cong \theta(4, 4, 4, 4, 4, u)$ and is χ-unique by Lemma 2.9.

If $u = 4$ then $G \cong \theta(4, 4, 4, 4, 4, 4)$ and is χ-unique by Theorem 2.1.

This completes the proof of theorem. □

4 Conclusion

The coloring of graphs and hypergraphs is one of the most studied and interesting topic in graph theory. G proper coloring of the graph is a mapping from the vertex set of the graph to the set of k-colors such that every adjacent vertices have different labeling. The chromatic polynomial of graph is the number of all proper coloring of graph. The chromatically equivalent graph is the family of graphs which have same chromatic polynomial. A graph is said to be chromatically unique if no other graph shares its chromatic polynomial. The chromaticity of graph is the study of chromatically equivalent and chromatically unique graphs. The chromaticity of k-bridge graph was initiated by Dong et al. [5] and since then many result about the chromaticity of k-bridge graphs are obtained. In this paper, this study has been extended and the chromatic uniqueness of a new family of 6-bridge graph $\theta(r,r,r,s,t,u)$, where $2 \le r \le s \le t \le u$ is investigated.

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