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Comultiplication-like modules and related results

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Abstract Let R be a commutative ring with identity. The main purpose of this paper is to introduce the notions of comultiplication-like and virtually codivisible R-modules as generalizations of comultiplication and codivisible R-modules, respectively. Also, we explore some of theirs basis properties.

1 Introduction

Throughout this paper, R will denote a commutative ring with identity and "$\subset$" will denote the strict inclusion. Further, $\mathbb{Z}$ will denote the ring of integers.

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [8].

An R-module M is said to be a *multiplication-like module* if for each non-zero submodule N of M, $Ann_R(M) \subset Ann_R(M/N)$ [9]. More information concerning this class of modules can be found in [12], [7], and [16].

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [2].

A non-zero submodule N of an R-module M is said to be *second* if for each $a \in R$, the homomorphism $N \xrightarrow{a} N$ is either surjective or zero [17].

An R-module M is said to be a *virtually divisible module*, if $Ann(M/N) = Ann(M)$ for each proper submodule N of M [9].

Let R be an integral domain. An R module M is called *codivisible* if $(0 :_M r) = 0$, for all $0 \neq r \in R$ [6]. For example, every projective module is codivisible. Over $\mathbb{Z}$, or more generally over any principal ideal domain, the codivisible modules are exactly the projective modules [6].

The main purpose of this paper is to introduce the notion of comultiplication-like R-modules (this can be regarded as a dual notion of multiplication-like modules) as a generalization of comultiplication modules. Also, we investigate the first properties of this class of modules. Moreover, we introduce the notion of virtually codivisible R-modules (this can be regarded as a dual notion of virtually divisible modules) as a generalization of codivisible R-modules and obtain some related results.

2 Comultiplication-like modules

Definition 2.1. We say that an R-module M is a *comultiplication-like module* if for each proper submodule N of M, $Ann_R(M) \subset Ann_R(N)$.

Clearly, every comultiplication R-module is comultiplication-like, but we have not found any example where M is comultiplication-like R-module and M is not a comultiplication R-module. Therefore, we have the following question.

Question 2.2. Let M be a comultiplication-like R-module. Is M a comultiplication R-module?

Proposition 2.3. Let M be an R-module. Then we have the following.

(a) If every submodule of M is a comultiplication-like R-module, then M is a comultiplication R-module.

(b) If every homomorphic image of M is a multiplication-like R-module, then M is a multiplication R-module.

Proof. (a) Suppose that every submodule of M is a comultiplication-like R-module. First note that always for each submodule N of an R-module M, we have $Ann_R((0:_M Ann_R(N))) = Ann_R(N)$. Assume that M is not a comultiplication R-module. Then there is a submodule N of M such that $N \subset (0:_M Ann_R(N))$. By assumption, $(0:_M Ann_R(N))$ is a comultiplication-like R-module. Therefore, $Ann_R((0:_M Ann_R(N))) \subset Ann_R(N)$, a contradiction. Thus M is a comultiplication R-module.

(b) Let every homomorphic image of M be a multiplication-like R-module. First note that always for each submodule N of an R-module M, we have $Ann_R(M/(N:_R M)M) = Ann_R(M/N)$. If M is not a multiplication R-module, then there is a submodule N of M such that $(N:_R M)M \subset N$. By assumption, $M/(N:_R M)M$ is a multiplication-like R-module. Hence,

$$Ann_R(M/(N:_R M)M) \subset Ann_R((M/(N:_R M)M)/(N/(N:_R M)M))$$

$$= Ann_R(M/N).$$

This is a contradiction. Thus M is a multiplication R-module. □

A submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of R-submodules of M, then $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [11].

Remark 2.4. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$ [4].

In the following theorem, we provide a characterization for comultiplication-like modules.

Theorem 2.5. Let M be an R-module. Then M is a comultiplication-like module if and only if for each proper completely irreducible submodule L of M, we have $Ann_R(M) \subset Ann_R(L)$

Proof. The necessity is clear. Conversely, suppose that for each proper completely irreducible submodule L of M, we have $Ann_R(M) \subset Ann_R(L)$. Let N be a proper submodule of M. Then there exists a proper completely irreducible submodule L of M with $N \subseteq L \subset M$ by Remark 2.4. Thus $Ann_R(M) \subset Ann_R(L) \subseteq Ann_R(N)$ by assumption. This implies that $Ann_R(M) \subset Ann_R(N)$, as required. □

Proposition 2.6. Let M be a comultiplication-like R-module. Then M is a second module if and only if $Ann_R(M)$ is a prime ideal of R.

Proof. First suppose that M is a second module. Then by [17], $Ann_R(M)$ is a prime ideal of R. Now let $Ann_R(M)$ is a prime ideal of R, $r \in R$, $rM \neq 0$, and $rM \neq M$. As M is a comultiplication-like module, $Ann_R(M) \neq Ann_R(rM)$. Thus there exists $s \in Ann_R(rM) \setminus Ann_R(M)$. Hence $rsM = 0$ and $sM \neq 0$. Therefore, $rM = 0$, a contradiction. Hence M is a second module. □

An R-module M is called *indecomposable* if $M \neq 0$ and M cannot be written as a direct sum of non-zero submodules.

Theorem 2.7. Let M be a second comultiplication-like R-module. Then we have the following.

(a) M is an indecomposable R-module.

(b) If M has a maximal submodule, then M is a simple R-module.

Proof. (a) Let M be a decomposable second comultiplication-like R-module. Then $M = N \oplus K$ for some non-zero submodules N and K of M. Thus $Ann_R(M) = Ann_R(N) \cap Ann_R(K)$. As M is second, $Ann_R(M)$ is a prime ideal of R. Thus $Ann_R(M) = Ann_R(N)$ or $Ann_R(M) = Ann_R(K)$. This implies that $M = N$ or $M = K$ since M is a comultiplication-like R-module. This contradiction implies that M is an indecomposable R-module.

(b) Let K be a maximal submodule of M. Then $Ann_R(M/K)$ is a maximal ideal of R. As M is second, $Ann_R(M/K)M = M$ or $Ann_R(M/K)M = 0$. If $Ann_R(M/K)M = M$, then $M = Ann_R(M/K)M \subseteq K$, a contradiction. Thus $Ann_R(M/K)M = 0$. This implies that $Ann_R(M/K) = Ann_R(M)$ and so $Ann_R(M)$ is a maximal ideal of R. Thus $R/Ann_R(M)$ is a field. Therefore, M is a semisimple $R/Ann_R(M)$-module by [14, Proposition 3.7], and hence M as an R-module is semisimple. Now the result follows from part (a). □

Corollary 2.8. Let M be a finitely generated second comultiplication-like R-module. Then M is a simple module.

Proof. This immediately follows from Theorem 2.7 (b). □

Let M be an R-module. The subset $Z_R(M)$ of R, the set of zero divisors of M, is defined by $\{r \in R | \exists 0 \neq m \in M such\ that\ rm = 0\}$.

The dual notion of $Z_R(M)$ is denoted by $W_R(M)$ and defined by

$$W(M) = \{r \in R : rM \neq M\}.$$

Proposition 2.9. Let M be an R-module. Then $Z_R(R/Ann_R(M)) \subseteq W_R(M)$. Moreover, the reverse inequality holds when M is a comultiplication-like R-module.

Proof. Let $r \in Z_R(R/Ann_R(M))$. Then there exist $\bar{0} \neq s + Ann_R(M) \in R/Ann_R(M)$ such that $r(s + Ann_R(M)) = \bar{0}$. This implies that $rsM = 0$. If $rM = M$, then $0 = srM = sM \neq 0$, a contradiction. Therefore, $rM \neq M$. Thus $Z_R(R/Ann_R(M)) \subseteq W_R(M)$. Now let M be a comultiplication-like R-module and $r \in W_R(M)$. Then $rM \neq M$ and hence $Ann_R(M) \subset Ann_R(rM)$. Thus there exists $t \in Ann_R(rM) \setminus Ann_R(M)$. Therefore, $rtM = 0$ and $rM \neq 0$. It follows that $r \in Z_R(R/Ann_R(M))$, as required. □

The following example shows that the condition "M is a comultiplication-like R-module" cannot be omitted in Proposition 2.9.

Example 2.10. Let M be the $\mathbb{Z}$-module $\mathbb{Z}$. Then clearly, M is not a comultiplication-like $\mathbb{Z}$-module. We have $W_{\mathbb{Z}}(M) = \mathbb{Z} \setminus \{1, -1\}$. But $Z_{\mathbb{Z}}(\mathbb{Z}/Ann_{\mathbb{Z}}(M)) = \{0\}$.

An R-module M is said to be *co-Hopfian* if every injective endomorphism f of M is an isomorphism [13].

Proposition 2.11. Let M be a comultiplication-like R-module. Then we have the following.

(a) M is co-Hopfian.

(b) For each $r \in W_R(M) \setminus Ann_R(M)$ there exists $t \in W_R(M) \setminus Ann_R(M)$ such that $rt \in Ann_R(M)$.

Proof. (a) Let $f : M \to M$ be a monomorphism. Assume that $f(M) \neq M$. Then by assumption, there exists $r \in Ann_R(f(M)) \setminus Ann_R(M)$. Thus $f(rM) = 0$ and so $rM \subseteq Ker(f) = \{0\}$, a contradiction. It follows that M is a co-Hopfian R-module.

(b) Let $r \in W_R(M) \setminus Ann_R(M)$. Then $rM \neq M$. As M is a comultiplication-like R-module, $An_R(M) \subset Ann_R(rM)$. Thus there exists $t \in Ann_R(rM) \setminus Ann_R(M)$. Hence, $rtM = 0$ and $tM \neq 0$. If $tM = M$, then $rM = 0$, a contradiction. Hence $t \in W_R(M)$, $t \notin Ann_R(M)$ and $rt \in Ann_R(M)$. □

For a submodule N of an R-module M the *second radical* (or *second socle*) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) (see [10] and [3]).

Theorem 2.12. Let M be a finitely generated comultiplication-like R-module and N be a submodule of M. If $sec(M) \subseteq N$, then $Ann_R(N) \subseteq \sqrt{Ann_R(M/N)}$.

Proof. The result follows by Proposition 2.6 and similar arguments as in the proof for Theorem 2.21 of [5]. □

An R-module M is said to be *coreduced* if $(L :_M r) = M$ implies that $L + (0 :_M r) = M$, where $r \in R$ and L is a completely irreducible submodule of M [5].

Proposition 2.13. Let M be a coreduced comultiplication-like R-module. Then we have the following.

(a) If M is a finitely generated R-module, then $sec(M) = M$.

(b) If I is an ideal of R such that $I \subseteq P$, where P is a minimal prime ideal of $Ann_R(M)$, then $I \subseteq W_R(M)$.

Proof. (a) Let M be a finitely generated R-module and $sec(M) \neq M$. Then there exists a proper completely irreducible submodule L of M such that $sec(M) \subseteq L$ by Remark 2.4. Hence, by Theorem 2.12, $Ann_R(L) \subseteq \sqrt{Ann_R(M/L)}$. As M is a comultiplication-like R-module and L is proper, there exits $t \in Ann_R(L) \setminus Ann_R(M)$. Therefore, $t^n M \subseteq L$ for some $n \in \mathbb{N}$. This implies that $t^{n+1}M = 0$. But since M is coreduced, $tM = t^2M$ by [5, Theorem 2.13]. Therefore, $tM = 0$, which is a contradiction. Thus $sec(M) = M$.

(b) Let I be an ideal of R such that $I \subseteq P$, where P is a minimal prime ideal of $Ann_R(M)$. By [5, Lemma 2.15], $R/Ann_R(M)$ is a reduced R-module. Hence since $R/Ann_R(M)$ is a multiplication R-module, $I \subseteq Z_R(R/Ann_R(M))$ by [1, 2.3]. Now as M is a comultiplication-like R-module, $W_R(M) = Z_R(R/Ann_R(M))$ by Proposition 2.9. Therefore, $I \subseteq W_R(M)$. □

Proposition 2.14. Let R be a Noetherian ring and let M be a finitely generated R-module. If S is a multiplicatively closed subset of R such that for all ideals I, J of R with $I \subset J$, we have $(I :_R J) \cap S = \emptyset$, then M is a comultiplication-like R-module if and only if $S^{-1}M$ is a comultiplication-like $S^{-1}R$-module.

Proof. First note that as R is Noetherian and M is a finitely generated R-module, every submodule N of M is finitely generated. Therefore,

$$S^{-1}(Ann_R(N)) = Ann_{S^{-1}R}(S^{-1}N)$$

by [15, Lemma 9.12]. Assume that M is a comultiplication-like R-module and $S^{-1}N$ is a proper submodule of $S^{-1}M$. If $Ann_{S^{-1}R}(S^{-1}N) = Ann_{S^{-1}R}(S^{-1}M)$, then $S^{-1}(Ann_R(N)) = S^{-1}(Ann_R(M))$. This implies that $(Ann_R(N) :_R Ann_R(M)) \cap S \neq \emptyset$ since R is Noetherian and so $Ann_R(M)$ is finitely generated. This contradiction shows that $Ann_{S^{-1}R}(S^{-1}M) \subset Ann_{S^{-1}R}(S^{-1}N)$, as needed. Conversely, suppose that $S^{-1}M$ is a comultiplication-like $S^{-1}R$-module and N is a proper submodule of M. If $S^{-1}N = S^{-1}M$, then we can conclude that $(Ann_R(N) :_R Ann_R(M)) \cap S \neq \emptyset$, a contradiction. Thus $S^{-1}N \neq S^{-1}M$ and so by assumption, $Ann_{S^{-1}R}(S^{-1}N) \neq Ann_{S^{-1}R}(S^{-1}M)$. It follows that $Ann_R(M) = Ann_R(N)$ as requested. □

3 Virtually codivisible modules

Definition 3.1. Let M be a non-zero R-module. We say that M is a *virtually codivisible module*, if $Ann(N) = Ann(M)$ for each non-zero submodule N of M. Also, we say that M is a *weakly virtually codivisible module*, if $Ann(L) = Ann(M)$ for each non-zero completely irreducible submodule L of M.

Remark 3.2. It is clear that every virtually codivisible R-module is weakly virtually codivisible but the converse is not true. For example, $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$, where p is a prime number, is a weakly virtually codivisible $\mathbb{Z}$-module but it is not a virtually codivisible $\mathbb{Z}$-module.

Example 3.3. (a) Let R be an integral domain. If M is a non-zero codivisible R-module, then it is clear that for each non-zero submodule N of M, we have $Ann(N) = Ann(M) = 0$. Thus every codivisible R-module is virtually codivisible but the converse is not true (for example, every non-simple homogeneous cosemisimple $\mathbb{Z}$-module M is virtually codivisible but it is not a codivisible $\mathbb{Z}$-module).

(b) Now let R be a commutative ring (not necessarily a domain) and M be a homogeneous cosemisimple R-module. It is clear that $Ann(M)$ is a maximal ideal and so for each non-zero submodule N of M we have $Ann(N) = Ann(M)$. Hence, every homogeneous cosemisimple R-module is virtually codivisible.

Proposition 3.4. Let M be an R-module with $P = Ann(M)$. Then M is virtually codivisible if and only if P is a prime ideal and M is a codivisible R/P-module.

Proof. Let M be a virtually codivisible R-module. Let $ab \in P$, where $a, b \in R$. Assume that $aM \neq 0$. If $(0 :_M a) \neq 0$, then $Ann_R((0 :_M a)) = Ann_R(M) = P$ since M is virtually codivisible and so $a \in Ann_R((0 :_M a)) = Ann_R(M)$, a contradiction. Thus $(0 :_M a) = 0$ and so $(0 :_M b) = (0 :_M a) :_M b) = (0 :_M ab) = M$. It follows that $b \in Ann_R(M) = P$. Therefore, P is a prime ideal of R. Now, let $0 \neq r \in R \setminus P$. Then $rM \neq 0$. If $(0 :_M r) \neq 0$, then $r \in Ann_R((0 :_M r)) = Ann_R(M) = P$, a contradiction. Thus $(0 :_M r) = 0$ i.e., $(0 :_M r + P) = 0$ and so M is codivisible as a R/P-module.. The converse is clear. □

In the following theorem there are several characterizations for a virtually codivisible R-module.

Theorem 3.5. Let M be an R-module. Then the following are equivalent.

(a) M is virtually codivisible.

(b) $P = Ann(M)$ is a prime ideal and M is a codivisible R/P-module.

(c) Each direct summand of M is a virtually codivisible module.

(d) For each $a \in R$, we have $(0 :_M a) = 0$ or $aM = 0$.

(e) For each ideal I of R, we have $(0 :_M I) = 0$ or $IM = 0$.

Proof. The equivalence of (a) and (b) is from Proposition 3.4 and the equivalence of (d) and (e) is clear.

$(b) \Rightarrow (c)$ Let N be a direct summand of M. Then $M = N \oplus K$, for some submodule K of M. If $N = M$, then we are done. Let $N \neq M$. Since $P = Ann_R(M)$ is a prime ideal and M is a codivisible R/P-module, the submodule K is also a codivisible R/P-module. Now by $(a) \Rightarrow (b)$, $Ann_R(M/N) = Ann_R(M) = P$ and N a codivisible R/P-module (since, $M/N \cong K$). Thus N is a virtually codivisible R-module.

$(c) \Rightarrow (a)$ This is clear.

$(b) \Rightarrow (d)$ Let $a \in R$ and $aM \neq 0$. Then $a \notin Ann_R(M) = P$. As M is a codivisible R/P-module, $(0 :_M (a + P) = 0$ i.e., $(0 :_M a) = 0$.

$(d) \Rightarrow (b)$ Let $a, b \in R$ and $abM = 0$. If $bM \neq 0$ then by our hypothesis $(0 :_M b) = 0$. Now $abM = 0$ implies that

$$M = (0 :_M ab) = ((0 :_M b) :_M a) = (0 :_M a).$$

So $aM = 0$. Thus $P = Ann_R(M)$ is a prime ideal. Now let $r \in R \setminus P$. Then $(0 :_M r) = 0$ and so $(0 :_M r + P) = 0$. Thus M is a codivisible R/P-module. □

Next, we determine virtually codivisible modules over one-dimensional dimensional domains.

Corollary 3.6. Let R be an integral domain with $dim(R) = 1$ and let M be an R-module. Then M is a virtually codivisible R-module if and only if one of the following statements hold.

(a) M is a homogeneous cosemisimple module.

(b) M is a codivisible module.

Proof. $\Rightarrow$ Let M be a virtually codivisible R-module. By Proposition 3.4, $P = Ann_R(M)$ is a prime ideal of R and M is a codivisible R/P-module. If $P = 0$, then M is a codivisible R-module but, if $P \neq 0$, then P is a maximal ideal and so M is a homogeneous semisimple module.

$\Leftarrow$ This immediately follows from Theorem 3.5. □

Remark 3.7. Let R be a domain which is not a field. Then every codivisible R-module M has no minimal submodule, for otherwise if M is a codivisible R-module with a minimal submodule N, then $Ann_R(N) = P$ is a maximal ideal of R. This means that $N \subseteq (0 :_M P) = 0$, a contradiction.

The following proposition shows that if M is a finitely cogenerated module, then homogeneous cosemisimpility and virtually codivisibility of M coincide.

Proposition 3.8. Let M be a finitely cogenerated R-module. Then M is virtually codivisible if and only if M is a homogeneous cosemisimple module.

Proof. Let M be a finitely cogenerated virtually codivisible R-module. Then by Proposition 3.4, $P = Ann_R(M)$ is a prime ideal of R and M is a divisible R/P-module. If P is not a maximal ideal of R, then R/P is a domain which is not a field. By Remark 3.7, M as an R/P-module has no minimal submodule, this is a contradiction (since M is a finitely cogenerated R/P-module). Therefore, P is a maximal ideal of R and so, M is a homogeneous cosemisimple module. □

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