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Product of Traces of Symmetric Bi-Derivations in Rings

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Abstract In this paper, we prove that if the product of the traces of two symmetric bi-derivations on a ring maps the ring into a prime ideal, then one of the trace maps the ring into the prime ideal. We also extend the above result on the product of two or more traces under suitable characteristic restrictions. In fact our result is -for a semiprime ideal $\mathcal{P}$ of $\mathcal{R}$, $d^r(\mathcal{P}) \subseteq \mathcal{P}$ if and only if $d(\mathcal{P}) \subseteq \mathcal{P}$ for any positive integer r.

1 Introduction

Throughout this paper, $\mathcal{R}$ will denote an associative ring. An ideal $\mathcal{P}$ of a ring $\mathcal{R}$ is said to be semiprime ideal (resp. prime ideal) if it is the intersection of prime ideals or, alternatively, if $a\mathcal{R}a \subseteq \mathcal{P}$ implies that $a \in \mathcal{P}$ for any $a \in \mathcal{R}$ (resp. $a\mathcal{R}b \subseteq \mathcal{P}$ implies that $a \in \mathcal{P}$ or $b \in \mathcal{P}$ for any $a, b \in \mathcal{R}$). A ring is characteristic free if $nx = 0$ implies that $x = 0$ for any element x of the ring and for any positive integer n. If a ring is $n!$-torsion free, then it is d-torsion free for any divisor d of $n!$. An additive mapping $\delta : \mathcal{R} \to \mathcal{R}$ is called a derivation on $\mathcal{R}$ if $\delta(xy) = \delta(x)y + x\delta(y)$ holds for all $x, y \in \mathcal{R}$. A bi-additive map $D : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is said to be symmetric if $D(x_1, x_2) = D(x_2, x_1)$ and $x_1, x_2 \in \mathcal{R}$. D. A symmetric bi-additive map is said to be symmetric bi-derivation if $D(x_1x_2, x_3) = x_1D(x_2, x_3) + D(x_1, x_3)x_2$ for all $x_1, x_2, x_3 \in \mathcal{R}$. A map $d : \mathcal{R} \to \mathcal{R}$ defined by $d(x) = D(x, x)$ is called the trace of D. The trace d of $D(.,.)$ satisfies the relation $d(x_1 + x_2) = d(x_1) + 2D(x_1, x_2) + d(x_2)$ for all $x_1, x_2 \in \mathcal{R}$. The concept of symmetric bi-derivation in rings was introduced by G.Maksa [8] in the year 1987. More details about bi-derivations and their applications can be found in references [1] and [16].

In the year 1957, a remarkable result due to E.C.Posner [13] came into existence which states that if the product of two derivations on a prime ring of characteristic not two is a derivation, then one of the derivations must be zero. Since then, there has been substantial interest in examining the same result in different context, generally that of prime rings and semiprime rings admitting suitable constraints. Over and above, Vukman [14, 15] extended the above result for bi-derivations. Derivations and bi-derivations in rings have been studied by several algebraists in various directions. It is very important that the analogous properties of derivation which is one of the requisite theory in analysis and applied mathematics are also satisfied in the ring theory. Later, in the year 1998, T.Creedon [5] proved a variation of Posner's result. He proved that if the product of two derivations on a ring $\mathcal{R}$ maps $\mathcal{R}$ into a prime ideal $\mathcal{P}$, where the characteristic of $\mathcal{R}/\mathcal{P}$ is not two, then one of the derivations must map $\mathcal{R}$ into $\mathcal{P}$. Also, under same characteristic restrictions he also proved that, if a derivation d satisfies $d^k(\mathcal{P}) \subseteq \mathcal{P}$, for any fixed positive integer k, then $d(\mathcal{P}) \subseteq \mathcal{P}$.

In the present paper, we extend the result of Creedon on the traces of symmetric bi-derivation under suitable characteristic restrictions on a ring.

2 Main Results

We facilitate our discussion with the following known lemma which is needed for developing the proof of our main results.

Lemma 2.1. *([7], Lemma 3.10) If $\mathcal{P}$ is a prime ideal of a ring $\mathcal{R}$ such that $\mathcal{R}/\mathcal{P}$ has characteristic not equal to two and a and b are elements of $\mathcal{R}$ such that $arb + bra \in \mathcal{P}$, for all $r \in R$, then $a \in \mathcal{P}$ or $b \in \mathcal{P}$.*

Lemma 2.2. *([10], Theorem 4.4.3) Suppose $\mathcal{P}$ is a semiprime ideal of a ring $\mathcal{R}$ and $\mathcal{L}$ is a left ideal satisfying $\mathcal{L}^r \subseteq \mathcal{P}$, for some positive integer r. Then $\mathcal{L} \subseteq \mathcal{P}$.*

Lemma 2.3. *Let $\mathcal{R}$ be a ring and $\mathcal{P}$ be a prime ideal of $\mathcal{R}$, such that $\mathcal{R}/\mathcal{P}$is $n!$-torsion free. Suppose $y_1, y_2, \cdots y_n \in \mathcal{R}$ satisfy $\alpha y_1 + \alpha^2 y_2 + \cdots + \alpha^n y_n \in \mathcal{P}$ for $\alpha = 1, 2, 3, ...n$. Then $y_i \in \mathcal{P}$ for all i.*

Proof. Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^n \\ 3 & 3^2 & \cdots & 3^n \\ \cdots & \cdots & \cdots & \cdots \\ n & n^2 & \cdots & n^n \end{bmatrix}.$$

Then, by our assumption,

$$A \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ . \\ . \\ . \\ p_n \end{bmatrix}$$

where $p_i = iy_1 + i^2 y_2 + \cdots + i^n y_n \in \mathcal{P}$, for $i = 1, 2, \cdots, n$. Premultiplying by the adjoint of A yields

$$(detA) \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix} = \begin{bmatrix} p'_1 \\ p'_2 \\ . \\ . \\ . \\ p'_n \end{bmatrix}$$

where $p'_1, p'_2 \cdots, p'_n \in \mathcal{P}$, since $\mathcal{P}$ is a prime ideal, or, $detA \cdot y_i = p'_i \in \mathcal{P}$ for $i = 1, 2, \cdots, n$. The determinant of A is Vandermonde determinant, is equal to the product of positive integers, each of which is equal or less than n say m_i for $i = 1, 2, \cdots, n$ and hence we get $m_i y_i \in \mathcal{P}$, and so $m_i(y_i + \mathcal{P}) = 0 + \mathcal{P}$, for each i. Since $Char(\mathcal{R}/\mathcal{P}) \neq n!$, it follows that $y_i \in \mathcal{P}$ for all i. □

Now we give the main theorem.

Theorem 2.4. *Let D_1 and D_2 be two nonzero symmetric bi-derivations of a ring $\mathcal{R}$ with traces d_1 and d_2 respectively such that $d_1 d_2(\mathcal{R}) \subseteq \mathcal{P}$, where $\mathcal{P}$ is a prime ideal for which the characteristic of $\mathcal{R}/\mathcal{P}$ is not $3!$, then $d_1(\mathcal{R}) \subseteq \mathcal{P}$ or $d_2(\mathcal{R}) \subseteq \mathcal{P}$.*

Proof. Given that $d_1 d_2(\mathcal{R}) \subseteq \mathcal{P}$, where $\mathcal{P}$ is a prime ideal. For $x \in \mathcal{R}$ we have $d_1 d_2(x) \in \mathcal{P}$. Replace x by $x + ky$, for $1 \leq k \leq 3$, for any $y \in \mathcal{R}$, we get $d_1(d_2(x + ky)) \in \mathcal{P}$, which further reduces in $d_1\big(d_2(x)+d_2(ky)+2D_2(x,ky)\big)\in\mathcal{P}$. Or,

$$\begin{aligned} d_1\big(d_2(x)+d_2(ky)\big)+d_1\big(2D_2(x,ky)\big)+2D_1\big(d_2(x)+d_2(ky),2D_2(x,ky)\big) &\in \mathcal{P}\\ d_1(d_2(x))+d_1(k^2d_2(y))+2D_1\big(d_2(x),k^2d_2(y)\big)+4k^2d_1(D_2(x,y)) &\\ +4kD_1\big(d_2(x),D_2(x,y)\big)+4k^3D_1\big(d_2(y),D_2(x,y)\big) &\in \mathcal{P}\\ d_1(d_2(x))+k^4d_1(d_2(y))+2k^2D_1\big(d_2(x),d_2(y)\big)+4k^2d_1(D_2(x,y)) &\\ +4kD_1\big(d_2(x),D_2(x,y)\big)+4k^3D_1\big(d_2(y),D_2(x,y)\big) &\in \mathcal{P}\end{aligned}$$

and thus,

$$k^1Q_1(x,y)+k^2Q_2(x,y)+k^3Q_3(x,y)\in\mathcal{P},\ \text{for all } x,y\in\mathcal{R},$$

where $Q_i(x,y)$, denotes the sum of the terms in which y appears i times where $i=1,2,3$. By application of Lemma 2.3, we get $Q_i(x,y)\in\mathcal{P}$ for each i, which implies $4D_1\big(d_2(x),D_2(x,y)\big)\in\mathcal{P}$. Since the characteristic of $\mathcal{R}/\mathcal{P}$ is not two, it reduces to

$$D_1\big(d_2(x),D_2(x,y)\big)\in\mathcal{P}\ \text{ for any } x,y\in\mathcal{R}. \tag{2.1}$$

Now on replacing y by yz, for any $z\in\mathcal{R}$, we get $D_1\big(d_2(x),D_2(x,yz)\big)\in\mathcal{P}$. Therefore,

$$\begin{aligned} D_1(d_2(x),y)D_2(x,z)+yD_1\big(d_2(x),D_2(x,z)\big)+D_1\big(d_2(x),D_2(x,y)\big)z &\\ +D_2(x,y)D_1(d_2(x),z) &\in \mathcal{P}.\end{aligned}$$

On using equation (2.1) for any $x,y,z\in\mathcal{R}$, we get

$$D_1(d_2(x),y)D_2(x,z)+D_2(x,y)D_1(d_2(x),z)\in\mathcal{P}. \tag{2.2}$$

On replacing z by zr in (2.2) and using (2.2), we get

$$D_1(d_2(x),y)zD_2(x,r)+D_2(x,y)zD_1(d_2(x),r)\in\mathcal{P}\ \text{ for all } x,y,z\in\mathcal{R}. \tag{2.3}$$

For $r=y$, above equation becomes

$$D_1(d_2(x),y)zD_2(x,y)+D_2(x,y)zD_1(d_2(x),y)\in\mathcal{P}\ \text{ for all } x,y,z\in\mathcal{R}. \tag{2.4}$$

By Lemma 2.1, $D_1(d_2(x),y)\in\mathcal{P}$ or $D_2(x,y)\in\mathcal{P}$ for fixed $x,y\in\mathcal{R}$. Let us assume $D_1(d_2(x),y)\notin\mathcal{P}$. Then $D_2(x,y)\in\mathcal{P}$ for fixed $x,y\in\mathcal{R}$. In view of relation (2.3) and using that $\mathcal{P}$ is a prime ideal of $\mathcal{R}$, we get

$$D_2(x,r)\in\mathcal{P}. \tag{2.5}$$

Now again in (2.2) replace x by $x+ka$ for $1\le k\le 3$, and for any $a\in\mathcal{R}$, we obtain

$$D_1(d_2(x+ka),y)D_2(x+ka,z)+D_2(x+ka,y)D_1(d_2(x+ka),z)\in\mathcal{P}.$$

On simple calculation we obtain,

$$k^1Q_1(x,y,z,a)+k^2Q_2(x,y,z,a)+k^3Q_3(x,y,z,a)\in\mathcal{P},\ \text{for all } x,y,z,a\in\mathcal{R},$$

where $Q_i(x,y,z,a)$, denotes the sum of the terms in which a appears i times where $i=1,2,3$. Application of Lemma 2.3, we get $Q_i(x,y,z,a)\in\mathcal{P}$ for each i, hence

$$\begin{aligned} D_1(d_2(x),y)D_2(a,z)+2D_1(D_2(x,a),y)D_2(x,z)+2D_2(x,y)D_1(D_2(x,a),z) &\\ +D_2(a,y)D_1(d_2(x),z) &\in \mathcal{P}.\end{aligned}$$

On replacing z by zr for any $r \in \mathcal{R}$, in the above equation and using the same, we get

$$D_1(d_2(x), y)zD_2(a, r) + 2D_1(D_2(x, a), y)zD_2(x, r) + 2D_2(x, y)zD_1(D_2(x, a), r)$$

$$+D_2(a, y)zD_1(d_2(x), r) \in \mathcal{P}. \tag{2.6}$$

On using (2.5), the relation (2.6) becomes

$$D_1(d_2(x), y)zD_2(a, r) + D_2(a, y)zD_1(d_2(x), r) \in \mathcal{P}. \tag{2.7}$$

Hence for $r = y$ yields $D_1(d_2(x), y)zD_2(a, y) + D_2(a, y)zD_1(d_2(x), y) \in \mathcal{P}$. Therefore, $D_1(d_2(x), y) \in \mathcal{P}$ or $D_2(a, y) \in \mathcal{P}$. Let us assume $D_1(d_2(x), y) \notin \mathcal{P}$. Then $D_2(a, y) \in \mathcal{P}$. Relation (2.7) reduces to $D_1(d_2(x), y)zD_2(a, r) \in \mathcal{P}$ for any $x, y, a, r \in \mathcal{R}$. Therefore primeness of $\mathcal{P}$ results in either $D_1(d_2(x), y) \in \mathcal{P}$ or $D_2(a, r) \in \mathcal{P}$. If $D_2(a, r) \in \mathcal{P}$ for any $a, r \in \mathcal{R}$ we can easily conclude that $d_2(\mathcal{R}) \subseteq \mathcal{P}$.
Now if $D_1(d_2(x), y) \in \mathcal{P}$. Replace x by $x+z$, for all $z \in \mathcal{R}$, we get $D_1(d_2(x), y)+D_1(d_2(z), y)+2D_1(D_2(x, z), y) \in \mathcal{P}$. On using hypothesis above relation becomes

$$D_1(D_2(x, z), y) \in \mathcal{P}. \tag{2.8}$$

Now for $z = zz'$ for any $z' \in \mathcal{R}$ and using (2.8) we get

$$D_2(x, z)D_1(z', y) + D_1(z, y)D_2(x, z') \in \mathcal{P}.$$

Now again replacing z by zy' for any $y' \in \mathcal{R}$ in the above relation and using the same, we obtain

$$D_2(x, z)y'D_1(z', y) + D_1(z, y)y'D_2(x, z') \in \mathcal{P}. \tag{2.9}$$

For $z' = z$, above relation reduces to

$$D_2(x, z)y'D_1(z, y) + D_1(z, y)y'D_2(x, z) \in \mathcal{P}.$$

By Lemma 2.1, we get $D_1(z, y) \in \mathcal{P}$ or $D_2(z, x) \in \mathcal{P}$. Let us assume that $D_1(z, y) \notin \mathcal{P}$ then $D_2(z, x) \in \mathcal{P}$ for a fixed $z \in \mathcal{R}$. In view of relation (2.9), and by using primeness of $\mathcal{P}$, we get $D_2(x, z') \in \mathcal{P}$. Therefore we have either $D_1(z, y) \in \mathcal{P}$ or $D_2(x, z') \in \mathcal{P}$ for any $x, y, z, z' \in \mathcal{P}$. Also, we can clearly say that $d_1(\mathcal{R}) \subseteq \mathcal{P}$ or $d_2(\mathcal{R}) \subseteq \mathcal{P}$. □

Now our next theorem is on semiprime ideals.

Theorem 2.5. *Let D be nonzero symmetric bi-derivation of a ring $\mathcal{R}$ with trace d and $\mathcal{P}$ is a semiprime ideal of $\mathcal{R}$ for which the characteristic of $\mathcal{R}/\mathcal{P}$ is not* $3!$ *and d is onto, then $d^r(\mathcal{P}) \subseteq \mathcal{P}$ if and only if $d(\mathcal{P}) \subseteq \mathcal{P}$ for any positive integer r.*

Proof. It is obvious to show that if $d(\mathcal{P}) \subseteq \mathcal{P}$, then $d^r(\mathcal{P}) \subseteq \mathcal{P}$ for any positive integer r. We will prove the reverse implication. We proceed the proof with certain claims.
Claim 1. For $r = 2$, if $d^2(\mathcal{P}) \subseteq \mathcal{P}$, then $d(\mathcal{P}) \subseteq \mathcal{P}$.

Let us suppose that for any $x \in \mathcal{P}$, $d^2(x) \in \mathcal{P}$, or $d(d(x)) \in \mathcal{P}$. On replacing x by $x + ky$ for $1 \leq k \leq 3$, for any $y \in \mathcal{P}$, we get $d(d(x)) + k^4 d(d(y)) + 2k^2 D(d(x), d(y)) + 2k^2 d\big(D(x, y)\big) + 4kD(d(x), D(x, y)) + 4k^3 D\big(d(y), D(x, y)\big) \in \mathcal{P}$, which implies that,

$$k^1 Q_1(x, y) + k^2 Q_2(x, y) + k^3 Q_3(x, y) \in \mathcal{P}, \text{ for all } x, y \in \mathcal{R},$$

where $Q_i(x, y)$, denotes the sum of the terms in which y appears i times where $i = 1, 2, 3$. Application of Lemma 2.3, we get $Q_i(x, y) \in \mathcal{P}$ for each i, which gives

$$D(d(x), D(x, y)) \in \mathcal{P} \text{ for all } x, y \in \mathcal{P}. \tag{2.10}$$

For any $z \in \mathcal{R}$, on replacing y by yz, above relation reduces to

$$D(d(x), D(x,y))z + D(x,y)D(d(x),z) + D(d(x),y)D(x,z) + yD(d(x), D(x,z)) \in \mathcal{P}.$$

Since $y \in \mathcal{P}$ and by (2.10) we get

$$D(x,y)D(d(x),z) + D(d(x),y)D(x,z) \in \mathcal{P} \quad \text{for all } \ x,y \in \mathcal{P}, z \in \mathcal{R}.$$

For $z = d(x) \in \mathcal{R}$ above relation yields

$$D(x,y)d^2(x) + D(d(x),y)D(x,d(x)) \in \mathcal{P} \quad \text{for all } \ x,y \in \mathcal{P},$$

by hypothesis, we get

$$D(d(x),y)D(x,d(x)) \in \mathcal{P} \quad \text{for all } \ x,y \in \mathcal{P}.$$

Put $y = x$, in the above relation, we get $(D(d(x),x))^2 \in \mathcal{P}$ and since $\mathcal{P}$ is semiprime ideal we obtain $D(d(x),x) \in \mathcal{P}$ for all $x \in \mathcal{P}$. Now put $x = x + ka$, for $1 \leq k \leq 3$ and for all $a \in \mathcal{P}$, and by using Lemma 2.3, we get

$$D(d(x),a) + 2D(D(x,a),x) \in \mathcal{P} \quad \text{for all } \ x,a \in \mathcal{P}.$$

On replacing a by ax, and since $a, x \in \mathcal{P}$, the above relation becomes $D(x,a)d(x) + D(a,x)d(x) \in \mathcal{P}$, which further reduces to $2d(x)^2 \in \mathcal{P}$, and therefore $d(x)^2 \in \mathcal{P}$ for all $x \in \mathcal{P}$ and hence $d(\mathcal{P})^2 \subseteq \mathcal{P}$. Consider the ideal $d(\mathcal{P}) + \mathcal{P}$ of $\mathcal{R}$. We have $\big(d(\mathcal{P}) + \mathcal{P}\big)^2 \subseteq \mathcal{P}$. Therefore, in view of Lemma 2.2, we find $d(\mathcal{P}) + \mathcal{P} \subseteq \mathcal{P}$, and hence $d(\mathcal{P}) \subseteq \mathcal{P}$.

Claim 2. For $r = 2$, if $d^2(\mathcal{R}) \subseteq \mathcal{P}$, then $d(\mathcal{R}) \subseteq \mathcal{P}$.

The proof is quite similar as used in Claim 1, let us suppose that for any $x \in \mathcal{R}$, $d^2(x) \in \mathcal{P}$, or $d(d(x)) \in \mathcal{P}$. On replacing x by $x + ky$ for $1 \leq k \leq 3$, for any $y \in \mathcal{R}$, we get $d(d(x)) + k^4 d(d(y)) + 2k^2 D(d(x), d(y)) + 2k^2 d\big(D(x,y)\big) + 4kD(d(x), D(x,y)) + 4k^3 D(d(y), D(x,y) \in \mathcal{P}$, which implies that,

$$k^1 Q_1(x,y) + k^2 Q_2(x,y) + k^3 Q_3(x,y) \in \mathcal{P}, \text{ for all } x,y \in \mathcal{R},$$

where $Q_i(x,y)$, denotes the sum of the terms in which y appears i times where $i = 1,2,3$. Application of Lemma 2.3, we get $Q_i(x,y) \in \mathcal{P}$ for each i, which gives

$$D(d(x), D(x,y)) \in \mathcal{P} \ \text{ for all } \ x,y \in \mathcal{R}. \tag{2.11}$$

For any $z \in \mathcal{R}$, on replacing y by yz in the above relation and using the same, we get

$$D(x,y)D(d(x),z) + D(d(x),y)D(x,z) \in \mathcal{P} \quad \text{for all } \ x,y,z \in \mathcal{R}.$$

For $z = d(x) \in \mathcal{R}$ above relation yields

$$D(x,y)d^2(x) + D(d(x),y)D(x,d(x)) \in \mathcal{P} \quad \text{for all } \ x,y \in \mathcal{R},$$

by hypothesis, we get

$$D(d(x),y)D(x,d(x)) \in \mathcal{P} \quad \text{for all } \ x,y \in \mathcal{R}.$$

Put $y = x$, in the above relation, we get $(D(d(x),x))^2 \in \mathcal{P}$ and since $\mathcal{P}$ is semiprime ideal we obtain $D(d(x),x) \in \mathcal{P}$ for all $x \in \mathcal{R}$. Now put $x = x + ka$, for $1 \leq k \leq 3$ and for all $a \in \mathcal{R}$, and by using Lemma 2.3, we get

$$D(d(x),a) + 2D(D(x,a),x) \in \mathcal{P} \quad \text{for all } \ x,a \in \mathcal{R}. \tag{2.12}$$

On replacing a by ax, and using $D(d(x),x) \in \mathcal{P}$ and (2.12), the above relation becomes $2D(x,a)d(x) + 2D(a,x)d(x) \in \mathcal{P}$, which further reduces to $2d(x)^2 \in \mathcal{P}$, and therefore $d(x)^2 \in \mathcal{P}$ for all $x \in \mathcal{R}$ and hence on similar lines as in Claim 1, Claim 2 is proved.

Claim 3. For $r = 3$, if $d^3(\mathcal{P}) \subseteq \mathcal{P}$, then $d(\mathcal{P}) \subseteq \mathcal{P}$.

For any $p \in \mathcal{P}$, we have $d^3(p) \in \mathcal{P}$, i.e., $d^2(d(p)) \in \mathcal{P}$. Let $x = d(p) \in \mathcal{R}$, then we have $d^2(x) \in \mathcal{P}$ for $x \in \mathcal{R}$. By Claim 2, we get $d(x) \in \mathcal{P}$ for $x \in \mathcal{R}$. Therefore, $d^2(p) \in \mathcal{P}$ for $p \in \mathcal{P}$, or $d^2(\mathcal{P}) \subseteq \mathcal{P}$ and hence by Claim 1, we obtain $d(\mathcal{P}) \subseteq \mathcal{P}$.

Claim 4. For any positive integer r, if $d^r(\mathcal{P}) \subseteq \mathcal{P}$, then $d(\mathcal{P}) \subseteq \mathcal{P}$.

For any $p \in \mathcal{P}$, we have $d^r(p) \in \mathcal{P}$, i.e., $d^2(d^{r-2}(p)) \in \mathcal{P}$. If $x = d^{r-2}(p) \in \mathcal{R}$, then we have $d^2(x) \in \mathcal{P}$ for $x \in \mathcal{R}$. By Claim 2, we get $d(x) \in \mathcal{P}$ for $x \in \mathcal{R}$. Therefore, $d^{r-1}(p) \in \mathcal{P}$ for $p \in \mathcal{P}$, or $d^2(d^{r-3}(p)) \in \mathcal{P}$. Proceeding in the similar manner we arrive at $d^2(\mathcal{P}) \subseteq \mathcal{P}$ and hence by Claim 1, we obtain $d(\mathcal{P}) \subseteq \mathcal{P}$. This completes the proof of our theorem. $\square$

3 Conjectures

For a fixed positive integer n, a map $D : \mathcal{R}^n \longrightarrow \mathcal{R}$ is said to be permuting if $D(x_1, x_2, ..., x_n) = D(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$ for all $\pi \in S_n$ and $x_i \in \mathcal{R}$ where $i = 1, 2, ..., n$. The notion of permuting n-derivation was defined by Park [11] as follows: a permuting map $D : \mathcal{R}^n \longrightarrow \mathcal{R}$ is said to be permuting n-derivation if D is n-additive (i.e., additive in each argument) and $D(x_1, x_2, ..., x_i x_i', ..., x_n) = x_i D(x_1, x_2, ..., x_i', ..., x_n) + D(x_1, x_2, ..., x_i, ..., x_n) x_i'$ holds for all $x_i, x_i' \in \mathcal{R}$. Trace, $d : \mathcal{R} \longrightarrow \mathcal{R}$ of D is defined by $d(x) = D(x, x, \cdots, x)$. A 1-derivation is a derivation and a 2-derivation is a symmetric bi-derivation. In view of the concept on n-derivation one can easily think of the following conjectures.

Conjecture 3.1. Let D_1 and D_2 be two nonzero permuting n-derivations of a ring $\mathcal{R}$ with traces d_1 and d_2 respectively such that $d_1 d_2(\mathcal{R}) \subseteq \mathcal{P}$, where $\mathcal{P}$ is a prime ideal for which the characteristics of $\mathcal{R}/\mathcal{P}$ is not finite, then $d_1(\mathcal{R}) \subseteq \mathcal{P}$ or $d_2(\mathcal{R}) \subseteq \mathcal{P}$.

Conjecture 3.2. Let D be nonzero permuting n-derivation of a ring $\mathcal{R}$ with trace d and $\mathcal{P}$ is a semiprime ideal of $\mathcal{R}$ for which the characteristic of $\mathcal{R}/\mathcal{P}$ is not finite, then $d^r(\mathcal{P}) \subseteq \mathcal{P}$ if and only if $d(\mathcal{P}) \subseteq \mathcal{P}$ for any positive integer r.

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