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On e-Reversible Rings

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Abstract Let R be a ring and e be an idempotent element of R, then R is said to be an e-reversible ring if $ab = 0$ implies $bae = 0$ and we call R a *strongly e-reversible ring* if $ab = 0$ implies $bea = 0$, for all $a, b \in R$. We provide a number of examples of e-reversible and non e-reversible rings. We characterize (strongly) e-reversible rings. Also, we study various properties and extensions of (strongly) e-reversible rings.

1 Introduction and Preliminaries

The study of reversible rings, which are generalization of reduced rings, is meaningful in Ring Theory. Throughout all rings are associative and noncommutative with identity unless otherwise stated. We denote the center, the set of all nilpotent elements and the set of all idempotent elements of a ring R by $Z(R)$, $N(R)$ and $E(R)$, respectively. Let $M_n(R)$, $T_n(R)$ and $D_n(R)$ be the ring of all $n \times n$ matrices, upper triangular matrices and diagonal matrices over the ring R, respectively. $E_{ij} \in M_n(R)$ denotes the matrix with $(i, j)^{th}$ entry 1_R (the identity of R) and elsewhere 0_R (the zero of R). We refer readers to [8] for all undefined terms and notions.

We begin by defining the notions of (strongly) e-reversible rings.

Definition 1.1. Let R be a ring and $e \in E(R)$.

(i) If $ab = 0$ implies $bae = 0$ for all $a, b \in R$, then R is said to be *e-reversible*.

(ii) If $ab = 0$ implies $bea = 0$ for all $a, b \in R$, then R is said to be *strongly e-reversible*.

According to [2], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$.

Example 1.2. (i) Every reversible ring is e-reversible for any idempotent element e of the ring, but the converse need not be true. In support, let $R = T_2(D)$ where D is a domain. Then

a. R is an E_{11}-reversible ring by Corollary 2.3(i), and

b. R is not reversible because if, we take $A = E_{12}, B = E_{11} \in R$, then $AB = 0$ while $BA = E_{12} \neq 0$.

(ii) Every strongly e-reversible ring is e-reversible for any idempotent element e of the ring; it follows from Theorem 2.4 − 2.5, as we will subsequently prove. But the converse need not be true. In support, consider the ring R from Example 1.2(i). Then

a. R is an E_{11}-reversible ring; and

b. R is not strongly E_{11}-reversible because if, we take $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in R$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but $BE_{11}A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$.

In [9], a ring R is said to be *symmetric* if $abc = 0$ implies $acb = 0$, for all $a, b, c \in R$. It is clear that symmetric rings are reversible but not conversely (see [10, Example 5]). Recently, F. Meng and J. Wei [11] defined the notion of an (strongly) e-symmetric ring where e is an idempotent element of R, which is a generalization of a symmetric ring. A ring R is said to be *(strongly) e-symmetric* if for any $a, b, c \in R$, $abc = 0$ implies $(aceb = 0)$ $acbe = 0$. A ring R is called *reduced* if it has no nonzero nilpotent elements. A ring R is said to be a *right e-reduced* ring if $N(R)e = 0$. We now find some relations between (strongly) e-reversible rings and these rings.

Example 1.3. (i) Every e-symmetric ring R is e-reversible, where $e \in E(R)$. Let R be a ring which is reversible but not symmetric (for such a ring see [10, Example 5]). Then $T_2(R)$ is E_{11}-reversible by Corollary 2.3(i). But $T_2(R)$ is not E_{11}-symmetric by [12, Proposition 4.1(1)].

(ii) Every strongly e-symmetric ring R is strongly e-reversible for $e \in E(R)$. Consider a ring R (see [10, Example 5]), it is strongly 1_R-reversible but not strongly 1_R-symmetric.

(iii) It follows from [11, Corollary 4.3] that for $e \in E(R)$, a right e-reduced ring is e-reversible.

Recall [7], a ring R is said to be *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. Recall [11], a ring R is said to be *abelian* if all idempotents of R are central.

Lemma 1.4. *A semiprime and semicommutative ring R is (strongly) e-reversible for any $e \in E(R)$.*

Proof. Clear. □

Remark 1.5. The converse of Lemma 1.4 need not be true. Consider, a ring R from Example 2.8(ii). Then R is e-reversible but not abelian. Since semicommutative rings are abelian by [4, Lemma 1] , therefore R is not semicommutative.

Example 1.6. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is any field and let $I = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ be an ideal of R. Then

(i) R/I is (strongly) $\bar{e}$-reversible for any $\bar{e} \in E(R/I)$ because $R/I \simeq F$.

(ii) R is not an e-reversible ring for $e = E_{22}$ as if, we take $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $AB = 0$ while $BAe = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \neq 0$.

(iii) R is not strongly e-reversible.

Lemma 1.7. *Let D be a division ring and $R = D < x, y >$ be the free D-algebra in two noncommutating variables x and y. Then*

(i) R is an (strongly) 1_R-reversible ring.

(ii) For ideal $I =< xy >, R/I$ is not (strongly) $\bar{1}_R$-reversible.

Example 1.8. Let $S = R/I$, where $R = D < x, y >$ be the free D-algebra in two noncommutating variables x and y, D be a division ring and $I =< xy >$. Let $a = x + I$, $b = y + I$ and $e = I$, then $ab = 0$. Also, $bea = (y + I)I(x + I) = 0$ in R/I but $ba = (y + I)(x + I) \neq 0$. Then S is a strongly e-reversible ring but not a reversible ring.

Recall [11], an idempotent e of a ring R is said to be *left (right) semicentral* if $ae = eae$ $(ea = eae)$ for each $a \in R$.

Proposition 1.9. *Let I be a reduced ideal of a ring R and $e \in E(R)$.*

(i) For any $\bar{e} \in E(R/I)$, if R/I is an $\bar{e}$-reversible and e is left semicentral, then R is an e-reversible ring.

(ii) For any $\bar{e} \in E(R/I)$, if R/I is a strongly $\bar{e}$-reversible ring, then R is a strongly e-reversible ring.

Proof. (1). Let $a, b \in R$ such that $ab = 0$. Then $ab \in I$ and so $bae \in I$ because R/I is $\bar{e}$-reversible. As $ab = 0$ and e is left semicentral, we have $(bae)^2 = baebae = babae = 0$. Since I is a reduced ring, therefore $bae = 0$. Hence R is an e-reversible ring.

(2). It is analogous to (1). □

Lemma 1.10. *Let S be any subring of a ring R and $e \in E(S)$. If R is e-reversible (strongly e-reversible), then S is also e-reversible (strongly e-reversible).*

Lemma 1.11. *Let $(R_i)_{i \in I}$ be a family of rings and $(e_i)_{i \in I} \in E(\Pi_{i \in I} R_i)$. Then $\Pi_{i \in I} R_i$ is an $(e_i)_{i \in I}$-reversible ring if and only if for each $i \in I$, R_i is an e_i-reverssible ring.*

Corollary 1.12. *For a central idempotent element e of the ring R, eR and $(1-e)R$ are e-reversible rings if and only if R is an e-reversible ring.*

2 e-Reversible and Strongly e-reversible Rings

In this section, we provide characterizations of reversible rings; (strongly) e-reversible rings; (strongly)left minimal abelian rings; left quasi-duo rings. Finally, we discuss some equivalent classes of rings over a semiprime ring.

Theorem 2.1. *Given a ring R containing an idempotent $e \in E(R)$, for any $r_1, r_2, r_3, \cdots, r_{n-1} \in R$ define the idempotent X of the ring $T_n(R)$ of upper triangular n by n matrices over R to be* $X = \begin{bmatrix} e & er_1 & \cdots & er_{n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Then R is e-reversible if and only if $T_n(R)$ is X-reversible.*

Proof. Let $a, b \in R$ such that $ab = 0$. If we take $A = aE_{11}, B = bE_{11} \in T_n(R)$, then $AB = 0$ and so $BAX = 0$, because $T_n(R)$ is X-reversible. This implies that $bae = 0$. Thus, R is e-reversible. Conversely, let $A = [a_{ij}], B = [b_{ij}] \in T_n(R)$ such that $AB = 0$, then we have $a_{ii}b_{ii} = 0 \,\forall\, 1 \leq i \leq n$ and so $b_{ii}a_{ii}e = 0 \,\forall\, 1 \leq i \leq n$, due to R is e-reversible. So,

$$BAX = \begin{bmatrix} b_{11}a_{11}e & b_{11}a_{11}er_1 & \cdots & b_{11}a_{11}er_{n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = 0.$$

Thus, $T_n(R)$ is X-reversible.

Remark 2.2. (i) If $e = 1$, then R is a reversible ring if and only if $T_n(R)$ is a X-reversible ring for any $n \geq 1$.

(ii) If $r_1, r_2, r_3, \cdots, r_{n-1} = 0$, then R is e-reversible if and only if $T_n(R)$ is eE_{11}-reversible for any $n \geq 1$.

□

Corollary 2.3. *Let R be a ring and $e \in E(R)$.*

(i) R is reversible if and only if $T_n(R)$ is E_{11}-reversible for any $n \geq 1$.

(ii) R is (strongly) e-reversible if and only if $D_n(R)$ is (strongly) eI_n-reversible for any $n \geq 1$.

Theorem 2.4. *The following are equivalent for a ring R and $e \in E(R)$:*

(i) R is an e-reversible ring.

(ii) eRe *is a reversible ring and* e *is left semicentral.*

Proof. Suppose R is an e-reversible ring. For $x \in R$, $e(1-e)x = 0$. It follows that $(1-e)xe = (1-e)xe^2 = 0$, this implies that $xe = exe$. Thus, e is left semicentral. Now, we have to show that eRe is a reversible ring. Let $a, b \in eRe$ such that $ab = 0$. Since eRe is a subring of R and R is an e-reversible ring, therefore we have $bae = 0$. Also, $ae = a$ implies that $ba = 0$. Thus, eRe is a reversible ring.

Conversely, suppose eRe is a reversible ring and e is left semicentral. Let $a, b \in R$ such that $ab = 0$. It follows that $0 = abe = eaebe$ and $bae = ebeae = 0$. Hence, R is an e-reversible ring. □

Theorem 2.5. *The following are equivalent for a ring* R *and* $e \in E(R)$*:*

(i) R *is a strongly* e*-reversible ring.*

(ii) eRe *is a reversible ring and* $e \in Z(R)$*.*

Proof. Assume that R is a strongly e-reversible ring. For each $a \in R$, $a(1-e)e = 0$. It follows that $ea(1-e) = e^2a(1-e) = 0$. Thus, $ea = eae$. Since R is a strongly e-reversible ring, therefore e-reversible. Then by Theorem 2.4, e is left semicentral. It follows that $e \in Z(R)$. Again by Theorem 2.4, eRe is a reversible ring.

Conversely, suppose that $e \in Z(R)$ and eRe is a reversible ring. It follows that $eae = aee = ae$. This implies that e is left semicentral. By Theorem 2.4, R is an e-reversible ring. Due to e as a central element, R is a strongly e-reversible ring.

□

Corollary 2.6. *The following are equivalent for a ring* R *and* $e \in E(R)$*:*

(i) R *is a strongly* e*-reversible ring.*

(ii) R *is an* e*-reversible ring and* $e \in Z(R)$*.*

Proof. It directly follows from Theorem 2.4 and 2.5. □

Recall [12], a ring R is said to be *left* e*-reflexive* if $aRe = 0 \implies eRa = 0$ for any $a \in R$.

Proposition 2.7. *The following are equivalent for a ring* R *and* $e \in E(R)$*:*

(i) R *is strongly* e*-reversible ring.*

(ii) R *is* e*-reversible and left* e*-reflexive.*

Proof. Assume that R is strongly e-reversible. It follows by Theorem 2.5 that e is central and R is e-reversible. Let $a \in R$ such that $aRe = 0$. It gives $ae = 0$ and so, $eRa = aRae = 0$ due to e is central. Thus, R is left e-reflexive.

Conversely, suppose R is e-reversible and left e-reflexive. Since R is e-reversible, it follows by Theorem 2.4 that e is left semicentral. So, we have $(1-e)Re = 0$ which implies that $eR(1-e) = 0$ as R is left e-reflexive. Hence, e is central and thus R is strongly e-reversible by Theorem 2.4 and 2.5. □

According to [7], a ring R is called *central reversible* if for any $a, b \in R$, $ab = 0$ implies ba belongs to the center of R. Recall from [11] that an element $e \in E(R)$ is said to be *left minimal idempotent* of R if Re is a minimal left ideal of R. The set of all left minimal idempotents of R is denoted by $ME_l(R)$. By [15], a ring R is called *left minimal abelian* if either $ME_l(R) = \phi$ or every $e \in ME_l(R)$ is left semicentral.

Example 2.8. (i) In general, an abelian ring need not be e-reversible for some $e \in E(R)$. Consider the following ring from [7],

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a \equiv d(mod2), b \equiv c \equiv 0(mod2), a, b, c, d \in \mathbb{Z} \right\}.$$

Then

a. R is an abelian ring because $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are the only idempotents in R.

b. R is not an e-reversible ring for $e = I_2$ as if, we take $x = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, y = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \in R$, then $xy = 0$ but $yxe = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \neq 0$.

(ii) An e-reversible ring need not be a central reversible ring for some $e \in E(R)$. Consider a ring R from Example 1.2(i). Then

a. R is an E_{11}-reversible ring, and

b. R is not central reversible because if, we take $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then $AB = 0$ but $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A(BA) \neq (BA)A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ which implies that $BA \notin Z(T_2(R))$.

c. It is also observed that R is not abelian as if, we take $A = E_{12}, B = E_{11}$ then $0 = AB \neq BA = E_{12}$. Thus, an e-reversible ring need not be abelian.

Remark 2.9. If R is a central reversible ring, then R is e-reversible for any $e \in ME_l(R)$.

In the following, we characterize a left minimal abelian ring:

Proposition 2.10. *The following are equivalent for a ring R:*

(i) R is a left minimal abelian ring.

(ii) R is e-symmetric for any $e \in ME_l(R)$.

(iii) R is e-reversible for any $e \in ME_l(R)$.

Proof. $(1) \implies (2)$. It follows from [11, Theorem 2.5].
$(2) \implies (3)$. It follows from Example 1.3(i).
$(3) \implies (1)$. Let $e \in ME_l(R)$. By the assumption, R is e-reversible. So, e is left semicentral by Theorem 2.4. Hence, R is left minimal abelian. □

Corollary 2.11. *The following are equivalent for a ring R:*

(i) R is a left quasi-duo ring,

(ii) R is a MELT ring and e-symmetric for any $e \in ME_l(R)$,

(iii) R is a MELT ring and e-reversible for any $e \in ME_l(R)$.

Proof. $(1) \iff (2)$. It follows from [11, Corollary 2.6].
$(2) \iff (3)$. It follows from Proposition 2.10. □

Recall [15], a ring R is called a *strongly left minimal abelian* if $ME_l(R) \subseteq Z(R)$.

Proposition 2.12. *The following are equivalent for a ring R:*

(i) R is a strongly left minimal abelian ring,

(ii) R is strongly e-symmetric for any $e \in ME_l(R)$,

(iii) R is strongly e-reversible for any $e \in ME_l(R)$.

Proof. $(1) \implies (2)$. It follows from [11, Theorem 3.4].
$(2) \implies (3)$. It follows from Example 1.3(ii).
$(3) \implies (1)$. Let $e \in ME_l(R)$. By the assumption, R is strongly e-reversible. It follows by Theorem 2.5 that $e \in Z(R)$. Hence, R is strongly left minimal abelian. □

Corollary 2.13. *If R is an abelian ring, then R is (strongly) e-reversible for every $e \in ME_l(R)$.*

Lemma 2.14. *The following are equivalent for a ring R and $e \in E(R)$:*

(i) R is a reversible ring.

(ii) R is both an e-reversible and $(1-e)$-reversible ring.

Proof. (1) $\implies$ (2). Clear.

(2) $\implies$ (1). Let $a, b \in R$ such that $ab = 0$. Since R is $(1-e)$-reversible ring, therefore $ba(1-e) = 0$. This implies that $ba = 0$ as R is an e-reversible ring. Hence, R is a reversible ring.

Corollary 2.15. *The following are equivalent for a ring R and $e \in E(R)$:*

(i) R is a reversible ring.

(ii) R is a strongly e-reversible ring and $(1-e)R(1-e)$ is a reversible ring.

Proof. Suppose that R is a reversible ring. It follows that R is an e-reversible ring and an abelian ring. This implies that $e \in Z(R)$. By Corollary 2.6, R is a strongly e-reversible ring. Since $(1-e)R(1-e)$ is a subring of R, therefore $(1-e)R(1-e)$ is also a reversible ring.

Conversely, suppose that R is a strongly e-reversible ring and $(1-e)R(1-e)$ is a reversible ring. Then by Theorem 2.5, R is a strongly $(1-e)$-reversible as $(1-e) \in Z(R)$. It follows that R is both an e-reversible and $(1-e)$-reversible ring. Hence by Lemma 2.14, R is a reversible ring. □

□

3 Some extensions

In this section, first we discuss some extensions of the class of (strongly) e-reversible rings and after that various properties related to these classes of rings with $*$-rings.

Recall [3], let R be an algebra over a commutative ring S. The *Dorroh extension* of R by S is the ring $R \times S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1 r_2 + s_1 r_2 + s_2 r_1, s_1)s_2$, where $r_i \in R$ and $s_i \in S$. According to Rege and Chhawchharia [14], a ring R is called *Armendariz* if for any $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \ inR[x]$, $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for all i and j.

Theorem 3.1. *Let R be an e-reversible ring and $e \in E(R)$.*

(i) If R is an algebra over a commutative domain S, and D is the Dorroh extension of R by S, then D is an e-reversible ring.

(ii) If S is a multiplicatively closed subset of R consisting of central regular elements, then $S^{-1}R$ is an e-reversible ring.

(iii) If R is an Armendariz ring, then $R[x]$ is an e-reversible ring.

Proof. (1). It is easy to prove by following the proof of [5, Proposition 1.14(2)].

(2). It is analogous to [5, Proposition 1.13(2)].

(3). Clear. □

Corollary 3.2. *Let R be a ring. If $R[x]$ is an e-reversible ring, then $R[x; x^{-1}]$ is also an e-reversible ring.*

Proof. It follows from Theorem 3.1(ii). □

Recall [12], two idempotents e and f in R are said to be *left (respectively, right) isomorphic* if $Re \cong Rf$ as left R-modules (respectively, $eR \cong fR$ as right R-modules).

Proposition 3.3. *Let R be an e-reversible ring and $e, f \in E(R)$.*

(i) If e and f are left isomorphic, then R is f-reversible.

(ii) If e and f are left isomorphic, then $eR = fR$.

Proof. Since R is e-reversible, by Theorem 2.4 e is left semicentral.

(1). Let $\phi : Re \to Rf$ be the left R-module isomorphism. Then $\exists\, a \in R$ such that $f = \phi(ae)$, so we have $ef = e\phi(ae) = \phi(eae) = \phi(ae) = f$. Let $a, b \in R$ such that $ab = 0$ then $bae = 0$ as R is e-reversible. So, we have $baf = baef = 0$. Thus, R is f-reversible.

(2). Suppose e and f are left isomorphic. Then by (1), R is a f-reversible ring. Hence, f is left semicentral by Theorem 2.4. by observing the proof of (1), we have $f = ef$ and $e = fe$ which implies that $eR = fR$. □

Proposition 3.4. *Let R be a strongly e-reversible ring and $e, f \in E(R)$. If e and f are left isomorphic, then $e = f$.*

Proof. Since R is strongly e-reversible, therefore by Corollary 2.6 e is central and R is e-reversible. It follows by Proposition 3.3(ii) that $eR = fR$. This implies that $e = fe$ and $f = ef$. So, $f = fe = e$ due to e is central.

□

Proposition 3.5. *Let R be an e-reversible ring and $e, f \in E(R)$. If R satisfies any one of the following conditions, then R is f-reversible:*

(i) $eR + (1 - f)R = R$.

(ii) $ea + 1 - f \in U(R)$ *for some* $a \in R$.

(iii) $Re + R(1 - f) = R$.

(iv) $ae + 1 - f \in U(R)$ *for some* $a \in R$.

Proof. Let R be an e-reversible ring. Then by Theorem 2.4, e is left semicentral. In each case first, we show that $f = ef$ which implies by the proof of Proposition 3.3(i) that R is f-reversible.

(1). Since $eR + (1 - f)R = R$ and e is left semicentral, $fR = feR = efeR \subseteq eR$ and this implies that $f = ef$.

(2). Consider $ea + 1 - f = u \in U(R)$. This implies that $fu = f(ea + 1 - f) = fea$ and therefore $f = feau^{-1}$. So $f = efeau^{-1} = ef$, as e is left semicentral.

(3). Since $Re + R(1 - f) = R$, $Rf = Ref$. Consider $f = xef$ for some $x \in R$. It gives $f = exef = ef$, as e is left semicentral.

(4). Consider $ae + 1 - f = v \in U(R)$. It gives $fv = f(ae + 1 - f) = fae$ which implies that $f = faev^{-1}$. So $f = faev^{-1} = efaev^{-1} = ef$, as e is left semicentral. □

Recall [12], an *involution* $a \mapsto a^*$ in a ring R is a map with the properties: $(a^*)^* = a, (a + b)^* = a^* + b^*, (ab)^* = b^*a^*$ for all $a, b \in R$. A ring R with an involution $*$ is called a $*$-*ring*. Let R be a $*$-ring and $e \in E(R)$. If $e^* = e$, then e is called *projection*.

Proposition 3.6. *If $e \in E(R)$ is a projection element in a $*$-ring R, then R is strongly e-reversible if and only if R is e-reversible.*

Proof. The projection element in a $*$-ring R is left semicentral if and only if it is central (see the proof of [12, Proposition 3.2]). Now, the result follows from Theorem 2.4 and 2.5. □

Proposition 3.7. *Let R be a $*$-ring and e-reversible with $e \in E(R)$. Then*

(i) $e^*e \in E(R)$.

(ii) The following conditions are equivalent:

- *(a)* R *is* e^*e*-reversible.*
- *(b)* R *is strongly* e^*e*-reversible.*
- *(c)* e^*e *is central.*
- *(d)* $e^*xe = xe^*e$ *for each* $x \in R$.
- *(e)* $ee^*e = e^*e$.

Proof. (i) Since R is an e-reversible ring, therefore e is left semicentral by Theorem 2.4. It follows that $(e^*e)^2 = e^*ee^*e = e^*e^*e = e^*e$.

(ii) (a)⇒(b). Assume the condition (a). From (1), it follows that e^*e is a projection. By the Proposition 3.6, R is strongly e^*e-reversible.

(b)⇒(c). It follows from Theorem 2.5.

(c)⇒(d). Since e^*e is central, we have $xe^*e = e^*ex$ for each $x \in R$ and so, $xe^*e = e^*exe$. This implies that $xe^*e = e^*xe$, e is left semicentral.

(d)⇒(e). Take $x = e$ in the assumption, then we have $ee^*e = e^*e$.

(e)⇒(a). Let $a, b \in R$ such that $ab = 0$. Then $bae = 0$, as R is e-reversible. So, $bae^*e = baee^*e = 0$. Thus, R is e^*e-reversible.

□

Proposition 3.8. *If R is a $*$-ring and e-reversible, $e \in E(R)$, then the following conditions are equivalent:*

(1) $ee^* \in E(R)$.

(2) $e^*xe = xee^*$ *for each* $x \in R$.

(3) $ee^* = e^*e$.

(4) ee^* *is central.*

Proof. (1)⇒(2). Assume the condition (1). It follows that R is ee^*-reversible (since $bae = 0 \implies baee^* = 0$) and so, ee^* is left semicentral by Theorem 2.4. Hence, $xee^* = ee^*xee^*$ for each $x \in R$ which implies that $xee^* = e^*xe$ for each $x \in R$, as e is left semicentral. So, e^* is right semicentral.

(2)⇒(3). Choose $x = e$ in the assumption, then we have $ee^* = e^*e$.

(3)⇒(4). Since R is e-reversible and $ee^*e = e^*ee = e^*e$, by Proposition 3.7(ii), $ee^* = e^*e$ is central.

(4)⇒(1). Since e is left semicentral and ee^* is central, we have $(ee^*)^2 = ee^*ee^* = e^*ee^* = ee^*e^* = ee^*$. This implies that $ee^* \in E(R)$. □

Proposition 3.9. *Let R be a $*$-ring and e-reversible, $e \in E(R)$. If $1 + (e^* - e)^*(e^* - e) \in U(R)$, then R is a strongly e-reversible ring and e is a projection.*

Proof. Consider $u = 1 + (e^* - e)^*(e^* - e)$ and $v = u^{-1}$, then we have $u^* = u, eu = ue = ee^*e, e^*u = ue^*$ and $v^* = v, ev = ve, e^*v = ve^*$. Choose $f = ee^*v = vee^*$, then $f^2 = (vee^*)(ee^*v) = v(ee^*e)e^*v = veue^*v = veuve^* = vee^* = f$ and $f^* = (vee^*)* = (ee^*) * v^* = ee^*v = f$ which implies that f is a projection. Since R is e-reversible and $ef = e(ee^*v) = ee^*v = f$, R is ef-reversible (because $bae = 0 \implies baf = baef = 0$). By the Proposition 3.6, R is strongly ef-reversible and so, f is central by the Theorem 2.5. This implies that $f = ef = fe = vee^*e = vue = e$. Thus, R is an strongly e-reversible ring and e is a projection. □

Recall [6], let R be a $*$-ring and $e \in E(R)$, then $p \in R$ is called a *range projection* if, p is a projection satisfying $pe = e$ and $ep = p$. The projection of e is denoted by $e^{\perp}$.

Proposition 3.10. *If R is a $*$-ring and e-reversible for $e \in E(R)$, then the following conditions are equivalent:*

(1) $1 + (e^* - e)^*(e^* - e) \in U(R)$.

(2) $e + e^* - 1 \in U(R)$.

(3) $e^{\perp}$ *exists.*

Proof. (1)⇒(2). Suppose $1 + (e^* - e)^*(e^* - e) \in U(R)$. By the Proposition 3.9, e is projection and so $e + e^* - 1 = 2e - 1 \in U(R)$.

(2)⇒(3). It follows from [6, Theorem 2.1].

(3)⇒(1). Let $p = e^{\perp}$. Since R is e-reversible and $ep = p$, R is p-reversible and so, p is central by the Proposition 3.6 and Theorem 2.5. It follows that $e = pe = ep = p$. Since e is central, $(e^* - e)^*(e^* - e) = 0$ and so, $1 + (e^* - e)^*(e^* - e) = 1 \in U(R)$. □

Recall [13], an element $a^\dagger$ in a $*$-ring is said to be the *Moore-Penrose inverse (or MP inverse)* if $aa^\dagger a = a, a^\dagger aa^\dagger = a^\dagger, aa^\dagger = (aa^\dagger)^*, a^\dagger a = (a^\dagger a)^*$. In this case, we say that a is *MP-invertible*. The set of all MP-invertible elements of R is denoted by $R^\dagger$.

Lemma 3.11. *[6, Theorem 3.1] Let R be a $*$-ring and let $e \in E(R)$. Then $e \in R^\dagger$ if and only if $e + e^* - 1 \in U(R)$.*

Proposition 3.12. *Let R be a $*$-ring and let $e \in E(R)$ such that R is an e-reversible ring. Then $e \in R^\dagger$ if and only if e is a projection.*

Proof. It follows from Proposition 3.9-3.10 and Lemma 3.11. □

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