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On Property ($\mathcal{A}$) with respect to an ideal

A. Ait Ouahi, Y. Arssi and S. Bouchiba

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Abstract. The main goal of this paper is to introduce and study the notion of Property ($\mathcal{A}$) of a ring R or an R-module M with respect to an ideal I of R. The new notion turns out to be a weak form of the classic notion of an $\mathcal{A}$-ring, in the sense that, any R-module satisfying the Property ($\mathcal{A}$) satisfies as well the Property ($\mathcal{A}$) with respect to any ideal I of R. Moreover, we prove that if I is contained in the nilradical of R, then the notion of $\mathcal{A}$-module with respect to I and the notion of $\mathcal{A}$-module collapse. Also, we present an example of a ring R possessing an ideal $I \subseteq \mathrm{Z}(R)$ such that R is an $\mathcal{A}$-ring with respect to I while R is not an $\mathcal{A}$-ring. Finally, we totally characterize the rings R and the R-modules M satisfying the Proprety ($\mathcal{A}$) with respect to an ideal I as well as we investigate the behavior of the Property ($\mathcal{A}$) with respect to an ideal vis-à-vis the direct products of rings and modules.

1 Introduction

Throughout this paper, all rings are supposed to be commutative with unit element and all R-modules are unital. Let R be a commutative ring and M an R-module. We denote by $\mathrm{Z}_R(M) = \{r \in R : rm = 0 \text{ for some nonzero element } m \in M\}$ the set of zero divisors of R on M and by $\mathrm{Z}(R) := \mathrm{Z}_R(R)$ the set of zero divisors of the ring R. In [4], the notions of $\mathcal{A}$-module and $\mathcal{SA}$-module are extensively studied. In fact, an R-module M satisfies Property ($\mathcal{A}$), or M is an $\mathcal{A}$-module over R (or $\mathcal{A}$-module if no confusion is likely), if for every finitely generated ideal I of R with $I \subseteq \mathrm{Z}_R(M)$), there exists a nonzero $m \in M$ with $Im = 0$, or equivalently, $\mathrm{ann}_M(I) \neq 0$. M is said to satisfy strong Property ($\mathcal{A}$), or is an $\mathcal{SA}$-module over R (or an $\mathcal{SA}$-module if no confusion is likely), if for any $r_1, \cdots, r_n \in \mathrm{Z}_R(M)$, there exists a nonzero $m \in M$ such that $r_1 m = \cdots = r_n m = 0$. The ring R is said to satisfy Property ($\mathcal{A}$), or an $\mathcal{A}$-ring, (respectively, $\mathcal{SA}$-ring) if R is an $\mathcal{A}$-module (resp., an $\mathcal{SA}$-module). One may easily check that M is an $\mathcal{SA}$-module if and only if M is an $\mathcal{A}$-module and $\mathrm{Z}_R(M)$ is an ideal of R. It is worthwhile reminding the reader that the Property ($\mathcal{A}$) for commutative rings was introduced by Quentel in [18] who called it Property (C) and Huckaba used the term Property ($\mathcal{A}$) in [13, 14]. In [11], Faith called rings satisfying Property ($\mathcal{A}$) McCoy rings. The Property ($\mathcal{A}$) for modules was introduced by Darani [9] who called such modules F-McCoy modules (for Faith McCoy terminology). He also introduced the strong Property ($\mathcal{A}$) under the name super coprimal and called a module M coprimal if $\mathrm{Z}_R(M)$ is an ideal. In [17], the strong Property ($\mathcal{A}$) for commutative rings was independently introduced by Mahdou and Hassani in [17] and further studied by Dobbs and Shapiro in [10]. Note that a finitely generated module over a Noetherian ring is an $\mathcal{A}$-module (for example, see [15, Theorem 82]) and thus a Noetherian ring is an $\mathcal{A}$-ring. Also, it is well known that a zero-dimensional ring R is an $\mathcal{A}$-ring as well as any ring R whose total quotient ring $Q(R)$ is zero-dimensional. In fact, it is easy to see that R is an $\mathcal{A}$-ring if and only if so is $Q(R)$ [9, Corollary 2.6]. Any polynomial ring $R[X]$ is an $\mathcal{A}$-ring [13] as well as any reduced ring with a finite number of minimal prime ideals [13]. In [6], we generalize a result of T.G. Lucas which states that if R is a reduced commutative ring and M is a flat R-module, then the idealization $R \ltimes M$ is an $\mathcal{A}$-ring if and only if R is an $\mathcal{A}$-ring [16, Proposition 3.5]. In effect, we drop the reduceness hypotheses and prove that, given an arbitrary commutative ring R and any submodule M of a flat R-module F, $R \ltimes M$ is an $\mathcal{A}$-ring (resp., $\mathcal{SA}$-ring) if and only if R is an $\mathcal{A}$-ring (resp., $\mathcal{SA}$-ring). In [7], we present an answer to a problem raised by D.D. Anderson and S. Chun in [4] on characterizing when is the idealization $R \ltimes M$ of a ring R on an R-module M an $\mathcal{A}$-ring (resp., an $\mathcal{SA}$-ring) in terms of module-theoretic properties of R and M. Also, we were concerned with presenting a complete answer to an open question asked by these two authors which reads the following: What modules over a given ring R are homomorphic images of modules satisfying the strong Property ($\mathcal{A}$)? [4, Question 4.4 (1)]. The main theorem of [8] extends a result of Hong, Kim, Lee and Ryu in [12] which proves that a direct product $\prod R_i$ of rings is an $\mathcal{A}$-ring if and only if so is any R_i. In this regard, we show that if $\{R_i\}_{i\in I}$ is a family of rings and $\{M_i\}_{i\in I}$ is a family of modules such that each M_i is an R_i-module, then the direct product $\prod\limits_{i\in I} M_i$ of the M_i is an $\mathcal{A}$-module over $\prod\limits_{i\in I} R_i$ if and only if each M_i is an $\mathcal{A}$-module over R_i, $i \in I$. Finally, our main concern in [1] is to introduce and investigate a new class of rings lying properly between the class of $\mathcal{A}$-rings and the class of $\mathcal{SA}$-rings. The new class of rings, termed the class of $\mathcal{PSA}$-rings, turns out to share common characteristics with both $\mathcal{A}$-rings and $\mathcal{SA}$-rings. Numerous properties and characterizations of this class are given as well as the module-theoretic version of $\mathcal{PSA}$-rings is introduced and studied. For further works related to the Property ($\mathcal{A}$) and ($\mathcal{SA}$), we refer the reader to [2, 3, 4, 5, 12, 16].

The main goal of this paper is to introduce and investigate the new notions of an $\mathcal{A}$-ring and $\mathcal{A}$-module with respect to an ideal I of R. The new notion turns out to be a weak form of the classic notion of an $\mathcal{A}$-ring, in the sense that, any R-module satisfying the Property ($\mathcal{A}$) satisfies as well the Property ($\mathcal{A}$) with respect to any ideal I of R. Also, the introduced property stems from the lack of stability of $\mathrm{Z}(R)$ under the first operation "+" of R. In particular, we examine the ideals of R which satisfy the Property ($\mathcal{A}$) with respect to themselves. For instance, if R is Noetherian, then any ideal I is an $\mathcal{A}$-module with respect to itself. Also, if R is a ring and I is an ideal of R such that I is contained in the nilradical $\mathrm{Rad}(R)$ of R, then an R-module M is an $\mathcal{A}$-module with respect to I if and only if M is an $\mathcal{A}$-module. Moreover, through Example 4.5, we present an example of a ring R possessing an ideal $I \subseteq \mathrm{Z}(R)$ such that R is an $\mathcal{A}$-ring with respect to I while R is not an $\mathcal{A}$-ring. The main theorem of Section 3 totally characterizes when a ring R (resp., an R-module M) is an $\mathcal{A}$-ring (resp., an $\mathcal{A}$-module) with respect to a given ideal I. Finally, in Section 4, we investigate the behavior of the Property ($\mathcal{A}$) with respect to an ideal vis-à-vis the direct products of rings and modules. This allows us to generalize, via Theorem 4.1, a proposition of Hong-Kim-Lee-Ryu stating that the direct product $\prod R_i$ of a family of rings $(R_i)_i$ is an $\mathcal{A}$-ring if and only if each R_i is an $\mathcal{A}$-ring [12, Proposition 1.3].

2 The set of zero divisors with respect to an ideal

This section introduces and studies the set of zero divisors of a ring R (resp., an R-module M) with respect to a given ideal I of R denoted by $\mathrm{Z}^I(R)$ (resp., $\mathrm{Z}^I_R(M)$). We seek conditions under which the complement of $\mathrm{Z}^I(R)$ is a saturated multiplicative subset of R. In this regard, we prove that if R admits a finite number of maximal prime ideals, in particular if R is Noetherian, then $R \setminus \mathrm{Z}^I(R)$ is a saturated multiplicative subset of R. Also, we characterize the set of zero divisors of a direct product $R = \prod\limits_{i\in\Lambda} R_i$ of the rings R_i with respect to an ideal I of R in terms of the set of zero divisors of R_i with respect to the projections of I on the rings R_i.

We begin by giving the definitions of the new concepts.

Definition 2.1. Let R be a ring and I an ideal of R. Let M be an R-module.

(i) An element x of R is said to be a zero divisor with respect to I if $x + I \subseteq \mathrm{Z}(R)$.

(ii) The set of all zero divisors with respect to I is denoted by $\mathrm{Z}^I(R) = \{x \in R : x+I \subseteq \mathrm{Z}(R)\}$.

(iii) An element x of R is said to be a zero divisor of M with respect to I if $x + I \subseteq \mathrm{Z}_R(M)$.

(iv) The set of all zero divisors of M with respect to I is denoted by $\mathrm{Z}^I_R(M)$.

Let R be a ring and M an R-module. Let $\mathrm{Spec}(\mathrm{Z}(R))$ (resp., $\mathrm{Max}(\mathrm{Z}(R))$) denote the set of prime ideals (resp., maximal ideals) of R contained in $\mathrm{Z}(R)$ and $\mathrm{Spec}(\mathrm{Z}_R(M))$ (resp., $\mathrm{Max}(\mathrm{Z}_R(M))$) denote the set of prime ideals (resp., maximal ideals) of R contained in $\mathrm{Z}_R(M)$. According to [15], $\mathrm{Max}(\mathrm{Z}_R(M))$ stands for the set of the maximal primes of the R-module M. Also, let I be an ideal of R. Note that $\mathrm{Z}^I(R) \subseteq \mathrm{Z}(R)$ and $\mathrm{Z}^I_R(M) \subseteq \mathrm{Z}_R(M)$. If $I \subseteq \mathrm{Z}(R)$, then we denote by $\mathrm{Max}_I(\mathrm{Z}(R))$ the set of the maximal primes of R containing I, that is,

$$\mathrm{Max}_I(\mathrm{Z}(R)) = \{m \in \mathrm{Max}(\mathrm{Z}(R)) : I \subseteq m\}.$$

Our first theorem characterizes the set of zero divisors with respect to an ideal and it provides conditions under which $R \setminus \mathrm{Z}^I(R)$ is a saturated multiplicative subset of a ring R.

Theorem 2.2. *Let R be a ring and I an ideal of R. Then*
1) $\bigcup_{m\in \mathrm{Max}_I(\mathrm{Z}(R))} m \subseteq \mathrm{Z}^I(R)$.
2) If R admits a finite number of maximal primes, then $\mathrm{Z}^I(R) = \bigcup_{m\in \mathrm{Max}_I(\mathrm{Z}(R))} m$.

Proof. 1) It is clear.
2) Assume that $\mathrm{Max}(\mathrm{Z}(R)) = \{P_1, P_2, \cdots, P_n\}$ is a finite set. Then $\mathrm{Z}(R) = P_1 \cup P_2 \cup \cdots \cup P_n$. Let $\mathrm{Max}_I(\mathrm{Z}(R)) = \{P_1, P_2, \cdots, P_t\}$. Then $I \subseteq P_1 \cap P_2 \cap \cdots \cap P_t$ and $I \nsubseteq P_{t+1} \cup \cdots \cup P_n$. Suppose that $\mathrm{Z}^I(R) \nsubseteq P_1 \cup \cdots \cup P_t$ and let $x \in \mathrm{Z}^I(R) \setminus P_1 \cup \cdots \cup P_t$. Let $i \in I \setminus P_{t+1} \cup \cdots \cup P_n$. Let $x \in P_{t+1} \cap \cdots \cap P_r \setminus P_{r+1} \cup \cdots \cup P_n$ for some $r \in \{t+1, \cdots, n\}$. Note that, if $P_{r+1} \cap \cdots \cap P_n = (0)$, then, in particular, $P_{r+1} \cap \cdots \cap P_n \subseteq P_1$ and thus there exists $k \in \{r+1, \cdots, n\}$ such that $P_k \subseteq P_1$ so that $P_k = P_1$ which is absurd as $I \subseteq P_1$ while $I \nsubseteq P_k$. Hence $P_{r+1} \cap \cdots \cap P_n \neq (0)$. Let $y \in P_{r+1} \cap \cdots \cap P_n \setminus P_{t+1} \cup \cdots \cup P_r$. Consider $z := x + yi$. Then $z \in \mathrm{Z}(R)$ as $x \in \mathrm{Z}^I(R)$. If $z \in P_1 \cup P_2 \cup \cdots \cup P_t$, then $x \in P_1 \cup \cdots \cup P_t$ as $I \subseteq P_1 \cap \cdots \cap P_t$. This leads to a contradiction since $x \notin P_1 \cup \cdots \cup P_t$. If $z \in P_{t+1} \cup \cdots \cup P_r$, then $yi \in P_{t+1} \cup \cdots \cup P_r$ as $x \in P_{t+1} \cap \cdots \cap P_r$ which is absurd since $i, y \notin P_{t+1} \cup \cdots \cup P_r$. If $z \in P_{r+1} \cup \cdots \cup P_n$, then $x \in P_{r+1} \cup \cdots \cup P_n$ since $y \in P_{r+1} \cap \cdots \cap P_n$. This is absurd as $x \notin P_{r+1} \cup \cdots \cup P_n$. It follows that $Z^I(R) \subseteq P_1 \cup \cdots \cup P_t$ and thus the desired equality holds comlpleting the proof. □

Next, we deduce that in the setting of a Noetherian ring R, $R \setminus \mathrm{Z}^I(R)$ is a saturated multiplicative subset of R.

Corollary 2.3. *If R is a Noetherian ring and I an ideal of R, then*

$$\mathrm{Z}^I(R) = \bigcup_{m\in \mathrm{Max}_I(\mathrm{Z}(R))} m.$$

Proof. It follows from Theorem 2.2 since by [15, Theorem 80], if R is Noetherian, then it admits a finite number of maximal primes. □

The following theorem characterizes the set of zero divisors of a direct product $R = \prod_{i\in\Lambda} R_i$ of the rings R_i with respect to an ideal I of R in terms of the set of zero divisors of R_i with respect to the projections of I on the rings R_i.

Theorem 2.4. *Let $(R_j)_{j\in\Lambda}$ be a family of rings and let $R := \prod_{j\in\Lambda} R_j$. For each $j \in \Lambda$, let I_j be an ideal of R_j and let $I := \prod_{j\in\Lambda} I_j$ be the resulting ideal of R. Let $x = (x_j)_j \in R$. Then $x \in \mathrm{Z}^I(R)$ if and only if there exists $j \in \Lambda$ such that $x_j \in \mathrm{Z}^{I_j}(R_j)$.*

Proof. Assume that $x \in \mathrm{Z}^I(R)$. Then $x + I \subseteq \mathrm{Z}(R)$. Assume, by way of contradiction, that for each $j \in \Lambda$, there exists $i_j \in I_j$ such that $x_j + i_j \notin \mathrm{Z}(R_j)$. Let $z = (x_j + i_j)_j = x + (i_j)_j$ with $(i_j)_j \in I$. Then $z \in x + I$ and $z \notin \mathrm{Z}(R)$ which is absurd. It follows that there exists $j \in \Lambda$ such that $x_j + I_j \subseteq \mathrm{Z}(R_j)$, that is, $x_j \in \mathrm{Z}^{I_j}(R_j)$, as desired. The converse is direct completing the proof. □

Next, we describe the finitely generated ideals contained in the set of zero divisors of a direct products $\prod_{i\in\Lambda} R_i$ of a family of rings R_i with respect to an ideal I of this direct product.

Theorem 2.5. *Let $(R_t)_{t\in\Lambda}$ be a family of rings and $R = \prod_{t\in\Lambda} R_t$. Let I_t be an ideal of R_t for each $t \in \Lambda$ and $I = \prod_t I_t$ the resulting ideal of R. Let M_t be an R_t-module for each $t \in \Lambda$ and $M = \prod_{t\in\Lambda} M_t$. Let $J = (a_1, a_2, \cdots, a_n)R$ be a finitely generated ideal of R and $J_t = (a_{1t}, a_{2t}, \cdots, a_{nt})R_t$ the projection of J on R_t for each $t \in \Lambda$, where $a_k = (a_{kt})_{t\in\Lambda}$ for each $k = 1, 2, \cdots, n$. Then $J \subseteq \mathrm{Z}^I_R(M)$ if and only if there exists $t \in \Lambda$ such that $J_t \subseteq \mathrm{Z}^{I_t}_{R_t}(M_t)$.*

Proof. The sufficient condition is direct. Let us now prove the necessary one. Assume that $J_t \nsubseteq \mathrm{Z}_{R_t}^{I_t}(M_t)$ for each $t \in \Lambda$. Then, for each $t \in \Lambda$, there exists $b_t = \alpha_{1t}a_{1t} + \alpha_{2t}a_{2t} + \cdots + \alpha_{nt}a_{nt} \in J_t$ such that $b_t \notin \mathrm{Z}_{R_t}^{I_t}(M_t)$. Put $r_k = (\alpha_{kt})_{t\in\Lambda}$ for $k = 1, \cdots, n$. Now, take $x = r_1a_1 + r_2a_2 + \cdots + r_na_n$. Then $x \in J$, as J is an ideal, and $x_t = b_t$ for each $t \in \Lambda$. Therefore $x_t \notin \mathrm{Z}_{R_t}^{I_t}(M_t)$ for each $t \in \Lambda$ and thus, by Theorem 2.4, $x \notin \mathrm{Z}_R^I(\prod_{t\in\Lambda} M_t)$. It follows that $J \nsubseteq \mathrm{Z}_R^I(\prod_i M_i)$. This proves the necessary condition completing the proof.

□

3 Property ($\mathcal{A}$) with respect to an ideal

This section introduces and investigates the new notions of an $\mathcal{A}$-ring and $\mathcal{A}$-module with respect to an ideal I of R. It turns out that any $\mathcal{A}$-ring (resp., any $\mathcal{A}$-module) is, in particular, an $\mathcal{A}$-ring (resp., an $\mathcal{A}$-module) with respect to any ideal I of R. The new property stems from the lack of stability of $\mathrm{Z}(R)$ under the first operation "+" of R. In particular, we examine those ideals of R which satisfy the Property ($\mathcal{A}$) with respect to themselves. For instance, if R is Noetherian, then any ideal I is an $\mathcal{A}$-module with respect to itself. The new notion turns out to be a weak form of the classic notion of an $\mathcal{A}$-ring, in the sense that, any R-module satisfying the Property ($\mathcal{A}$) satisfies as well the Property ($\mathcal{A}$) with respect to any ideal I of R. Moreover, we prove that if I is contained in the nilradical of R, then the notion of $\mathcal{A}$-module with respect to I and the notion of $\mathcal{A}$-module collapse. Moreover, through Example 4.5, we present an example of a ring R admitting an ideal $I \subseteq \mathrm{Z}(R)$ such that R is an $\mathcal{A}$-ring with respect to I while R is not an $\mathcal{A}$-ring.

We begin by giving the definitions of the new concepts.

Definition 3.1. Let R be a ring and I an ideal of R. Let M be an R-module. Then

(i) R is said to be an $\mathcal{A}$-ring with respect to I if for each finitely generated ideal such that $J \subseteq \mathrm{Z}^I(R)$, we have $\mathrm{ann}_R(J) \neq (0)$.

(ii) M is said to be an $\mathcal{A}$-module with respect to I if for each finitely generated such that $J \subseteq \mathrm{Z}_R^I(M)$, we have $\mathrm{ann}_M(J) \neq (0)$.

The following proposition presents examples of (vacuous) $\mathcal{A}$-rings and $\mathcal{A}$-modules with respect to particular ideals.

Proposition 3.2. *Let R be a ring, I an ideal of R and M an R-module.*

(i) If $I \nsubseteq \mathrm{Z}(R)$, then R is (vacuously) an $\mathcal{A}$-ring with respect to I.

(ii) If $I \nsubseteq \mathrm{Z}_R(M)$, then M is (vacuously) an $\mathcal{A}$-module with respect to I.

(iii) R is (vacuously) an $\mathcal{A}$-ring with respect to R and M is (vacuously) an $\mathcal{A}$-module with respect to R.

Proof. 1) It suffices to note that, when $I \nsubseteq \mathrm{Z}(R)$, then $J + I \nsubseteq \mathrm{Z}(R)$ for any ideal J of R. Hence the statement (1) follows vacuously from Definition 3.1.
2) The proof is similar to that of (1).
3) It is direct from (2). □

Let R be any ring which is not an $\mathcal{A}$-ring. Then R is an example of an $\mathcal{A}$-ring with respect to R which is not an $\mathcal{A}$-ring. Ahead, via Example 4.5, we provide a ring R and an ideal I with $I \subseteq \mathrm{Z}(R)$ such that R is an $\mathcal{A}$-ring with respect to I while R is not an $\mathcal{A}$-ring. This discussion proves that the notion of an $\mathcal{A}$-ring with respect to an ideal is weak form of the notion of an $\mathcal{A}$-ring.

We deduce from Proposition 3.2(2) the first case of an ideal of a ring R which is an $\mathcal{A}$-module with respect to itself.

Corollary 3.3. *Let R be a ring and I an ideal of R such that $I \nsubseteq \mathrm{Z}_R(I)$. Then I is an $\mathcal{A}$-module with respect to itself.*

The following two propositions records the fact that the Property $(\mathcal{A})$ of a ring R is a particular case of the Property $(\mathcal{A})$ of R with respect to an ideal and that in the Noetherian setting any ideal I of R is an $\mathcal{A}$-module with respect to itself.

Proposition 3.4. *Let R be a ring and M be an R-module. Then*
1) The following assertions are equivalent:
a) R is an $\mathcal{A}$-ring;
b) R is an $\mathcal{A}$-ring with respect to (0).
2) The following assertions are equivalent:
a) M is an $\mathcal{A}$-module;
b) M is an $\mathcal{A}$-module with respect to (0).

Proof. It is direct from Definition 3.1. □

Proposition 3.5. *Let R be a ring. Then*

(i) Any $\mathcal{A}$-module M over R is an $\mathcal{A}$-module with respect to any ideal I of R. In particular, if R is an $\mathcal{A}$-ring, then R is an $\mathcal{A}$-ring with respect to any ideal I of R.

(ii) If R is Noetherian, then any ideal I of R is an $\mathcal{A}$-module and thus an $\mathcal{A}$-module with respect to itself.

Proof. 1) It is clear from Definition 3.1.
2) Assume that R is Noetherian. Then I is a Noetherian module over R. Therefore, by [4, Theorem 2.2(5)], I is an $\mathcal{A}$-module. Hence, by (1), I is an $\mathcal{A}$-module with respect to itself. □

It is known that a ring R (resp., an R-module M) is an $\mathcal{A}$-ring (resp., an $\mathcal{A}$-module over R) if and only if the total quotient ring $Q(R)$ (resp., the total quotient module $Q(M)$) is an $\mathcal{A}$-ring (resp., an $\mathcal{A}$-module over $Q(R)$). Next, we handle the transfer of this result to the Property $(\mathcal{A})$ with respect to an ideal. Given a ring R and an R-module M, put $S := R \setminus \mathrm{Z}(R)$ and $S_R(M) := R \setminus \mathrm{Z}_R(M)$.

Proposition 3.6. *Let R be a ring and I an ideal of R. Let M be an R-module.*
1) Assume that $I \subseteq \mathrm{Z}(R)$. Then the following assertions are equivalent.
a) R is an $\mathcal{A}$-ring with respect to I;
b) $Q(R)$ is an $\mathcal{A}$-ring with respect to $S^{-1}I$.
2) Assume that $I \subseteq \mathrm{Z}_R(M)$. Let $Q(M) = S_R(M)^{-1}M$ denote the total quotient module of M. Then the following assertions are equivalent.
a) M is an $\mathcal{A}$-module with respect to I;
b) $Q(M)$ is an $\mathcal{A}$-module with respect to $S_R(M)^{-1}I$.

Proof. 1) a) $\Rightarrow$ b) Assume that R is an $\mathcal{A}$-ring with respect to I. Let K be a proper finitely generated ideal of $Q(R)$ such that $K + S^{-1}I \subseteq \mathrm{Z}(Q(R))$. Then there exists a finitely generated ideal $J \subseteq \mathrm{Z}(R)$ of R such that $K = S^{-1}J$. Hence $S^{-1}(J+I) \subseteq \mathrm{Z}(Q(R))$ and thus $J+I \subseteq \mathrm{Z}(R)$. Therefore, as R is an $\mathcal{A}$-ring with respect to I, $\mathrm{ann}(J) \neq (0)$. It follows, since $K = S^{-1}J$, that $\mathrm{ann}_{Q(R)}(K) \neq (0)$. Consequently, $Q(R)$ is an $\mathcal{A}$-ring with respect to $S^{-1}I$, as desired.
b) $\Rightarrow$ a) Assume that $Q(R)$ is an $\mathcal{A}$-ring with respect to $S^{-1}I$. Let $J \subseteq \mathrm{Z}(R)$ be a finitely generated ideal of R such that $J + I \subseteq \mathrm{Z}(R)$. Then $S^{-1}(J+I) = S^{-1}J + S^{-1}I$ is a proper ideal of $Q(R)$. Hence, as $Q(R)$ is an $\mathcal{A}$-ring with respect to $S^{-1}I$, we get $\mathrm{ann}_{Q(R)}(S^{-1}J) \neq (0)$. It follows, as S consists of regular elements of R, that $\mathrm{ann}(J) \neq (0)$. Consequently, R is an $\mathcal{A}$-ring with respect to I, as desired.
2) The proof is similar to that of (1).

□

Corollary 3.7. *Let R be a ring and I an ideal of R such that $I \subseteq \mathrm{Z}_R(I)$. Then I is an $\mathcal{A}$-module with respect to itself if and only if the ideal $S_R(I)^{-1}I$ of $Q_R(I)$ is an $\mathcal{A}$-module with respect to itself.*

Through the next bunch of results we seek conditions under which an R-module M is an $\mathcal{A}$-module with respect to an ideal I. Given a ring R, we denote by $\mathrm{Rad}(R)$ the nilradical of R.

Proposition 3.8. *Let R be a ring and I an ideal of R. Assume that $I \subseteq \mathrm{Rad}(R)$. Let M be an R-module. Then M is an $\mathcal{A}$-module with respect to I if and only if M is an $\mathcal{A}$-module.*

Proof. By Proposition 3.5, it suffices to prove the necessary statement. Assume that M is an $\mathcal{A}$-module with respect to I. Let J be a finitely generated ideal of R such that $J \subseteq \mathrm{Z}_R(M)$. Let $j \in J$ and $i \in I$. Then there exists $0 \neq m \in M$ such that $jm = 0$ and there exists $n \in \mathbb{N}$ such that $i^n = 0$. Let $r := \max\{t \in \mathbb{N} : i^t m \neq 0\}$. Note that $0 \leq r \leq n-1$. Hence $(j+i)i^r m = 0$ and $i^r m \neq 0$ so that $j + i \in \mathrm{Z}_R(M)$. It follows that $J + I \subseteq \mathrm{Z}_R(M)$ and thus $J \subseteq \mathrm{Z}_R^I(M)$. Hence $\mathrm{ann}_M(J) \neq (0)$. Consequently, M is an $\mathcal{A}$-module. □

Corollary 3.9. *Let R be a ring and I a nilpotent ideal of R, that is, there exists $n \geq 1$ such that $I^n = (0)$. Let M be an R-module. Then M is an $\mathcal{A}$-module with respect to I if and only if M is an $\mathcal{A}$-module.*

Proof. I suffices to note that $I \subseteq \mathrm{Rad}(R)$ and then to apply Proposition 3.8.

□

Proposition 3.10. *Let R be an $\mathcal{SA}$-ring. Put $I := \mathrm{Z}(R)$. Then I is an $\mathcal{A}$-module and thus I is an $\mathcal{A}$-module with respect to itself.*

Proof. Let $J \subseteq \mathrm{Z}_R(I)$ be a nonzero finitely generated ideal of R. Then, as $J \subseteq \mathrm{Z}_R(I) \subseteq \mathrm{Z}(R)$ and R is an $\mathcal{A}$-ring, we get $\mathrm{ann}(J) \neq (0)$ and thus there exists $a \in R \setminus \{0\}$ such that $aJ = (0)$. As $J \neq (0)$, we get $a \in \mathrm{Z}(R) = I$. Then $\mathrm{ann}_I(J) \neq (0)$. It follows that I is an $\mathcal{A}$-module and thus I is an $\mathcal{A}$-module with respect to itself, as desired.

□

Proposition 3.11. *Let R be an $\mathcal{A}$-ring and I an ideal of R. Assume that $\mathrm{ann}(I) \subseteq I$. Then I is an $\mathcal{A}$-module with respect to itself.*

Proof. Let $J \subseteq \mathrm{Z}_R^I(I)$ be a nonzero finitely generated ideal of R. As $\mathrm{Z}_R^I(I) \subseteq \mathrm{Z}_R(I) \subseteq \mathrm{Z}(R)$ and R is an $\mathcal{A}$-ring, we get $\mathrm{ann}(J) \neq (0)$ and thus there exists $a \in R$ such that $a \neq 0$ and $aJ = (0)$. If $a \in I$, then $\mathrm{ann}_I(J) \neq (0)$. Assume that $a \notin I$. Then, as $\mathrm{ann}(I) \subseteq I$, we get $a \notin \mathrm{ann}(I)$. Hence there exists $i \in I$ such that $j := ai \neq 0$. It follows that $jJ = (0)$ and $j \in I \setminus \{0\}$, so that $\mathrm{ann}_I(J) \neq (0)$. Consequently, I is an $\mathcal{A}$-module with respect to itself.

□

Corollary 3.12. *Let R be an $\mathcal{A}$-ring and I an ideal of R such that $\mathrm{ann}(I) = (0)$. Then I is an $\mathcal{A}$-module with respect to itself.*

Corollary 3.13. *Let R be an $\mathcal{A}$-ring and I an ideal of R. Assume that $\mathrm{Z}_R(I) \subseteq I$. Then I is an $\mathcal{A}$-module with respect to itself.*

Proof. It is direct from Proposition 3.11 as $\mathrm{ann}(I) \subseteq \mathrm{Z}_R(I)$.

□

Next, we prove a sort of ascent behavior of the Property ($\mathcal{A}$) with respect to an ideal.

Proposition 3.14. *Let R be a ring and M an R-module. Let $I_1 \subseteq I_2$ be ideals of R. Then*

(i) If R is an $\mathcal{A}$-ring with respect to I_1, then R is an $\mathcal{A}$-ring with respect to I_2.

(ii) If M is an $\mathcal{A}$-module with respect to I_1, then M is an $\mathcal{A}$-module with respect to I_2.

Proof. 1) Assume that R is an $\mathcal{A}$-ring with respect to I_1. Let J be a finitely generated ideal of R such that $J \subseteq \mathrm{Z}^{I_2}(R)$. Then, as $I_1 \subseteq I_2$, $J + I_1 \subseteq J + I_2 \subseteq \mathrm{Z}(R)$ and thus $J \subseteq \mathrm{Z}^{I_1}(R)$. Now, since R is an $\mathcal{A}$-ring with respect to I_1, it follows that $\mathrm{ann}(J) \neq (0)$. Therefore R is an $\mathcal{A}$-ring with respect to I_2, as desired.
2) It is similar to (1).

□

The following theorem and corollary characterize the $\mathcal{A}$-rings R (resp., $\mathcal{A}$-modules M) with respect to a given ideal I of R in the crucial case when $I \subseteq \mathrm{Z}(R)$ (resp., $I \subseteq \mathrm{Z}_R(M)$). Given a ring R and an ideal I of R, we denote by $\mathrm{Max}_I(R)$ the set of maximal ideals of R containing I and we denote by $\mathrm{Max}_I(\mathrm{Z}(R))$ the set of prime ideals of R which are maximal among the prime ideals in $\mathrm{Z}(R)$ and which contain I, in other words, the elements of $\mathrm{Max}_I(\mathrm{Z}(R))$ are the maximal primes of R containing I. Also, given an R-module M, let $\mathrm{Max}_I(\mathrm{Z}_R(M))$ denote the set of prime ideals of R which are maximal among the prime ideals in $\mathrm{Z}_R(M)$ and which contain I.

Theorem 3.15. *Let R be a ring and I an ideal of R.*
1) Assume that $I \subseteq \mathrm{Z}(R)$ and that $Q(R) = R$. Then the following assertions are equivalent.
a) R is an $\mathcal{A}$-ring with respect to I;
b) For each proper finitely generated ideal J of R such that $I + J$ is a proper ideal of R, $\mathrm{ann}(J) \neq (0)$;
c) For each proper finitely generated ideal J of R such that $J \subseteq \bigcup_{m\in \mathrm{Max}_I(R)} m$, $\mathrm{ann}(J) \neq (0)$.
2) Let M be an R-module such that $I \subseteq \mathrm{Z}_R(M)$. Assume that $Q_R(M) = R$. Then the following assertions are equivalent.
a) M is an $\mathcal{A}$-module with respect to I;
b) For each finitely generated ideal J of R such that $I + J$ is a proper ideal of R, $\mathrm{ann}_M(J) \neq (0)$;
c) For each finitely generated ideal J of R such that $J \subseteq \bigcup_{m\in \mathrm{Max}_I(R)} m$, $\mathrm{ann}_M(J) \neq (0)$.

Lemma 3.16. *Let R be a ring such that $Q(R) = R$. Let I be a proper ideal of R. Then, for each ideal J of R, $I + J$ is a proper ideal of R if and only if $J \subseteq \bigcup_{m\in \mathrm{Max}_I(R)} m$.*

Proof. Let J be an ideal of R. Assume that $I + J$ is a proper ideal of R. Then there exists a maximal ideal of R such that $I + J \subseteq m$. Hence $m \in \mathrm{Max}_I(R)$ such that $J \subseteq m$. Therefore $J \subseteq \bigcup_{m\in \mathrm{Max}_I(R)} m$. Conversely, suppose that $J \subseteq \bigcup_{m\in \mathrm{Max}_I(R)} m$. Then $I + J \subseteq \bigcup_{m\in \mathrm{Max}_I(R)} m$ as $I \subseteq m$ for each $m \in \mathrm{Max}_I(R)$. Hence $I + J$ is a proper ideal of R since $1 \notin \bigcup_{m\in \mathrm{Max}_I(R)} m$. This completes the proof of the lemma. □

Proof of Theorem 3.15. 1) a) $\Leftrightarrow$ b) It is clear from Definition 3.1 as $\mathrm{Z}(R)$ is the set of non invertible elements of R.
b) $\Leftrightarrow$ c) It is straighforward by Lemma 3.16.
c) $\Rightarrow$ a) Assume that (c) holds. Let J be a finitely generated ideal of R such that $I + J \subseteq \mathrm{Z}(R)$. It follows, applying (c), that $\mathrm{ann}(J) \neq (0)$. Consequently, R is an $\mathcal{A}$-ring with respect to I, as desired.
2) The proof is similar to the treatment of (1). □

Our final result gives a characterization of $\mathcal{A}$-rings and $\mathcal{A}$-modules with respect to an ideal in the general setting.

Corollary 3.17. *Let R be a ring and I an ideal of R.*
1) Assume that $I \subseteq \mathrm{Z}(R)$. Then the following assertions are equivalent.
a) R is an $\mathcal{A}$-ring with respect to I;
b) For each finitely generated ideal $J \subseteq \mathrm{Z}(R)$ of R such that $J \subseteq \bigcup_{m\in \mathrm{Max}_I(\mathrm{Z}(R))} m$, $\mathrm{ann}(J) \neq (0)$.
2) Let M be an R-module such that $I \subseteq \mathrm{Z}_R(M)$. Then the following assertions are equivalent.
a) M is an $\mathcal{A}$-module with respect to I;
b) For each finitely generated ideal $J \subseteq \mathrm{Z}_R(M)$ of R such that $J \subseteq \bigcup_{m\in \mathrm{Max}_I(\mathrm{Z}_R(M))} m$, $\mathrm{ann}_M(J) \neq (0)$.

Proof. It follows easily from the combination of Theorem 3.15 and Proposition 3.6. □

4 Property ($\mathcal{A}$) with respect to an ideal and direct product of rings and modules

This section investigates the behavior of the Property ($\mathcal{A}$) with respect to an ideal vis-à-vis the direct products of rings and modules. Given a family of rings $(R_k)_{k\in\Lambda}$, we characterize when a direct product $\prod_k M_k$ is an $\mathcal{A}$-module with respect to the ideal $\prod_k I_k$ with each M_k is an R_k-module and each I_k is an ideal of R_k for any $k \in \Lambda$. This allows to generalize, via Theorem 4.1, a result of Hong-Kim-Lee-Ryu stating that the direct product $\prod R_i$ of a family of rings $(R_i)_i$ is an $\mathcal{A}$-ring if and only if each R_i is an $\mathcal{A}$-ring [12, Proposition 1.3].

We begin by announcing the main theorem of this section.

Theorem 4.1. *Let $(R_k)_{k\in\Lambda}$ be a family of commutative rings. Let $R = \prod_{k\in\Lambda} R_k$. Let I_k be an ideal of R_k for each $k \in \Lambda$ and let $I := \prod I_k$. Let M_k be an R_k-module for each $k \in \Lambda$ and $M := \prod_k M_k$. Then the following assertions are equivalent.*

(i) M is an $\mathcal{A}$-module with respect to I;

(ii) M_k is an $\mathcal{A}$-module with respect to I_k for each $k \in \Lambda$.

Proof. 1) $\Rightarrow$ 2) Assume that M is an $\mathcal{A}$-module with respect to I. Fix $t \in \Lambda$ and let $J \subseteq \mathrm{Z}_{R_t}(M_t)$ be a finitely generated ideal of R_t such that $J + I_t \subseteq \mathrm{Z}_{R_t}(M_t)$. Consider the ideal $K = JR + (\cdots, 1, 1, 0_{R_t}, 1, 1, \cdots)R$ of R. Then K is a finitely generated ideal of R and it is easily checked that $K \subseteq \mathrm{Z}^I_R(M)$ since $J \subseteq \mathrm{Z}^{I_t}_{R_t}(M_t)$. Hence, since M is an $\mathcal{A}$-module with respect to I, there exists $0 \neq m' \in M$ such that $Km' = 0$. Put $m' = (m'_k)_k$. Then $(\cdots, 1, 1, 0_{R_t}, 1, 1, \cdots)m' = 0$, as $(\cdots, 1, 1, 0_{R_t}, 1, 1, \cdots) \in K$, and thus $m'_k = 0$ for each $k \neq t$. It follows that $m'_t \neq 0$ and $Jm'_t = (0)$, so that, $\mathrm{ann}_{M_t}(J) \neq (0)$. Consequently, M_k is an $\mathcal{A}$-module with respect to I_t, as desired.
2) $\Rightarrow$ 1) Assume that each M_k is an $\mathcal{A}$-module with respect to I_k. Let $J = (a_1, a_2, \cdots, a_n)R \subseteq \mathrm{Z}_R(M)$ be a finitely generated ideal of R such that $J \subseteq \mathrm{Z}^I_R(M)$. Let $a_k = (a_{ki})_{i\in\Lambda}$ for each $k = 1, \cdots, n$ and let $J_i := (a_{1i}, a_{2i}, \cdots, a_{ni})R_i$ the ith projection of J for each $i \in \Lambda$. Then, by Theorem 2.5, there exists $t \in \Lambda$ such that $J_k \subseteq \mathrm{Z}^{I_t}_{R_t}(M_t)$. Since M_k is an $\mathcal{A}$-module with respect to I_k, we get that $\mathrm{ann}_{M_k}(J_k) \neq (0)$, that is, there exists $0 \neq m_k \in M_k$ such that $J_k m_k = (0)$. Hence, it is easily verified that

$$J(\cdots, 0, 0, m_k, 0, 0, \cdots) \subseteq (\prod_{i\in\Lambda} J_i)(\cdots, 0, 0, m_k, 0, 0, \cdots)$$

$$= \cdots \times (0) \times (0) \times J_k m_k \times (0) \times (0) \times \cdots = (0),$$

that is, $\mathrm{ann}_M(J) \neq (0)$. Therefore M is an $\mathcal{A}$-module with respect to I completing the proof of the theorem.

□

Corollary 4.2. *Let $(R_k)_{k\in\Lambda}$ be a family of commutative rings and $R = \prod_{k\in\Lambda} R_k$. Let I_k be an ideal of R_k for each $k \in \Lambda$ and $I := \prod I_k$. Then R is an $\mathcal{A}$-ring with respect to I if and only if R_k is an $\mathcal{A}$-ring with respect to I_k for each $k \in \Lambda$.*

Corollary 4.3. *Let R_1 and R_2 be rings. Let I_1 and I_2 be ideals of R_1 and R_2, respectively. Then $R_1 \times R_2$ is an $\mathcal{A}$-ring with respect to $I_1 \times I_2$ if and only if R_1 is an $\mathcal{A}$-ring with respect to I_1 and R_2 is an $\mathcal{A}$-ring with respect to I_2.*

Corollary 4.4. *Let $(R_k)_{k\in\Lambda}$ be a family of commutative rings. Let $R = \prod_{k\in\Lambda} R_k$. Let I_k be an ideal of R_k for each $k \in \Lambda$ and let $I := \prod I_k$. Then I is an $\mathcal{A}$-module with respect to itself if and only if I_k is an $\mathcal{A}$-module with respect to itself for each $k \in \Lambda$.*

We close this paper by giving an example of a ring R and an ideal I such that $I \subseteq \mathrm{Z}(R)$ and R is an $\mathcal{A}$-ring with respect to I while R is not an $\mathcal{A}$-ring.

Example 4.5. Let S be a ring which is not an $\mathcal{A}$-ring. Note that, by Proposition 3.2(3), S is an $\mathcal{A}$-ring with respect to S. Let T be a zero-dimensional ring and m a maximal ideal of T. Then T is an $\mathcal{A}$-ring, and in particular an $\mathcal{A}$-ring with respect to m, and $m \subseteq \mathrm{Z}(T)$. Let $R := S \times T$ and $I := S \times m$. Note that $I \subseteq \mathrm{Z}(R)$. Moreover, by Corollary 4.3, R is an $\mathcal{A}$-ring with respect to I while, by [12, Proposition 1.3], R is not an $\mathcal{A}$-ring as S is not so, as desired.

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Author information

A. Ait Ouahi, Y. Arssi and S. Bouchiba, Department of Mathematics, Faculty of Sciences, Moulay Ismail University of Meknes, Morocco.
E-mail: `s.bouchiba@fs.umi.ac.ma`

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