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On Complex Neutrosophic Lie Algebras

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Abstract Complex neutrosophic Lie subalgebras and complex neutrosophic ideals of Lie algebras are defined in this paper. Each component in complex neutrosophic Lie algebra has magnitude and phase terms. Some characteristics of complex neutrosophic Lie subalgebras (ideals) and some of their operations like intersection and Cartesian product are also discussed. Moreover, the relationship between complex neutrosophic Lie subalgebras (ideals) and neutrosophic Lie subalgebras (ideals) is investigated. Finally, the image and the inverse image of complex neutrosophic Lie subalgebra under Lie algebra homomorphisms are defined and the properties of complex neutrosophic Lie subalgebras and complex neutrosophic ideals under homomorphisms of Lie algebras are studied.

1 Introduction

L. Zadeh's [18] fuzzy sets and fuzzy logic have been implemented in vague, unclear situations of real world problems. Atanassov's Intuitionistic fuzzy set [3] have been developed from fuzzy set by including one more component called non-membership function into fuzzy set. His theory gained an extensive recognition as a very valuable tool in area of science, Technology, Engineering, Medicine, etc. Smarandache [14] further extended Atanassov's theory and he named it as neutrosophic theory, in which he included a third component called indeterminacy into Atanassov's theory. Smarandache's neutrosophic theory deals with imprecision, indeterminacy, and inconsistent data. Later, Ali and Smarandache [1] developed novel complex neutrosophic sets and this theory extends the range of components from unit interval to the unit disc in complex plane. Each of its components has amplitude values and phase values. Simultaneously, complex neutrosophic set has been appLied in science and engineering field. Lie algebras are a special case of general linear algebra and was named after being developed by Sophus Lie (1842-1899). Lie groups classifies the smooth subgroups. After the development of this theory, it was appLied in mathematics and physics. Lie subalgebras and their properties were developed and investigated further in [2, 6, 12, 13, 15].

This paper is concerned about complex neutrosophic sets in Lie algebras and it is constructed as follows: After an Introduction, in Section 2, we present some definitions that are used throughout the paper. In Section 3, we extend neutrosophic Lie algebra by including some components into complex neutrosophic Lie algebra and further we extend each component range from unit interval to unit disc in complex plane. Additionally, we introduce complex neutrosophic Lie subalgebras (ideals) and investigate their properties such as their intersection and their Cartesian product. Finally, in Section 4, we study complex neutrosophic Lie subalgebras (ideals) under homomorphism of Lie algebras.

2 Preliminaries

We include some descriptions, comments and findings in this section, that are important and are used all over the paper regularly.
A description of complex neutrosophic structure was introduced by M. Ali and F. Smarandache [1] and is as follows.

Definition 2.1. [1] An object $\mathfrak{S}$ defined on a universe of discourse $\mathfrak{U}$ is called complex neutrosophic set (CNS), if it can be expressed as $\mathfrak{S} = \{(\zeta, \langle \mathfrak{M}(\zeta), \mathfrak{I}(\zeta), \mathfrak{F}(\zeta)\rangle) : \zeta \in \mathfrak{U}\}$. The values $\mathfrak{M}(\zeta), \mathfrak{I}(\zeta), \mathfrak{F}(\zeta)$ and their number can be in the complex plane all inside the unit circle, and so is in the following form, $\mathfrak{M}(\zeta) = p(\zeta)e^{j\mu(\zeta)}, \mathfrak{I}(\zeta) = q(\zeta)e^{j\nu(\zeta)}, \mathfrak{F}(\zeta) = r(\zeta)e^{j\omega(\zeta)}$ where $p(\zeta), q(\zeta), r(\zeta)$ and $\mu(\zeta), \nu(\zeta), \omega(\zeta)$ are respectively the amplitude terms and the phase terms, $\mu(\zeta), \nu(\zeta), \omega(\zeta) \in [0, 1]$, with $^-0 \leq p(\zeta)+q(\zeta)+r(\zeta) \leq 3^+$ and $\mu(\zeta), \nu(\zeta), \omega(\zeta)$ are real valued with $j = \sqrt{-1}$. The scaling factors μ, ν and $\omega \in [0, 2\pi]$.

Definition 2.2. A vector space $\mathfrak{L}$ over a field $\mathfrak{G}$ (equal to $\mathfrak{R}$ or $\mathfrak{D}$) on which $\mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ denoted by $(\alpha, \beta) \to [\alpha, \beta]$ is defined as a Lie algebra, if the following axioms are satisfied:

(i) $[\alpha, \beta]$ is bilinear,

(ii) $[\alpha, \alpha] = 0$ for all $\alpha \in \mathfrak{L}$,

(iii) $[[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = 0$ for all $\alpha, \beta, \gamma \in \mathfrak{L}$, (Jacobi identity).

$\mathfrak{L}$ is used to denote a Lie algebra(LA). It is noted that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $[[\alpha, \beta], \gamma] = [\alpha, [\beta, \gamma]]$. But it is anti commutative, i.e. $[\alpha, \beta] = -[\beta, \alpha]$.
A subspace $\mathfrak{H}$ of $\mathfrak{L}$ that is closed under $[', ']$ is a Lie subalgebra. We define a subspace $\mathfrak{G}$ of $\mathfrak{L}$ as a Lie ideal of $\mathfrak{L}$, if $\mathfrak{G}$ is with the property $[\mathfrak{G}, \mathfrak{L}] \subseteq \mathfrak{G}$. Clearly, any Lie ideal is a Lie subalgebra.

3 Complex Neutrosophic Lie Algebra

In this section, we introduce new concepts related to complex neutrosophic sets. In particular, we define and study complex neutrosophic Lie subalgebras as well as complex neutrosophic Lie ideals of Lie algebra.

Definition 3.1. A complex neutrosophic triplet set $\mathfrak{C} = (\mathfrak{M}, \mathfrak{I}, \mathfrak{F})$ on $\mathfrak{L}$ is said to be a complex neutrosophic Lie subalgebra if it satisfies the following conditions:

(i) $\mathfrak{M}_\mathfrak{C}(\alpha+\beta) \geq \wedge(\mathfrak{M}_\mathfrak{C}(\alpha), \mathfrak{M}_\mathfrak{C}(\beta)), \mathfrak{I}_\mathfrak{C}(\alpha+\beta) \leq \vee(\mathfrak{I}_\mathfrak{C}(\alpha), \mathfrak{I}_\mathfrak{C}(\beta)), \mathfrak{F}_\mathfrak{C}(\alpha+\beta) \leq \vee(\mathfrak{F}_\mathfrak{C}(\alpha), \mathfrak{F}_\mathfrak{C}(\beta))$,

(ii) $\mathfrak{M}_\mathfrak{C}(\zeta\alpha) \geq \mathfrak{M}_\mathfrak{C}(\alpha)$, $\mathfrak{I}_\mathfrak{C}(\zeta\alpha) \leq \mathfrak{I}_\mathfrak{C}(\alpha)$, $\mathfrak{F}_\mathfrak{C}(\zeta\alpha) \leq \mathfrak{F}_\mathfrak{C}(\alpha)$,

(iii) $\mathfrak{M}_\mathfrak{C}([\alpha, \beta]) \geq \wedge\{\mathfrak{M}_\mathfrak{C}(\alpha), \mathfrak{M}_\mathfrak{C}(\beta)\}, \mathfrak{I}_\mathfrak{C}([\alpha, \beta]) \leq \vee\{\mathfrak{I}_\mathfrak{C}(\alpha), \mathfrak{I}_\mathfrak{C}(\beta)\}, \mathfrak{F}_\mathfrak{C}([\alpha, \beta]) \leq \vee\{\mathfrak{F}_\mathfrak{C}(\alpha), \mathfrak{F}_\mathfrak{C}(\beta)\}$,

where,

$$\begin{aligned}\wedge(\mathfrak{M}_\mathfrak{C}(\alpha), \mathfrak{M}_\mathfrak{C}(\beta)) &= [p_\mathfrak{C}(\alpha) \wedge p_\mathfrak{C}(\beta)]e^{j[\mu_\mathfrak{C}(\alpha) \wedge \mu_\mathfrak{C}(\beta)]}\\ \vee(\mathfrak{I}_\mathfrak{C}(\alpha), \mathfrak{I}_\mathfrak{C}(\beta)) &= [q_\mathfrak{C}(\alpha) \vee q_\mathfrak{C}(\beta)]e^{j[\nu_\mathfrak{C}(\alpha) \vee \nu_\mathfrak{C}(\beta)]}\\ \vee(\mathfrak{F}_\mathfrak{C}(\alpha), \mathfrak{F}_\mathfrak{C}(\beta)) &= [r_\mathfrak{C}(\alpha) \vee r_\mathfrak{C}(\beta)]e^{j[\omega_\mathfrak{C}(\alpha) \vee \omega_\mathfrak{C}(\beta)]}\end{aligned}$$

for all $\alpha, \beta \in \mathfrak{L}$ and $\zeta \in \mathcal{F}$

Definition 3.2. A complex neutrosophic triplet set $\mathfrak{C} = (\mathfrak{M}, \mathfrak{I}, \mathfrak{F})$ on $\mathfrak{L}$ is said to be a complex neutrosophic Lie subalgebra if it satisfies the following conditions:

(i) $\mathfrak{M}_\mathfrak{C}(\alpha+\beta) \geq \wedge(\mathfrak{M}_\mathfrak{C}(\alpha), \mathfrak{M}_\mathfrak{C}(\beta)), \mathfrak{I}_\mathfrak{C}(\alpha+\beta) \leq \vee(\mathfrak{I}_\mathfrak{C}(\alpha), \mathfrak{I}_\mathfrak{C}(\beta)), \mathfrak{F}_\mathfrak{C}(\alpha+\beta) \leq \vee(\mathfrak{F}_\mathfrak{C}(\alpha), \mathfrak{F}_\mathfrak{C}(\beta))$,

(ii) $\mathfrak{M}_\mathfrak{C}(\zeta\alpha) \geq \mathfrak{M}_\mathfrak{C}(\alpha)$, $\mathfrak{I}_\mathfrak{C}(\zeta\alpha) \leq \mathfrak{I}_\mathfrak{C}(\alpha)$, $\mathfrak{F}_\mathfrak{C}(\zeta\alpha) \leq \mathfrak{F}_\mathfrak{C}(\alpha)$,

(iii) $\mathfrak{M}_\mathfrak{C}([\alpha, \beta]) \geq \wedge\{\mathfrak{M}_\mathfrak{C}(\alpha), \mathfrak{M}_\mathfrak{C}(\beta)\}, \mathfrak{I}_\mathfrak{C}([\alpha, \beta]) \leq \vee\{\mathfrak{I}_\mathfrak{C}(\alpha), \mathfrak{I}_\mathfrak{C}(\beta)\}, \mathfrak{F}_\mathfrak{C}([\alpha, \beta]) \leq \vee\{\mathfrak{F}_\mathfrak{C}(\alpha), \mathfrak{F}_\mathfrak{C}(\beta)\}$,

where,

$$\begin{aligned}\wedge(\mathfrak{M}_\mathfrak{C}(\alpha), \mathfrak{M}_\mathfrak{C}(\beta)) &= [p_\mathfrak{C}(\alpha) \wedge p_\mathfrak{C}(\beta)]e^{j[\mu_\mathfrak{C}(\alpha) \wedge \mu_\mathfrak{C}(\beta)]}\\ \vee(\mathfrak{I}_\mathfrak{C}(\alpha), \mathfrak{I}_\mathfrak{C}(\beta)) &= [q_\mathfrak{C}(\alpha) \vee q_\mathfrak{C}(\beta)]e^{j[\nu_\mathfrak{C}(\alpha) \vee \nu_\mathfrak{C}(\beta)]}\\ \vee(\mathfrak{F}_\mathfrak{C}(\alpha), \mathfrak{F}_\mathfrak{C}(\beta)) &= [r_\mathfrak{C}(\alpha) \vee r_\mathfrak{C}(\beta)]e^{j[\omega_\mathfrak{C}(\alpha) \vee \omega_\mathfrak{C}(\beta)]}\end{aligned}$$

for all $\alpha, \beta \in \mathfrak{L}$ and $\zeta \in \mathcal{F}$.

Remark 3.3. If $\mathfrak{C}$ is a complex neutrosophic subalgebra of $\mathfrak{L}$ then it may not be a complex neutrosophic ideal of $\mathfrak{L}$. (See Example 3.4.)

Example 3.4. The set of all 3-dimensional real vectors $\mathbb{R}^3 = \{(\alpha,\beta,\gamma)|\alpha,\beta,\gamma \in \mathfrak{R}\}$ forms a Lie algebra over $\mathfrak{F} = \mathbb{R}$ and with the usual cross product $\times$. We define the set $\mathfrak{C} = (\mathfrak{M},\mathfrak{I},\mathfrak{F})$, where $\mathfrak{M},\mathfrak{I},\mathfrak{F} : \mathbb{R}^3 \to \mathcal{E}^2$ ($\mathcal{E}^2$ is the unit disc), by

$$\mathfrak{M}_{\mathfrak{C}}(\alpha) = \begin{cases} 0.8e^{j\frac{3\pi}{4}}, if \alpha = \beta = \gamma = 0 \\ 0.5e^{j\frac{\pi}{3}}, if \alpha \neq 0, \beta = \gamma = 0 \\ 0, otherwise \end{cases}$$

$$\mathfrak{I}_{\mathfrak{C}}(\alpha) = \begin{cases} 0, if \alpha = \beta = \gamma = 0 \\ 0.6e^{j\frac{\pi}{2}}, if \alpha \neq 0, \beta = \gamma = 0 \\ 07e^{j\frac{2\pi}{3}}, otherwise \end{cases}$$

$$\mathfrak{F}_{\mathfrak{C}}(\alpha) = \begin{cases} 0, if \alpha = \beta = \gamma = 0 \\ 0.6e^{j\frac{\pi}{2}}, if \alpha \neq 0, \beta = \gamma = 0 \\ 07e^{j\frac{2\pi}{3}}, otherwise \end{cases}$$

Then it is clear that $\mathfrak{C}$ is a complex neutrosophic subalgebra of $\mathfrak{L} = \mathbb{R}^3$. But it is not a complex neutrosophic Lie ideal since $\mathfrak{M}_{\mathfrak{C}} = ([(1,0,0),(1,1,1)]) = \mathfrak{M}_{\mathfrak{C}}(0,-1,1) = 0 \not\geq \mathfrak{I}_{\mathfrak{C}}(1,0,0)$,$\mathfrak{I}_{\mathfrak{C}} = ([(1,0,0),(1,1,1)]) = \mathfrak{I}_{\mathfrak{C}}(0,-1,1) = 1 \not\leq \mathfrak{I}_{\mathfrak{C}}(1,0,0)$, and $\mathfrak{F}_{\mathfrak{C}} = ([(1,0,0),(1,1,1)]) = \mathfrak{F}_{\mathfrak{C}}(0,-1,1) = 1 \not\leq \mathfrak{F}_{\mathfrak{C}}(1,0,0)$.

Remark 3.5. Every complex neutrosophic Lie ideal is a complex neutrosophic Lie subalgebra.

Theorem 3.6. *Let $\mathfrak{L}$ be a neutrosophic Lie algebra and $\mathfrak{C} = (\mathfrak{M},\mathfrak{I},\mathfrak{F})$ be a complex neutrosophic set on it. Then $\mathfrak{C} = (\mathfrak{M},\mathfrak{I},\mathfrak{F})$ is a complex neutrosophic Lie subalgebra $\mathfrak{L}$ if and only if the non-empty complex neutrosophic upper s-level cut(NCU s-lc)*

$$\mathfrak{U}_{\mathfrak{M}}(\mathfrak{s}) = \{\alpha \in \mathfrak{L}|\mathfrak{M}(\alpha) \geq \mathfrak{s}\}$$

and the non-empty complex neutrosophic lower t-level cut(NCL t-lc)

$$\mathfrak{V}_{\mathfrak{I}}(\mathfrak{t}) = \{\alpha \in \mathfrak{L}|\mathfrak{I}(\alpha) \leq \mathfrak{t}\},\ \mathfrak{V}_{\mathfrak{F}}(\mathfrak{t}) = \{\alpha \in \mathfrak{L}|\mathfrak{F}(\alpha) \leq \mathfrak{t}\}$$

are Lie subalgebras of $\mathfrak{L}$, for all $\mathfrak{s},\mathfrak{t}$ lies in the complex unit disk in the plane.

Proof: *Let $\mathfrak{C} = (\mathfrak{M},\mathfrak{I},\mathfrak{F})$ be a complex neutrosophic Lie subalgebra on $\mathfrak{L}$ and $\mathfrak{s},\mathfrak{t}$ lies in the complex unit disk in the plane, be such that $\mathfrak{U}_{\mathfrak{M}}(\mathfrak{s}) \neq \emptyset$. Let $\alpha,\beta \in \mathfrak{L}$ be such that $\alpha \in \mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$ and $\beta \in \mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$. It follows that*

$$\mathfrak{M}_{\mathfrak{C}}(\alpha+\beta) \geq \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha),\mathfrak{M}_{\mathfrak{C}}(\beta)) \geq \mathfrak{s},$$
$$\mathfrak{M}_{\mathfrak{C}}(\zeta\alpha) \geq \mathfrak{M}_{\mathfrak{C}}(\alpha) \geq \mathfrak{s},$$
$$\mathfrak{M}_{\mathfrak{C}}([\alpha,\beta]) \geq \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha),\mathfrak{M}_{\mathfrak{C}}(\beta)) \geq \mathfrak{s},$$

and hence , $\alpha+\beta \in \mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$, $\zeta\alpha \in \mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$ and $[\alpha,\beta] \in \mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$, Thus, $\mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$ forms a Lie subalgebra of $\mathfrak{L}$. For the case of $\mathfrak{V}_{\mathfrak{I}}(\mathfrak{t})$, and $\mathfrak{V}_{\mathfrak{F}}(\mathfrak{t})$ the proof is similar.
Conversely, suppose that $\mathfrak{U}_{\mathfrak{M}}(\mathfrak{s}) \neq \emptyset$ is a Lie subalgebra of $\mathfrak{L}$ for every $\mathfrak{s} \in [0,1]e^{j\pi[0,1]}$. Assume that $\mathfrak{M}_{\mathfrak{C}}(\alpha+\beta) < \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha),\mathfrak{M}_{\mathfrak{C}}(\beta))$, for some $\alpha,\beta \in \mathfrak{L}$. Now taking $\mathfrak{s}_{\mathfrak{o}} := \frac{1}{2}\{\mathfrak{M}_{\mathfrak{C}}(\alpha+\beta) + \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha),\mathfrak{M}_{\mathfrak{C}}(\beta))\}$.

Then we have that $\mathfrak{M}_{\mathfrak{C}}(\alpha+\beta) < \mathfrak{s}_{\mathfrak{o}} < \mathfrak{M}_{\mathfrak{C}}(\wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha)\beta))\}$. and hence $\alpha+\beta \notin \mathfrak{M}_{\mathfrak{C}}(\mathfrak{s})$, $\alpha \in \mathfrak{M}_{\mathfrak{C}}(\mathfrak{s})$ and $\beta \in \mathfrak{M}_{\mathfrak{C}}(\mathfrak{s})$. However, this is clearly a contradiction. Therefore $\mathfrak{M}_{\mathfrak{C}}(\alpha+\beta) \geq \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha),\mathfrak{M}_{\mathfrak{C}}(\beta))$
for all $\alpha,\beta \in \mathfrak{L}$. Similarly we can show that $\mathfrak{M}_{\mathfrak{C}}(\zeta\alpha) \geq \mathfrak{M}_{\mathfrak{C}}(\alpha)$,
$\mathfrak{M}_{\mathfrak{C}}([\alpha,\beta]) \geq \wedge(\mathfrak{M}_{\mathfrak{C}}(\alpha),\mathfrak{M}_{\mathfrak{C}}(\beta))$, hence $\mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$ is a complex neutrosophic Lie subalgebra of $\mathfrak{L}$
For the case of $\mathfrak{V}_{\mathfrak{I}}(\mathfrak{t})$, and $\mathfrak{V}_{\mathfrak{F}}(\mathfrak{t})$ the proof is similar. □

Theorem 3.7. *Let $\mathfrak{C} = (\mathfrak{M},\mathfrak{I},\mathfrak{F})$ be a complex neutrosophic subset of $\mathfrak{L}$. Then the following statements are equivalent:*

(i) $\mathfrak{C}$ is a complex neutrosophic ideal of $\mathfrak{L}$,

(ii) The complex neutrosophic upper s-level cut $\mathfrak{U}_{\mathfrak{M}}(\mathfrak{s})$ is an ideal of $\mathfrak{L}$ for every $\mathfrak{s} \in Im(\mathfrak{M}_{\mathfrak{C}})$.

(iii) The complex neutrosophic lower t-level cuts $\mathfrak{V}_{\mathfrak{I}}(\mathfrak{t})$ and $\mathfrak{V}_{\mathfrak{F}}(\mathfrak{t})$ are ideals of $\mathfrak{L}$ for every $\mathfrak{t} \in Im(\mathfrak{I}_{\mathfrak{C}})$ and $\mathfrak{t} \in Im(\mathfrak{F}_{\mathfrak{C}})$ respectively.

Theorem 3.8. *Let $\mathfrak{C}_1 = (\mathfrak{M}_1, \mathfrak{I}_1, \mathfrak{F}_1)$ and $\mathfrak{C}_2 = (\mathfrak{M}_2, \mathfrak{I}_2, \mathfrak{F}_2)$ be two neutrosophic complex Lie subalgebras over $\mathfrak{L}$, then the intersection $\mathfrak{C}_3 = \mathfrak{C}_1 \cap \mathfrak{C}_2 = (\mathfrak{M}_3, \mathfrak{I}_3, \mathfrak{F}_3)$ is a complex neutrosophic Lie subalgebra over $\mathfrak{L}$.*
Proof. *For each $\alpha, \beta \in \mathfrak{L}$ and $\zeta \in \mathcal{F}$.*

$$\begin{gathered}
\mathfrak{M}_{\mathfrak{C}_3}(\alpha+\beta) = \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha+\beta), \mathfrak{M}_{\mathfrak{C}_2}(\alpha+\beta)\} \\
\geq \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha), \mathfrak{M}_{\mathfrak{C}_1}(\beta)\}, \wedge\{\mathfrak{M}_{\mathfrak{C}_2}(\alpha), \mathfrak{M}_{\mathfrak{C}_2}(\beta)\}\} \\
= \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha), \mathfrak{M}_{\mathfrak{C}_2}(\alpha)\}, \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\beta), \mathfrak{M}_{\mathfrak{C}_2}(\beta)\}\} \\
= \wedge\{\mathfrak{M}_{\mathfrak{C}_3}(\alpha), \mathfrak{M}_{\mathfrak{C}_3}(\beta)\} \\
\mathfrak{I}_{\mathfrak{C}_3}(\alpha+\beta) = \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha+\beta), \mathfrak{I}_{\mathfrak{C}_2}(\alpha+\beta)\} \\
\leq \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha), \mathfrak{I}_{\mathfrak{C}_1}(\beta)\}, \vee\{\mathfrak{I}_{\mathfrak{C}_2}(\alpha), \mathfrak{I}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha), \mathfrak{I}_{\mathfrak{C}_2}(\alpha)\}, \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\beta), \mathfrak{I}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\mathfrak{I}_{\mathfrak{C}_3}(\alpha), \mathfrak{I}_{\mathfrak{C}_3}(\beta)\} \\
\mathfrak{F}_{\mathfrak{C}_3}(\alpha+\beta) = \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha+\beta), \mathfrak{F}_{\mathfrak{C}_2}(\alpha+\beta)\} \\
\leq \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha), \mathfrak{F}_{\mathfrak{C}_1}(\beta)\}, \vee\{\mathfrak{F}_{\mathfrak{C}_2}(\alpha), \mathfrak{F}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha), \mathfrak{F}_{\mathfrak{C}_2}(\alpha)\}, \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\beta), \mathfrak{F}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\mathfrak{F}_{\mathfrak{C}_3}(\alpha), \mathfrak{F}_{\mathfrak{C}_3}(\beta)\} \\
\mathfrak{M}_{\mathfrak{C}_3}(\zeta\alpha) = \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\zeta\alpha), \mathfrak{M}_{\mathfrak{C}_2}(\zeta\alpha)\} \geq \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha), \mathfrak{M}_{\mathfrak{C}_2}(\alpha)\} = \mathfrak{M}_{\mathfrak{C}_3}(\alpha) \\
\mathfrak{I}_{\mathfrak{C}_3}(\zeta\alpha) = \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\zeta\alpha), \mathfrak{I}_{\mathfrak{C}_2}(\zeta\alpha)\} \leq \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha), \mathfrak{I}_{\mathfrak{C}_2}(\alpha)\} = \mathfrak{I}_{\mathfrak{C}_3}(\alpha) \\
\mathfrak{F}_{\mathfrak{C}_3}(\zeta\alpha) = \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\zeta\alpha), \mathfrak{F}_{\mathfrak{C}_2}(\zeta\alpha)\} \leq \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha), \mathfrak{F}_{\mathfrak{C}_2}(\alpha)\} = \mathfrak{F}_{\mathfrak{C}_3}(\alpha) \\
\mathfrak{M}_{\mathfrak{C}_3}([\alpha,\beta]) = \wedge\{\mathfrak{M}_{\mathfrak{C}_1}([\alpha,\beta]), \mathfrak{M}_{\mathfrak{C}_2}([\alpha,\beta])\} \\
\geq \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha), \mathfrak{M}_{\mathfrak{C}_1}(\beta)\}, \wedge\{\mathfrak{M}_{\mathfrak{C}_2}(\alpha), \mathfrak{M}_{\mathfrak{C}_2}(\beta)\}\} \\
= \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha), \mathfrak{M}_{\mathfrak{C}_2}(\alpha)\}, \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\beta), \mathfrak{M}_{\mathfrak{C}_2}(\beta)\}\} \\
= \wedge\{\mathfrak{M}_{\mathfrak{C}_3}(\alpha), \mathfrak{M}_{\mathfrak{C}_3}(\beta)\} \\
\mathfrak{I}_{\mathfrak{C}_3}([\alpha,\beta]) = \vee\{\mathfrak{I}_{\mathfrak{C}_1}([\alpha,\beta]), \mathfrak{I}_{\mathfrak{C}_2}([\alpha,\beta])\} \\
\geq \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha), \mathfrak{I}_{\mathfrak{C}_1}(\beta)\}, \vee\{\mathfrak{I}_{\mathfrak{C}_2}(\alpha), \mathfrak{I}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha), \mathfrak{I}_{\mathfrak{C}_2}(\alpha)\}, \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\beta), \mathfrak{I}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\mathfrak{I}_{\mathfrak{C}_3}(\alpha), \mathfrak{I}_{\mathfrak{C}_3}(\beta)\} \\
\mathfrak{F}_{\mathfrak{C}_3}([\alpha,\beta]) = \vee\{\mathfrak{F}_{\mathfrak{C}_1}([\alpha,\beta]), \mathfrak{F}_{\mathfrak{C}_2}([\alpha,\beta])\} \\
\geq \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha), \mathfrak{F}_{\mathfrak{C}_1}(\beta)\}, \vee\{\mathfrak{F}_{\mathfrak{C}_2}(\alpha), \mathfrak{F}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha), \mathfrak{F}_{\mathfrak{C}_2}(\alpha)\}, \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\beta), \mathfrak{F}_{\mathfrak{C}_2}(\beta)\}\} \\
= \vee\{\mathfrak{F}_{\mathfrak{C}_3}(\alpha), \mathfrak{F}_{\mathfrak{C}_3}(\beta)\} \ \square
\end{gathered}$$

Theorem 3.9. *Let $\{\mathfrak{C}_i | i \in \Delta\}$ be a collection of complex neutrosophic subalgebras of $\mathfrak{L}$ such that $\mathfrak{C}_i$ is homogenous with $\mathfrak{C}_\mathfrak{k}$ for all $j, \mathfrak{k} \in \Delta$. Then $\bigcap_{i\in\Delta} \mathfrak{C}_i = (\mathfrak{M}_{\cap_{i\in\Delta}\mathfrak{C}_i}, \mathfrak{I}_{\cap_{i\in\Delta}\mathfrak{C}_i}, \mathfrak{F}_{\cap_{i\in\Delta}\mathfrak{C}_i})$ is a complex neutrosophic subalgebra of $\mathfrak{L}$, where*
$\bigcap_{i\in\Delta} \mathfrak{C}_i = (\mathfrak{M}_{\cap_{i\in\Delta}\mathfrak{C}_i}, \mathfrak{I}_{\cap_{i\in\Delta}\mathfrak{C}_i}, \mathfrak{F}_{\cap_{i\in\Delta}\mathfrak{C}_i}) = ((\wedge_{i\in\Delta} p_{\mathfrak{C}_i}) e^{j\wedge_{i\in\Delta}\mu_{\mathfrak{C}_i}}, (\vee_{i\in\Delta} q_{\mathfrak{C}_i}) e^{j\vee_{i\in\Delta}\nu_{\mathfrak{C}_i}},$
$(\vee_{i\in\Delta} r_{\mathfrak{C}_i}) e^{j\vee_{i\in\Delta}\omega_{\mathfrak{C}_i}})$

We omit the proof as it is similar to the proof of Theorem 3.8. □

Theorem 3.10. *Let $\mathfrak{C}_1 = (\mathfrak{M}_1, \mathfrak{I}_1, \mathfrak{F}_1)$ and $\mathfrak{C}_2 = (\mathfrak{M}_2, \mathfrak{I}_2, \mathfrak{F}_2)$ be two neutrosophic complex Lie subalgebras over $\mathfrak{L}$, then the cartesian product $\mathfrak{C}_3 = \mathfrak{C}_1 \times \mathfrak{C}_2 = (\mathfrak{M}_3, \mathfrak{I}_3, \mathfrak{F}_3) = (\mathfrak{M}_1 \times \mathfrak{M}_2, \mathfrak{I}_1 \times \mathfrak{I}_2, \mathfrak{F}_1 \times \mathfrak{F}_2)$ is a complex neutrosophic Lie subalgebra over $\mathfrak{L} \times \mathfrak{L}$.*
Proof. *For each $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathfrak{L} \times \mathfrak{L}$ and $\zeta \in \mathcal{F}$. Then*

$$\begin{gathered}
\mathfrak{M}_{\mathfrak{C}_3}(\alpha+\beta) = (\mathfrak{M}_{\mathfrak{C}_1} \times \mathfrak{M}_{\mathfrak{C}_2})(\alpha+\beta) = (\mathfrak{M}_{\mathfrak{C}_1} \times \mathfrak{M}_{\mathfrak{C}_2})((\alpha_1,\alpha_2)+(\beta_1,\beta_2)) = \\
\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha_1+\beta_1), \mathfrak{M}_{\mathfrak{C}_2}(\alpha_2+\beta_2)\} \\
\geq \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha_1), \mathfrak{M}_{\mathfrak{C}_1}(\beta_1)\}, \wedge\{\mathfrak{M}_{\mathfrak{C}_2}(\alpha_2), \mathfrak{M}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha_1), \mathfrak{M}_{\mathfrak{C}_2}(\alpha_2)\}, \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\beta_1), \mathfrak{M}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \wedge\{\mathfrak{M}_{\mathfrak{C}_1} \times \mathfrak{M}_{\mathfrak{C}_2}(\alpha_1,\alpha_2), \mathfrak{M}_{\mathfrak{C}_1} \times \mathfrak{M}_{\mathfrak{C}_2}(\beta_1,\beta_2)\} \\
= \wedge\{\mathfrak{M}_{\mathfrak{C}_1} \times \mathfrak{M}_{\mathfrak{C}_2}(\alpha), \mathfrak{M}_{\mathfrak{C}_1} \times \mathfrak{M}_{\mathfrak{C}_2}(\beta)\} \\
\mathfrak{I}_{\mathfrak{C}_3}(\alpha+\beta) = (\mathfrak{I}_{\mathfrak{C}_1} \times \mathfrak{I}_{\mathfrak{C}_2})(\alpha+\beta) = (\mathfrak{I}_{\mathfrak{C}_1} \times \mathfrak{I}_{\mathfrak{C}_2})((\alpha_1,\alpha_2)+(\beta_1,\beta_2)) = \\
\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha_1+\beta_1), \mathfrak{I}_{\mathfrak{C}_2}(\alpha_2+\beta_2)\} \\
\leq \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha_1), \mathfrak{I}_{\mathfrak{C}_1}(\beta_1)\}, \vee\{\mathfrak{I}_{\mathfrak{C}_2}(\alpha_2), \mathfrak{I}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha_1), \mathfrak{I}_{\mathfrak{C}_2}(\alpha_2)\}, \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\beta_1), \mathfrak{I}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{(\mathfrak{I}_{\mathfrak{C}_1} \times \mathfrak{I}_{\mathfrak{C}_2})(\alpha_1,\alpha_2), (\mathfrak{I}_{\mathfrak{C}_1} \times \mathfrak{I}_{\mathfrak{C}_2})(\beta_1,\beta_2)\} \\
= \vee\{(\mathfrak{I}_{\mathfrak{C}_1} \times \mathfrak{I}_{\mathfrak{C}_2})(\alpha), (\mathfrak{I}_{\mathfrak{C}_1} \times \mathfrak{I})_{\mathfrak{C}_2}(\beta)\}
\end{gathered}$$

$$\begin{aligned}
\mathfrak{F}_{\mathfrak{C}_3}(\alpha+\beta) = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\alpha+\beta) = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})((\alpha_1,\alpha_2)+(\beta_1,\beta_2)) = \\
\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha_1+\beta_1),\mathfrak{F}_{\mathfrak{C}_2}(\alpha_2+\beta_2)\} \\
\leq \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{F}_{\mathfrak{C}_1}(\beta_1)\},\vee\{\mathfrak{F}_{\mathfrak{C}_2}(\alpha_2),\mathfrak{F}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{F}_{\mathfrak{C}_2}(\alpha_2)\},\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\beta_1),\mathfrak{F}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\alpha_1,\alpha_2),(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\beta_1,\beta_2)\} \\
= \vee\{(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\alpha),(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F})_{\mathfrak{C}_2}(\beta)\} \\
\mathfrak{M}_{\mathfrak{C}_3}(\zeta\alpha) = (\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})(\zeta\alpha) = (\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})(\zeta(\alpha_1,\alpha_2)) = \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\zeta\alpha_1),\mathfrak{M}_{\mathfrak{C}_2}(\zeta\alpha_2))\} \\
\geq \wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{M}_{\mathfrak{C}_2}(\alpha_2))\} = (\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})(\alpha_1,\alpha_2) = \mathfrak{M}_{\mathfrak{C}_3}(\alpha) \\
\mathfrak{I}_{\mathfrak{C}_3}(\zeta\alpha) = (\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})(\zeta\alpha) = (\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})(\zeta(\alpha_1,\alpha_2)) = \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\zeta\alpha_1),\mathfrak{I}_{\mathfrak{C}_2}(\zeta\alpha_2))\} \\
\leq \vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{I}_{\mathfrak{C}_2}(\alpha_2))\} = (\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})(\alpha_1,\alpha_2) = \mathfrak{I}_{\mathfrak{C}_3}(\alpha) \\
\mathfrak{F}_{\mathfrak{C}_3}(\zeta\alpha) = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\zeta\alpha) = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\zeta(\alpha_1,\alpha_2)) = \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\zeta\alpha_1),\mathfrak{F}_{\mathfrak{C}_2}(\zeta\alpha_2))\} \\
\leq \vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{F}_{\mathfrak{C}_2}(\alpha_2))\} = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\alpha_1,\alpha_2) = \mathfrak{F}_{\mathfrak{C}_3}(\alpha) \\
\mathfrak{M}_{\mathfrak{C}_3}([\alpha,\beta]) = (\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})([\alpha,\beta]) = (\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})([(\alpha_1,\alpha_2),(\beta_1,\beta_2)]) = \\
\wedge\{\mathfrak{M}_{\mathfrak{C}_1}([\alpha_1,\beta_1]),\mathfrak{M}_{\mathfrak{C}_2}([\alpha_2,\beta_2])\} \\
\geq \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{M}_{\mathfrak{C}_1}(\beta_1)\},\wedge\{\mathfrak{M}_{\mathfrak{C}_2}(\alpha_2),\mathfrak{M}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \wedge\{\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{M}_{\mathfrak{C}_2}(\alpha_2)\},\wedge\{\mathfrak{M}_{\mathfrak{C}_1}(\beta_1),\mathfrak{M}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \wedge\{(\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})([\alpha_1,\alpha_2]),(\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})([\beta_1,\beta_2])\} \\
= \wedge\{(\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})(\alpha),(\mathfrak{M}_{\mathfrak{C}_1}\times\mathfrak{M}_{\mathfrak{C}_2})(\beta)\} \\
\mathfrak{I}_{\mathfrak{C}_3}([\alpha,\beta]) = (\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})([\alpha,\beta]) = (\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})([(\alpha_1,\alpha_2),(\beta_1,\beta_2)]) = \\
\vee\{\mathfrak{I}_{\mathfrak{C}_1}([\alpha_1,\beta_1]),\mathfrak{I}_{\mathfrak{C}_2}([\alpha_2,\beta_2])\} \\
\leq \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{I}_{\mathfrak{C}_1}(\beta_1)\},\vee\{\mathfrak{I}_{\mathfrak{C}_2}(\alpha_2),\mathfrak{I}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{I}_{\mathfrak{C}_2}(\alpha_2)\},\vee\{\mathfrak{I}_{\mathfrak{C}_1}(\beta_1),\mathfrak{I}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{(\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})([\alpha_1,\alpha_2]),(\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})([\beta_1,\beta_2])\} \\
= \vee\{(\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})(\alpha),(\mathfrak{I}_{\mathfrak{C}_1}\times\mathfrak{I}_{\mathfrak{C}_2})(\beta)\} \\
\mathfrak{F}_{\mathfrak{C}_3}([\alpha,\beta]) = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})([\alpha,\beta]) = (\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})([(\alpha_1,\alpha_2),(\beta_1,\beta_2)]) = \\
\vee\{\mathfrak{F}_{\mathfrak{C}_1}([\alpha_1,\beta_1]),\mathfrak{F}_{\mathfrak{C}_2}([\alpha_2,\beta_2])\} \\
\leq \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{F}_{\mathfrak{C}_1}(\beta_1)\},\vee\{\mathfrak{F}_{\mathfrak{C}_2}(\alpha_2),\mathfrak{F}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\alpha_1),\mathfrak{F}_{\mathfrak{C}_2}(\alpha_2)\},\vee\{\mathfrak{F}_{\mathfrak{C}_1}(\beta_1),\mathfrak{F}_{\mathfrak{C}_2}(\beta_2)\}\} \\
= \vee\{(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})([\alpha_1,\alpha_2]),(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})([\beta_1,\beta_2])\} \\
= \vee\{(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\alpha),(\mathfrak{F}_{\mathfrak{C}_1}\times\mathfrak{F}_{\mathfrak{C}_2})(\beta)\}
\end{aligned}$$

This shows that $\mathfrak{C}_1\times\mathfrak{C}_2$ is a complex neutrosophic Lie subalgebra of $\mathfrak{L}\times\mathfrak{L}$. □

4 On complex neutrosophic Lie algebra homomorphisms

In this section, we investigate the properties of complex neutrosophic Lie subalgebras and complex neutrosophic ideals under homomorphisms of Lie algebras.

Definition 4.1. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be two Lie algebras over a field $\mathfrak{F}$. Then a linear transformation $\mathfrak{f}:\mathfrak{L}_1\to\mathfrak{L}_2$ is called a Lie homomorphism if $\mathfrak{f}([\alpha,\beta]) = [\mathfrak{f}(\alpha),\mathfrak{f}(\beta)]$ holds for all $\alpha,\beta\in\mathfrak{L}_1$.

For the Lie algebras $\mathfrak{L}_1$ and $\mathfrak{L}_2$, it can be easily observed that if $\mathfrak{f}:\mathfrak{L}_1\to\mathfrak{L}_2$ is a Lie homomorphism and $\mathfrak{C}=(\mathfrak{M},\mathfrak{I},\mathfrak{F})$ is a complex neutrosophic Lie subalgebra of $\mathfrak{L}_2$, then the complex neutrosophic set $\mathfrak{f}^{-1}(\mathfrak{C})$ of $\mathfrak{L}_1$ is also a neutrosophic Lie subalgebra, where

$$\mathfrak{f}^{-1}(\mathfrak{M}_{\mathfrak{C}})(\alpha) = \mathfrak{M}_{\mathfrak{C}}(\mathfrak{f}(\alpha)) = \mathfrak{p}_{\mathfrak{C}}(\mathfrak{f}(\alpha))e^{j\mu(\mathfrak{f}(\alpha))},\ \mathfrak{f}^{-1}(\mathfrak{I}_{\mathfrak{C}})(\alpha) = \mathfrak{I}_{\mathfrak{C}}(\mathfrak{f}(\alpha)) = \mathfrak{q}_{\mathfrak{C}}(\mathfrak{f}(\alpha))e^{j\nu(\mathfrak{f}(\alpha))}$$
$$\mathfrak{f}^{-1}(\mathfrak{F}_{\mathfrak{C}})(\alpha) = \mathfrak{F}_{\mathfrak{C}}(\mathfrak{f}(\alpha)) = \mathfrak{r}_{\mathfrak{C}}(\mathfrak{f}(\alpha))e^{j\omega(\mathfrak{f}(\alpha))}$$

Theorem 4.2. *Let $\xi:\mathfrak{L}\to\mathfrak{L}'$ be a Lie algebra homomorphism. If $\mathfrak{C}=(\mathfrak{M},\mathfrak{I},\mathfrak{F})$ is a complex neutrosophic Lie subalgebra of $\mathfrak{L}'$ with a membership, indeterminacy and non-membership functions are $\mathfrak{M}_{\mathfrak{C}}(\beta)=\mathfrak{p}_{\mathfrak{C}}(\beta)e^{j\mu_{\mathfrak{C}}(\beta)}$, $\mathfrak{I}_{\mathfrak{C}}(\beta)=\mathfrak{q}_{\mathfrak{C}}(\beta)e^{j\nu_{\mathfrak{C}}(\beta)}$, and $\mathfrak{F}_{\mathfrak{C}}(\beta)=\mathfrak{r}_{\mathfrak{C}}(\beta)e^{j\omega_{\mathfrak{C}}(\beta)}$, respectively , then the complex neutrosophic set $\xi^{-1}(\mathfrak{C})$ is also a complex neutrosophic Lie subalgebra of $\mathfrak{L}$.*

Proof. *First, we need to show that $\xi^{-1}(\mathfrak{C})$ is homogeneous. Note that if $\alpha\in\mathfrak{L}$, then $\mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha) = \mathfrak{M}_{\mathfrak{C}}(\xi(\alpha)) = \mathfrak{p}_{\mathfrak{C}}(\xi(\alpha))e^{j\mu_{\mathfrak{C}}(\xi(\alpha))} = (\mathfrak{p}_{\mathfrak{C}}\xi(\alpha))e^{j\mu_{\mathfrak{C}}(\xi(\alpha))}$, $\mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha) = \mathfrak{I}_{\mathfrak{C}}(\xi(\alpha)) = \mathfrak{q}_{\mathfrak{C}}(\xi(\alpha))e^{j\nu_{\mathfrak{C}}(\xi(\alpha))} = (\mathfrak{q}_{\mathfrak{C}}\xi(\alpha))e^{j\nu_{\mathfrak{C}}(\xi(\alpha))}$, and $\mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha) = \mathfrak{F}_{\mathfrak{C}}(\xi(\alpha)) = \mathfrak{r}_{\mathfrak{C}}(\xi(\alpha))e^{j\omega_{\mathfrak{C}}(\xi(\alpha))} = (\mathfrak{r}_{\mathfrak{C}}\xi(\alpha))e^{j\omega_{\mathfrak{C}}(\xi(\alpha))}$. Now, if $\alpha_1,\alpha_2\in\mathfrak{L}$ with $(\mathfrak{p}_{\mathfrak{C}}\xi)(\alpha_1)\leq(\mathfrak{p}_{\mathfrak{C}}\xi)(\alpha_2)$, that is $\mathfrak{p}_{\mathfrak{C}}(\xi(\alpha_1))\leq\mathfrak{p}_{\mathfrak{C}}(\xi(\alpha_2))$, $(\mathfrak{q}_{\mathfrak{C}}\xi)(\alpha_1)\geq(\mathfrak{q}_{\mathfrak{C}}\xi)(\alpha_2)$, that is $\mathfrak{q}_{\mathfrak{C}}(\xi(\alpha_1))\geq\mathfrak{q}_{\mathfrak{C}}(\xi(\alpha_2))$, $(\mathfrak{r}_{\mathfrak{C}}\xi)(\alpha_1)\geq(\mathfrak{r}_{\mathfrak{C}}\xi)(\alpha_2)$, that is $\mathfrak{r}_{\mathfrak{C}}(\xi(\alpha_1))\geq\mathfrak{r}_{\mathfrak{C}}(\xi(\alpha_2))$,* *then from the homogeneity of* $\mathfrak{C}$*, we have* $(\mu_{\mathfrak{C}}\xi)(\alpha_1) \leq (\mu_{\mathfrak{C}}\xi)(\alpha_2)$*, that is* $\mu_{\mathfrak{C}}(\xi(\alpha_1)) \leq \mu_{\mathfrak{C}}(\xi(\alpha_2))$*,* $(\nu_{\mathfrak{C}}\xi)(\alpha_1) \geq (\nu_{\mathfrak{C}}\xi)(\alpha_2)$*, that is* $\nu_{\mathfrak{C}}(\xi(\alpha_1)) \geq \nu_{\mathfrak{C}}(\xi(\alpha_2))$*,* $(\omega_{\mathfrak{C}}\xi)(\alpha_1) \geq (\omega_{\mathfrak{C}}\xi)(\alpha_2)$*, that is* $\omega_{\mathfrak{C}}(\xi(\alpha_1)) \geq \omega_{\mathfrak{C}}(\xi(\alpha_2))$*. Thus shows* $\xi^{-1}(\mathfrak{C})$ *is homogenous. Let* $\alpha_1, \alpha_2 \in \mathfrak{L}$ *and* $\zeta \in \mathcal{F}$*. Then*

$$\begin{aligned}
\mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha_1+\alpha_2) &= \mathfrak{M}_{\mathfrak{C}}(\xi(\alpha_1+\alpha_2))\\
&= \mathfrak{M}_{\mathfrak{C}}(\xi(\alpha_1)+\xi(\alpha_2))\\
&\geq \wedge\{\mathfrak{M}_{\mathfrak{C}}(\xi(\alpha_1)), \mathfrak{M}_{\mathfrak{C}}(\xi(\alpha_2))\}\\
&= \wedge\{\mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha_1), \mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha_2)\}\\
\mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha_1+\alpha_2) &= \mathfrak{I}_{\mathfrak{C}}(\xi(\alpha_1+\alpha_2))\\
&= \mathfrak{I}_{\mathfrak{C}}(\xi(\alpha_1)+\xi(\alpha_2))\\
&\leq \wedge\{\mathfrak{I}_{\mathfrak{C}}(\xi(\alpha_1)), \mathfrak{I}_{\mathfrak{C}}(\xi(\alpha_2))\}\\
&= \vee\{\mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha_1), \mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha_2)\}\\
\mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha_1+\alpha_2) &= \mathfrak{F}_{\mathfrak{C}}(\xi(\alpha_1+\alpha_2))\\
&= \mathfrak{F}_{\mathfrak{C}}(\xi(\alpha_1)+\xi(\alpha_2))\\
&\leq \wedge\{\mathfrak{F}_{\mathfrak{C}}(\xi(\alpha_1)), \mathfrak{F}_{\mathfrak{C}}(\xi(\alpha_2))\}\\
&= \vee\{\mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha_1), \mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha_2)\}, \text{ (ξ is linear).}\\
\mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\zeta\alpha) &= \mathfrak{M}_{\mathfrak{C}}(\xi(\zeta\alpha)) = \mathfrak{M}_{\mathfrak{C}}(\zeta\xi(\alpha))\\
&\geq \mathfrak{M}_{\mathfrak{C}}(\xi(\alpha)) = \mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha)\\
\mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\zeta\alpha) &= \mathfrak{I}_{\mathfrak{C}}(\xi(\zeta\alpha)) = \mathfrak{I}_{\mathfrak{C}}(\zeta\xi(\alpha))\\
&\leq \mathfrak{I}_{\mathfrak{C}}(\xi(\alpha)) = \mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha)\\
\mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\zeta\alpha) &= \mathfrak{F}_{\mathfrak{C}}(\xi(\zeta\alpha)) = \mathfrak{F}_{\mathfrak{C}}(\zeta\xi(\alpha))\\
&\leq \mathfrak{F}_{\mathfrak{C}}(\xi(\alpha)) = \mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha), \text{ (ξ is linear).}\\
\mathfrak{M}_{\xi^{-1}(\mathfrak{C})}([\alpha_1,\alpha_2]) &= \mathfrak{M}_{\mathfrak{C}}(\xi([\alpha_1,\alpha_2]))\\
&= \mathfrak{M}_{\mathfrak{C}}([\xi(\alpha_1),\xi(\alpha_2)])\\
&\geq \wedge\{\mathfrak{M}_{\mathfrak{C}}(\xi(\alpha_1)), \mathfrak{M}_{\mathfrak{C}}(\xi(\alpha_2))\}\\
&= \wedge\{\mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha_1), \mathfrak{M}_{\xi^{-1}(\mathfrak{C})}(\alpha_2)\},\\
\mathfrak{I}_{\xi^{-1}(\mathfrak{C})}([\alpha_1,\alpha_2]) &= \mathfrak{I}_{\mathfrak{C}}(\xi([\alpha_1,\alpha_2]))\\
&= \mathfrak{I}_{\mathfrak{C}}([\xi(\alpha_1),\xi(\alpha_2)])\\
&\leq \vee\{\mathfrak{I}_{\mathfrak{C}}(\xi(\alpha_1)), \mathfrak{I}_{\mathfrak{C}}(\xi(\alpha_2))\}\\
&= \vee\{\mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha_1), \mathfrak{I}_{\xi^{-1}(\mathfrak{C})}(\alpha_2)\},\\
\mathfrak{F}_{\xi^{-1}(\mathfrak{C})}([\alpha_1,\alpha_2]) &= \mathfrak{F}_{\mathfrak{C}}(\xi([\alpha_1,\alpha_2]))\\
&= \mathfrak{F}_{\mathfrak{C}}([\xi(\alpha_1),\xi(\alpha_2)])\\
&\leq \vee\{\mathfrak{F}_{\mathfrak{C}}(\xi(\alpha_1)), \mathfrak{F}_{\mathfrak{C}}(\xi(\alpha_2))\}\\
&= \vee\{\mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha_1), \mathfrak{F}_{\xi^{-1}(\mathfrak{C})}(\alpha_2)\}, \text{ (ξ is homomorphism).} \ \square
\end{aligned}$$

Theorem 4.3. *Let* $\xi : \mathfrak{L} \to \mathfrak{L}'$ *be a Lie algebra homomorphism. If* $\mathfrak{C} = (\mathfrak{M}, \mathfrak{I}, \mathfrak{F})$ *is a complex neutrosophic ideal of* $\mathfrak{L}'$ *with a membership, indeterminacy and non-membership functions are* $\mathfrak{M}_{\mathfrak{C}}(\beta) = \mathfrak{p}_{\mathfrak{C}}(\beta)e^{j\mu_{\mathfrak{C}}(\beta)}$*,* $\mathfrak{I}_{\mathfrak{C}}(\beta) = \mathfrak{q}_{\mathfrak{C}}(\beta)e^{j\nu_{\mathfrak{C}}(\beta)}$*, and* $\mathfrak{F}_{\mathfrak{C}}(\beta) = \mathfrak{r}_{\mathfrak{C}}(\beta)e^{j\omega_{\mathfrak{C}}(\beta)}$*, respectively , then the complex neutrosophic set* $\xi^{-1}(\mathfrak{C})$ *is also a complex fuzzy ideal of* $\mathfrak{L}$*.*
Proof. *The proof is similar to that of Theorem 4.2.*

Theorem 4.4. *Let* $\xi : \mathfrak{L} \to \mathfrak{L}'$ *be a surjective Lie algebra homomorphism. If* $\mathfrak{C} = (\mathfrak{M}, \mathfrak{I}, \mathfrak{F})$*, where* $\mathfrak{M}_{\mathfrak{C}}(\alpha) = \mathfrak{p}_{\mathfrak{C}}(\alpha)e^{j\mu_{\mathfrak{C}}(\alpha)}$*,* $\mathfrak{I}_{\mathfrak{C}}(\alpha) = \mathfrak{q}_{\mathfrak{C}}(\alpha)e^{j\nu_{\mathfrak{C}}(\alpha)}$*, and* $\mathfrak{F}_{\mathfrak{C}}(\alpha) = \mathfrak{r}_{\mathfrak{C}}(\alpha)e^{j\omega_{\mathfrak{C}}(\alpha)}$*, for any* $\alpha \in \mathfrak{L}$*, is a complex neutrosophic Lie subalgebra of* $\mathfrak{L}$*, then* $\xi(\mathfrak{C})$ *is also a complex neutrosophic Lie subalgebra of* $\mathfrak{L}'$*.*
Proof. *We prove that* $\xi(\mathfrak{C})$ *is homogenous. Suppose* $\beta \in \mathfrak{L}'$*. Then*

$$\begin{aligned}
\mathfrak{M}_{\xi(\mathfrak{C})}(\beta) &= sup_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{M}_{\mathfrak{C}}(\alpha)\} = sup_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{p}_{\mathfrak{C}}(\alpha)e^{j\mu_{\mathfrak{C}}(\alpha)}\}\\
&= sup_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{p}_{\mathfrak{C}}(\beta)\}e^{j(sup_{\alpha\in\xi^{-1}(\beta)}\{\mu_{\mathfrak{C}}(\beta)\}} = \mathfrak{p}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta)e^{j\mu_{\mathfrak{C}\xi(\mathfrak{C})}(\beta)}.\\
\mathfrak{I}_{\xi(\mathfrak{C})}(\beta) &= inf_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{I}_{\mathfrak{C}}(\alpha)\} = inf_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{q}_{\mathfrak{C}}(\alpha)e^{j\nu_{\mathfrak{C}}(\alpha)}\}\\
&= inf_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{q}_{\mathfrak{C}}(\beta)\}e^{j(sup_{\alpha\in\xi^{-1}(\beta)}\{\nu_{\mathfrak{C}}(\beta)\}} = \mathfrak{q}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta)e^{j\nu_{\mathfrak{C}\xi(\mathfrak{C})}(\beta)}.\\
\mathfrak{F}_{\xi(\mathfrak{C})}(\beta) &= inf_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{F}_{\mathfrak{C}}(\alpha)\} = inf_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{r}_{\mathfrak{C}}(\alpha)e^{j\omega_{\mathfrak{C}}(\alpha)}\}\\
&= inf_{\alpha\in\xi^{-1}(\beta)}\{\mathfrak{r}_{\mathfrak{C}}(\beta)\}e^{j(sup_{\alpha\in\xi^{-1}(\beta)}\{\omega_{\mathfrak{C}}(\beta)\}} = \mathfrak{r}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta)e^{j\omega_{\mathfrak{C}\xi(\mathfrak{C})}(\beta)}.
\end{aligned}$$

Now let $\beta_1, \beta_2 \in \mathfrak{L}'$ *with* $\mathfrak{p}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_1) \leq \mathfrak{p}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_2)$ *and* $\mu_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_2) < \mu_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_1)$*. Then there exist a* $\alpha_1 \in \xi^{-1}(\{\beta_1\})$*, such that* $\mu_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_2) < \mu_{\mathfrak{C}}(\alpha_1)$*. Therefore, If* $\alpha \in \xi^{-1}(\{\beta_2\})$*, then* $\mu_{\mathfrak{C}}(\alpha) < \mu_{\mathfrak{C}}(\alpha_1)$*, and so, from the homogeneity of* $\mathfrak{C}$*, we obtain* $\mathfrak{p}_{\mathfrak{C}}(\alpha) < \mathfrak{p}_{\mathfrak{C}}(\alpha_1)$*. Thus,*

$sup_{\alpha\in\xi^{-1}(\beta_2)}\{\mathfrak{p}_{\mathfrak{C}}(\alpha)\} < \mathfrak{p}_{\mathfrak{C}}(\alpha_1)$ and so, $\mathfrak{p}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_2) \leq \mathfrak{p}_{\mathfrak{C}\xi(\mathfrak{C})}(\beta_1)$, which is a contradiction. Similarly we can prove for indeterminacy and non-membership functions. This shows $\xi(\mathfrak{C})$ is homogenous.
Since $\mathfrak{C}$ is a complex neutrosophic subalgebra, $\overline{\mathfrak{C}} = \{(\alpha, \langle\mathfrak{F}_{\mathfrak{C}}(\alpha), 1-\mathfrak{I}_{\mathfrak{C}}(\alpha), \mathfrak{M}_{\mathfrak{C}}(\alpha)\rangle)|\alpha \in \mathfrak{L}\}$ is a neutrosophic subalgebra of $\mathfrak{L}$, and so the images of the components are neutrosophic subalgebra of $\mathfrak{L}'$. Hence, for $\beta_1, \beta_2 \in \mathfrak{L}'$ and $\zeta \in \mathcal{F}$, we have

(i) $\mathfrak{M}_{\xi(\mathfrak{C})}(\beta_1+\beta_2) \geq \wedge(\mathfrak{M}_{\xi(\mathfrak{C})}(\beta_1), \mathfrak{M}_{\xi(\mathfrak{C})}(\beta_2))$,
$\mathfrak{M}_{\xi(\mathfrak{C})}(\zeta\beta_1) \geq \wedge\mathfrak{M}_{\xi(\mathfrak{C})}(\beta_1)$,
$\mathfrak{M}_{\xi(\mathfrak{C})}([\beta_1,\beta_2]) \geq \wedge(\mathfrak{M}_{\xi(\mathfrak{C})}(\beta_1), \mathfrak{M}_{\xi(\mathfrak{C})}(\beta_2))$

(ii) $\mathfrak{I}_{\xi(\mathfrak{C})}(\beta_1+\beta_2) \leq \vee(\mathfrak{I}_{\xi(\mathfrak{C})}(\beta_1), \mathfrak{I}_{\xi(\mathfrak{C})}(\beta_2))$,
$\mathfrak{I}_{\xi(\mathfrak{C})}(\zeta\beta_1) \leq \vee\mathfrak{I}_{\xi(\mathfrak{C})}(\beta_1)$,
$\mathfrak{I}_{\xi(\mathfrak{C})}([\beta_1,\beta_2]) \leq \vee(\mathfrak{I}_{\xi(\mathfrak{C})}(\beta_1), \mathfrak{I}_{\xi(\mathfrak{C})}(\beta_2))$

(iii) $\mathfrak{F}_{\xi(\mathfrak{C})}(\beta_1+\beta_2) \leq \vee(\mathfrak{F}_{\xi(\mathfrak{C})}(\beta_1), \mathfrak{F}_{\xi(\mathfrak{C})}(\beta_2))$,
$\mathfrak{F}_{\xi(\mathfrak{C})}(\zeta\beta_1) \leq \vee\mathfrak{F}_{\xi(\mathfrak{C})}(\beta_1)$,
$\mathfrak{F}_{\xi(\mathfrak{C})}([\beta_1,\beta_2]) \leq \vee(\mathfrak{F}_{\xi(\mathfrak{C})}(\beta_1), \mathfrak{F}_{\xi(\mathfrak{C})}(\beta_2))$

Now our result follows from the homogeneity of $\xi(\mathfrak{C})$. □

Theorem 4.5. *Let $\xi : \mathfrak{L} \to \mathfrak{L}'$ be a surjective Lie algebra homomorphism. If $\mathfrak{C} = (\mathfrak{M}, \mathfrak{I}, \mathfrak{F})$, where $\mathfrak{M}_{\mathfrak{C}}(\alpha) = \mathfrak{p}_{\mathfrak{C}}(\alpha)e^{j\mu_{\mathfrak{C}}(\alpha)}$, $\mathfrak{I}_{\mathfrak{C}}(\alpha) = \mathfrak{q}_{\mathfrak{C}}(\alpha)e^{j\nu_{\mathfrak{C}}(\alpha)}$, and $\mathfrak{F}_{\mathfrak{C}}(\alpha) = \mathfrak{r}_{\mathfrak{C}}(\alpha)e^{j\omega_{\mathfrak{C}}(\alpha)}$, for any $\alpha \in \mathfrak{L}$, is a complex neutrosophic ideal of $\mathfrak{L}$, then $\xi(\mathfrak{C})$ is also a complex neutrosophic ideal of $\mathfrak{L}'$. .*

Theorem 4.6. *Let $\xi : \mathfrak{L} \to \mathfrak{L}'$ be a surjective Lie homomorphism. If $\mathfrak{C}_1 = (\mathfrak{M}_1, \mathfrak{I}_1, \mathfrak{F}_1)$ and $\mathfrak{C}_2 = (\mathfrak{M}_2, \mathfrak{I}_2, \mathfrak{F}_2)$ are complex neutrosophic ideals of $\mathfrak{L}$ such that $\mathfrak{C}_1$ is homogeneous of $\mathfrak{C}_2$, then $\xi(\mathfrak{C}_1+\mathfrak{C}_2) = \xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)$.*
Proof. *For $\beta \in \mathfrak{L}'$, we have*

(i) $\mathfrak{M}_{\xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)}(\beta) = sup_{\beta=\xi(\alpha)}\{\mathfrak{M}_{\mathfrak{C}_1+\mathfrak{C}_2}(\alpha)\}$
$= sup_{\beta=\xi(\alpha)}\{sup_{\alpha=\mathfrak{a}+\mathfrak{b}}\{\mathfrak{M}_{\mathfrak{C}_1}(\mathfrak{a}) \wedge \mathfrak{M}_{\mathfrak{C}_2}(\mathfrak{b})\}\}$
$= sup_{\beta=\xi(\mathfrak{a})+\xi(\mathfrak{b})}\{\mathfrak{M}_{\mathfrak{C}_1}(\mathfrak{a}) \wedge \mathfrak{M}_{\mathfrak{C}_2}(\mathfrak{b})\}$
$= sup_{\beta=\mathfrak{m}+\mathfrak{n}}\{sup_{\mathfrak{m}=\xi(\mathfrak{a})}\{\mathfrak{M}_{\mathfrak{C}_1}(\mathfrak{a})\} \wedge sup_{\mathfrak{m}=\xi(\mathfrak{a})}\{\mathfrak{M}_{\mathfrak{C}_2}(\mathfrak{b})\}\}$
$= sup_{\beta=\mathfrak{m}+\mathfrak{n}}\{\mathfrak{M}_{\xi(\mathfrak{C}_1)}(\mathfrak{m}) \wedge \mathfrak{M}_{\xi(\mathfrak{C}_1)}(\mathfrak{n})\}$
$=\mathfrak{M}_{\xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)}(\beta)$.

(ii) $\mathfrak{I}_{\xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)}(\beta) = inf_{\beta=\xi(\alpha)}\{\mathfrak{I}_{\mathfrak{C}_1+\mathfrak{C}_2}(\alpha)\}$
$= inf_{\beta=\xi(\alpha)}\{inf_{\alpha=\mathfrak{a}+\mathfrak{b}}\{\mathfrak{I}_{\mathfrak{C}_1}(\mathfrak{a}) \vee \mathfrak{I}_{\mathfrak{C}_2}(\mathfrak{b})\}\}$
$= inf_{\beta=\xi(\mathfrak{a})+\xi(\mathfrak{b})}\{\mathfrak{I}_{\mathfrak{C}_1}(\mathfrak{a}) \vee \mathfrak{I}_{\mathfrak{C}_2}(\mathfrak{b})\}$
$= inf_{\beta=\mathfrak{m}+\mathfrak{n}}\{inf_{\mathfrak{m}=\xi(\mathfrak{a})}\{\mathfrak{I}_{\mathfrak{C}_1}(\mathfrak{a})\} \vee inf_{\mathfrak{m}=\xi(\mathfrak{a})}\{\mathfrak{I}_{\mathfrak{C}_2}(\mathfrak{b})\}\}$
$= inf_{\beta=\mathfrak{m}+\mathfrak{n}}\{\mathfrak{I}_{\xi(\mathfrak{C}_1)}(\mathfrak{m}) \vee \mathfrak{I}_{\xi(\mathfrak{C}_1)}(\mathfrak{n})\}$
$=\mathfrak{I}_{\xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)}(\beta)$.

(iii) $\mathfrak{F}_{\xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)}(\beta) = inf_{\beta=\xi(\alpha)}\{\mathfrak{F}_{\mathfrak{C}_1+\mathfrak{C}_2}(\alpha)\}$
$= inf_{\beta=\xi(\alpha)}\{inf_{\alpha=\mathfrak{a}+\mathfrak{b}}\{\mathfrak{F}_{\mathfrak{C}_1}(\mathfrak{a}) \vee \mathfrak{F}_{\mathfrak{C}_2}(\mathfrak{b})\}\}$
$= inf_{\beta=\xi(\mathfrak{a})+\xi(\mathfrak{b})}\{\mathfrak{F}_{\mathfrak{C}_1}(\mathfrak{a}) \vee \mathfrak{F}_{\mathfrak{C}_2}(\mathfrak{b})\}$
$= inf_{\beta=\mathfrak{m}+\mathfrak{n}}\{inf_{\mathfrak{m}=\xi(\mathfrak{a})}\{\mathfrak{F}_{\mathfrak{C}_1}(\mathfrak{a})\} \vee inf_{\mathfrak{m}=\xi(\mathfrak{a})}\{\mathfrak{F}_{\mathfrak{C}_2}(\mathfrak{b})\}\}$
$= inf_{\beta=\mathfrak{m}+\mathfrak{n}}\{\mathfrak{F}_{\xi(\mathfrak{C}_1)}(\mathfrak{m}) \vee \mathfrak{F}_{\xi(\mathfrak{C}_1)}(\mathfrak{n})\}$
$=\mathfrak{F}_{\xi(\mathfrak{C}_1)+\xi(\mathfrak{C}_2)}(\beta)$. □

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