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Substructures with an Almost Division Algorithm in Euclidean Commutator Lattices

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Abstract In this article we introduce the notion of a Euclidean commutator lattice L and study some of its algebraic properties in a parallel fashion with Euclidean (rings, semirings, modules, and semimodules). The notion of an almost division algorithm and join absorptive subsets of order $t \geq 1$ (t a fixed integer) in L as a parallel extension of additively absorptive subsemirings of Euclidean semirings are discussed. For a fixed integer $t \geq 1$, a lower set D of a Euclidean commutator lattice L with Euclidean function ϕ, is said to be join absorptive (or simply, absorptive) of order t in L; if for each f in $L \setminus D$, there exists h in D and g in L such that $f = h \vee g$ with $1 \leq \phi(g) \leq t$. The main result of the paper states that if I is an ideal of an absorptive subset D of order t in L, then I can be generated by $(t + 1)$ or fewer elements. In addition, if I contains an element of ϕ value equals to $i + \phi(I)$ for some $1 \leq i \leq t$ with $\phi(I) = \inf\{\phi(f) \mid f \in I\}$, then I cannot be a principal ideal in D.

Dedicated to the Memory of Nick Vaughan

1 Introduction

In this article we introduce the notion of a *Euclidean commutator lattice* L (Definition 2.8) defining over a *commutator lattice* (Definition 2.1) in a parallel fashion with Euclidean rings or Euclidean semirings and study the *almost division algorithm on join absorptive subsets* of order $t \geq 1$ in L (Definition 3.1). We recall and review some basic properties and definitions together with some (simple) examples related to (Euclidean) commutator lattices and show that every ideal in a Euclidean commutator lattice is principal (Theorem 2.12). Finally, Section 3 is devoted on main results of the paper (see specially Theorem 4.4).

In [2], in connection to the work of Vaughan [16], Chapman defines and applies a weaker form of the division algorithm, namely *the almost division algorithm of index* m (m a positive integer) over a natural class of subrings R of $K[X]$ containing the field K and then shows that *the number of generators of an ideal* I of R can not be only bounded, but also provides examples of ideals that can be generated by n, but not $n-1$ elements. Further, besides [2], [3], and [16], a more general approach to *rings and semirings* satisfying an almost division algorithm can be found in [11] and [13]. Also, for the notion of *Euclidean modules and Euclidean semimodules* as a natural extension of *Euclidean rings and Euclidean semirings*, see [14], [10], and [7]. For a detailed study of *semirings, semimodules, and Euclidean semirings*, the reader is referred to [5].

• In [8], we introduce an *abstract ordered set (i.e., commutator poset and commutator lattice)* (imitating the lattice of ideals of a ring), equipped with a commutative (*not necessarily associative*) binary operation (imitating the product of ideals of a ring). We also give some examples of a large class of this type of lattices such as the lattice of ideals of a (commutative) ring, the lattice of normal subgroups of a group, and the lattice of all congruences on an algebra in a variety (congruence modular variety) by using the commutators as the multiplicative binary operation on these lattices. Note that in [8] and [9], we study some *algebraic combinatorial properties* of L and its associated graph with many examples related to both subjects, which investigates the interplay between some lattice-theoretic properties of L and graph-theoretic properties of its associated graph.

$(*)$ We assume that the reader is familiar with the basic notion and definitions of lattice theory. For the notation and definitions regarding lattice theory, the reader is referred to [12] or [1] or any standard text on lattice theory. Also the reader is referred to Section 1 of [8] for some necessary lattice-theoretic definitions and properties that are required in this paper.

• Through out this paper (unless otherwise indicated) *all lattices are bounded*. A lattice L is said to be bounded if it contains a least element 0 and greatest element 1. In other words, the lattice L is said to be bounded if there are elements 0 and 1 in L such that $0 \wedge p = 0$ and $p \vee 1 = 1$ for all $p \in L$.

2 Euclidean Commutator Lattices

In this section we give the definition of a (Euclidean) commutator lattice (Definitions 2.1 and (2.8)) and some of its basic properties. For many basic results and some motivational (simple) examples related to commutator posets and commutator lattices, the reader is referred to [8] and [9]. See also Examples 2.5, 2.6, and 2.7 below.

The familiar *group-theoretic* notion of a *commutator* has been generalized to various contexts of *universal algebra and category theory*. The universal-algebraic references to commutators usually begin with J. D. H. Smith [15], and then mention various further generalizations of Smith's definition (see e.g. [4] and references therein).

• For some motivational (simple) examples of commutator lattices, see Examples 2.6 and 2.8 of [8]. See also the *introduction* of [4] for a general knowledge on commutator theory.

Definition 2.1. A commutator lattice is a (bounded) lattice L equipped with a binary operation $[-,-]$, also written as $[x, y] = xy$ satisfying the following conditions:

L1 $xy \leq x \wedge y$

L2 $xy = yx$

L3 $x(y \vee z) = (xy) \vee (xz)$

for all $x, y, z \in L$.

Remark 2.2. The most surprising fact here is that the binary operation [= commutator operation] (on lattices) involved is not required to be *associative*, unlike the ring multiplication; this is important since the commutator operation is almost never associative, except the commutative ring case. Proposition 2.12 of [8] provides a *necessary and sufficient condition for the multiplicative operation of a commutator lattice to be associative*.

Remark 2.3. From **L1** of the above definition, it is clear that $xy \leq x$ since $x \wedge y \leq x$ for all $x, y \in L$. Also from **L3** above, we conclude $x \leq y$ implies $xz \leq yz$ for all z in L since $x \leq y$ implies $x \vee y = y$, which by **L3** it implies $zx \vee zy = zy$.

Remark 2.4. Let 0 and 1 be the least and greatest elements of a commutator lattice L, respectively. Clearly $0 \leq 0 \cdot x \leq 0$ implies $0 \cdot x = 0$ for each $x \in L$. Now suppose L contains a multiplicative identity element u. Then $u = 1$ since $1 = u \cdot 1 \leq u \leq 1$, which implies $u = 1$. For example, $R \in L = \text{ideal}(R)$, the commutator lattice of ideals (under multiplication of ideals) of a commutative ring R with identity, is the largest and multiplicative identity of L (see also Example 2.7). Note that $1 \in L$ need not be the multiplicative identity of L in general. That is, $1 \cdot x$ need not be equal to $x \in L$ (e.g. $1 \cdot 1$ need not be equal to 1).

We now define a simple property of commutator lattices, which will be useful for constructing new examples of the commutator lattices.

Example 2.5. Let $L = L_1 \times L_2 \times \cdots \times L_n$ $(n \geq 2)$ where L_i is a (commutator) lattice for each $1 \leq i \leq n$. Clearly, L is a (commutator) lattice by defining its operations *(multiplication), order* $\leq$*, meet, and join*, componentwise, respectively.

Let us mention two obvious examples:

Example 2.6. An arbitrary lattice L becomes a commutator lattice if we put either

(a) $xy = x \wedge y$ for all $x, y \in L$, provided L is distributive, or

(b) $xy = 0$ for all $x, y \in L$.

As suggested by commutator theory, we might call these two kinds of commutator lattices *arithmetical* and *Abelian*, respectively.

We now write an example of a commutator lattice whose multiplicative identity and its largest element, according to Remark 2.4, must coincide.

Example 2.7. Let X be a nonempty set with power set $P(X)$. Clearly, $L = (P(X), \cap, \cup, \subseteq)$ is a commutator lattice with commutator operation $\cap$ with $\emptyset$ and X the least and greatest elements, respectively. It can easily be seen that X is also the multiplicative identity of L.

• We now state the definition of a Euclidean commutator lattice, which is similar to the definition of a Euclidean ring as defined in [6].

Definition 2.8. Let $\mathbb{N}$ be the set of nonnegative integers and L a commutator lattice. L is a Euclidean commutator lattice if there is a function $\phi : L \setminus \{0\} \to \mathbb{N}$ such that: (i) If $a, b \in L$ and $ab \neq 0$, then $\phi(a) \leq \phi(ab)$; (ii) if $a, b \in L$ and $b \neq 0$, then there exist $q, r \in L$ such that $a = qb \vee r$ with $r = 0$, or $r \neq 0$ and $\phi(r) < \phi(b)$.

We now write an example of a Euclidean commutator lattice.

Example 2.9. Let $L = \{0, a, b, 1\}$ with $0 < a < b < 1$, $ab = a \cdot 1 = a^2 = a$, $b \cdot 1 = b^2 = b$, $x \cdot 0 = 0$, $xy = yx$ for all $x, y \in L$, and $1 \cdot 1 = 1$. Let us define $xy = x \wedge y$ for all $x, y \in L$. With this definition, L is a commutator lattice. Define the Euclidean function $\phi : L \setminus \{0\} \to \mathbb{N}$ by $\phi(1) = 1$, $\phi(b) = 2$, and $\phi(a) = 3$. Now, it is not difficult to show that L is a Euclidean commutator lattice under ϕ. For instance, $b = aa \vee b$; or $b = a \cdot 1 \vee b$ satisfies the definition since $\phi(b) = 2 < 3 = \phi(a)$.

We now write the definition of a lower set and an ideal in a commutator lattice before stating the next theorem.

Definition 2.10. Let L be a commutator lattice. A nonempty subset I of L is said to be a lower set of L provided that $x \leq a \in I$ implies $x \in I$ for each $x \in L$. A lower set I of a lattice L is said to be an ideal if it is closed under finite joins. A nonempty subset D of L is said to be *multiplicatively absorptive* in L provided $rx \in D$ for any $r \in L$ and $x \in d$. An ideal I of L is said to be principal if there exists an element $a \in I$ such that for each $b \in I$, there exists an element $r \in L$ with $b = ra$. In this case, we say that I is generated by a and is denoted by $I = (a)$.

Remark 2.11. By the above definition, a lower set D in L is *multiplicatively absorptive* in L since $rx \leq x$ implies $rx \in D$ for any $x \in D$ and $r \in L$. Clearly, $0 \in D$ whenever D is a lower set of a lattice L since $0 \leq a$ for all $a \in L$ and D is nonempty by definition. Obviously, L is a lower set of itself and $L = D$ whenever $1 \in D$ and D is a lower set of L. Thus, we always assume that a lower set of L is a proper subset of L. Also, if D is a lower set in L and I is a lower set in D, then I is a lower set in L.

Theorem 2.12. *Every Euclidean commutator lattice L is a principal ideal lattice (i.e., every ideal in L is principal (see Definition 2.10)).*

Proof. If I is a nonzero ideal in L, choose $a \in I$ such that $\phi(a)$ is the least integer in the set of nonnegative integers

$$\{\phi(x) \mid x \neq 0; x \in I\}.$$

If $b \in I$, then $b = qa \vee r$ with $r = 0$ or $r \neq 0$ and $\phi(r) < \phi(a)$. Since $b \in I$ and $r \leq qa \vee r \in I$, then $r \in I$ by the lower set property of I. Since $\phi(r) < \phi(a)$ would contradict the choice of a, we must have $r = 0$, whence $b = qa$ and hence $I = (a)$ is principal in L by definition. □

Question: Is it possible, similar to the case of Euclidean rings and Euclidean semirings, to characterize some substructures with an almost division algorithm in Euclidean modules and Euclidean semimodules? For the notion of Euclidean modules and Euclidean semimodules as a natural extension of Euclidean rings and Euclidean semirings, see [14], [10], and [7].

3 Join Absorptive Subsets

In this section we investigate the concept and some algebraic properties of *join absorptive subsets* of order $t \geq 1$ (t a fixed integer) in a Euclidean commutator lattice L in a parallel mode with additively absorptive subsemirings in Euclidean semirings.

Definition 3.1. Let L be a Euclidean commutator lattice with Euclidean function ϕ, and assume $t \geq 1$ is a fixed integer. A lower set D of L is said to be a join absorptive (or simply, absorptive) of order t in L, if for each f in $L \setminus D$, there exist h in D and g in L such that $f = h \vee g$ and $1 \leq \phi(g) \leq t$.

The following is an example of a join absorptive set of order 2.

Example 3.2. Let L be the Euclidean commutator lattice with Euclidean function ϕ as defined in Example 2.9. Then $D = (0, a) \subseteq L$ is a join absorptive set of order $t = 2$ in L. That is, $b = a \vee b$ and $1 = a \vee 1$ with $\phi(b) = 2$ and $\phi(1) = 1$.

- By a proper lower set of L, we mean a proper subset of L which is also a lower set in L.

Theorem 3.3. *Let L be a Euclidian commutator lattice with Euclidian function ϕ, and let $t \geq 1$ be a fixed integer. Assume D_1 and D_2 are two proper lower sets of L with $D_1 \subseteq D_2$. Then the following results are true.*

(i) If D_2 is not absorptive of order t in L, then D_1 is not absorptive of order t in L. Equivalently, if D is an absorptive subset of order t in L, then any proper lower set of L containing D is also absorptive of order t in L.

(ii) let $D_1 \subseteq D_2 \subseteq \cdots$ be an ascending chain of subsets in L, then $\bigcup D_i$ is an absorptive subset of order t in L provided that $\bigcup D_i$ is properly contained in L and at least one of the factors in the chain is absorptive of order t in L.

(iii) Let $\{D_i\}$ be a family of proper subsets of L. If $\bigcap_i D_i$ is an absorptive subset of order t in L, then each factor of the intersection is absorptive of order t in L.

Proof. We just give a proof for Part (i) and leave the other two parts to the reader. Assume to the contrary that D_1 is absorptive of order t in L. Let $f \in L \setminus D_2$. Now, by the assumption, there exist $h \in D_1$ and $g \in L$ such that $f = h \vee g$ and $1 \leq \phi(g) \leq t$. Hence, we can conclude that D_2 is absorptive of order t in L, which is a contradiction. □

4 Main Results

In this section we consider the number of generators of an ideal in a join absorptive subset D of a Euclidean commutator lattice by applying the almost division algorithm on D (Theorem 4.4).

Theorem 4.1. *Let L be a Euclidean commutator lattice with Euclidean function ϕ satisfying the following conditions:*
(1) $\phi(fg) = \phi(f) + \phi(g)$ for all $f, g \in L \setminus \{0\}$ and $fg \neq 0$.
(2) For all $r, g \in L$, if $\phi(r) \leq \phi(g)$, then $\phi(r \vee g) = \phi(g)$.
Assume $t \geq 1$ is a fixed integer and D is an absorptive subset of order t in L. Then for any $f, g \in D$ with $g \neq 0$, there exist $q, r \in D$ such that $f = qg \vee r$ with $r = 0$ or $\phi(r) < \phi(g)$ or $\phi(r) = i + \phi(g)$ for some $1 \leq i \leq t$.

Proof. Let $f, g \in D$ with $g \neq 0$. Since L is Euclidean, then there exist $q, r \in L$ such that $f = qg \vee r$ with $r = 0$ or $\phi(r) < \phi(g)$. If $q \in D$, we are done since $r \in D$ by the fact that $r \leq gq \vee r \in D$ and D is a lower set in L. Now, suppose $q \notin D$, then by hypothesis, there exist $h \in D$ and $q' \in L$ such that $q = h \vee q'$ and $1 \leq \phi(q') \leq t$. Consequently, $f = hg \vee r \vee q'g$ and since $r \vee q'g \in D$, by the lower set property of D, it remains only to show that

$$1 + \phi(g) \leq \phi(r \vee q'g) \leq t + \phi(g).$$

Clearly, $1 \leq \phi(q') \leq t$ implies

$$1 + \phi(g) \leq \phi(q') + \phi(g) \leq t + \phi(g).$$

Thus, from Condition (1), we obtain $1 + \phi(g) \leq \phi(q'g) \leq t + \phi(g)$. Therefore, by applying Condition (2), it follows that

$$1 + \phi(g) \leq \phi(r \vee q'g) \leq t + \phi(g)$$

since $\phi(r) < \phi(g) \leq \phi(q'g)$. □

In the proof of the above theorem, if $r = 0$ and $q \notin D$, condition (2) is not required.

Corollary 4.2. *Let L be a Euclidean commutator lattice with Euclidean function ϕ satisfying the following conditions:*
(1) $\phi(fg) = \phi(f) + \phi(g)$ for all $f, g \in L \setminus \{0\}$ and $fg \neq 0$.
(2) For all $r, g \in L$, if $\phi(r) \leq \phi(g)$, then $\phi(r \vee g) = \phi(g)$.
Assume $t \geq 1$ is a fixed integer and D is an absorptive subset of order t in L. Then for any nonzero proper ideal I of D and any $f, g \in I$, with $g \neq 0$, there exist $q \in D$ and $r \in I$ such that $f = qg \vee r$ with $r = 0$, or $\phi(r) < \phi(g)$, or $\phi(r) = i + \phi(g)$ for some $1 \leq i \leq t$.

Proof. By the above theorem, there exist $q, r \in D$ such that $f = qg \vee r$ with $r = 0$, or $\phi(r) < \phi(g)$, or $\phi(r) = i + \phi(g)$ for some $1 \leq i \leq t$. Consequently, $r \leq qg \vee r \in I \subseteq D$ implies $r \in I$ since I is an ideal in D and hence a lower set in D by the assumption. □

• Note that I as an ideal in D is also an ideal in L since it is a lower set in L by the lower set property of D in L. Thus, by Theorem 2.12, it is a principal ideal in L, but in the next theorem, we shall show that it can not be a principal ideal in D in general.

We shall use the following lemma in the proof of the first part of the next theorem.

Lemma 4.3. *Let L be a Euclidean commutator lattice with Euclidean function ϕ satisfying the condition that for all $r, g \in L$, $\phi(r \vee g) = \phi(g)$ whenever $\phi(r) \leq \phi(g)$. Assume $t \geq 1$ is a fixed integer and D is an absorptive subset of order t in L. Then for each a in L, the condition $\phi(a) = 0$ implies that a must be in D.*

Proof. Suppose to the contrary that a is not in D. Since D is absorptive, then there exists $h \in D$ and $b \in L$ such that $a = h \vee b$ with $1 \leq \phi(b) \leq t$. Thus, $0 = \phi(a) = \phi(h \vee b) < \phi(b)$. Now, we compare $\phi(h)$ and $\phi(b)$ in three different possible cases and show that it will lead to a contradiction in either case. If $\phi(h) \leq \phi(b)$, then by hypothesis, we will have

$$1 \leq \phi(b) = \phi(h \vee b) \leq t$$

which is a contradiction. In other case, again by hypothesis, $\phi(h) > \phi(b)$ implies

$$0 = \phi(a) = \phi(h \vee b) = \phi(h)$$

which is impossible since $\phi(b)$ is strictly larger than zero. We therefore must conclude that if $\phi(a) = 0$, then a must belong to D. □

Theorem 4.4. *Let L be a Euclidean commutator lattice with Euclidean function ϕ satisfying the following conditions:*
(1) $\phi(fg) = \phi(f) + \phi(g)$ for all $f, g \in L \setminus \{0\}$ and $fg \neq 0$.
(2) For all $r, g \in L$, if $\phi(r) \leq \phi(g)$, then $\phi(r \vee g) = \phi(g)$.
Assume $t \geq 1$ is a fixed integer and D is an absorptive subset of order t in L. Let I be a nonzero proper ideal of D (i.e., I is a lower set in D, that is, $r \leq a \in I$ implies $r \in I$ for all $r \in D$) with

$$\phi(I) = \inf\{\phi(f) \mid f \in I\} = j.$$

Then we have the following results:
(i) The ideal I as an ideal of D can be generated by $t+1$ or fewer elements.
(ii) Assume further that for each nonzero $a \in L$, the condition $1 \leq \phi(a) \leq t$ implies $a \notin D$; and also I contains an element h such that $\phi(h) = i + j for some 1 \leq i \leq t$. Then I is not a principal ideal of D.

Proof. Part (i): choose an element g in I with $\phi(g) = j$. By Corollary 4.2, for each f in I, there exist $q \in D$ and $r \in I$ such that $f = qg \vee r$ with the following three possibilities: (a) $r = 0$, or (b) $\phi(r) < \phi(g)$, or (c) $\phi(r) = i + \phi(g)$ for some $1 \leq i \leq t$.
Now, the minimality of $\phi(g)$ disregards the possibility of the case $\phi(r) < \phi(g)$. Also, for those f's in I such that $r = 0$, it is clear that f belongs to (g). Indeed, if for each f in I, Case (a) occurs, then we can conclude that $(g) = I$.

Next, suppose for some element f in I, Case (c) occurs. This implies the existence of an element of I (namely, r in I) with ϕ value equals to $i + \phi(g)$ for some $1 \leq i \leq t$ (see Corollary 4.2). Thus, $j < \phi(r) \leq t + j$. Therefore, it is clear that the set

$$C = \{m \in \mathbb{N} \mid I \text{contains elements that each of which having } \phi \text{value } m \; with \; j < m \leq t + j\}$$

is not empty. Assume that the cardinality of C is $|C| = k$. Now, label the elements of C as m_1, $m_2, \cdots$, and m_k, where $m_1 > m_2 > \cdots > m_k$. By construction of C, we can choose k elements f_{m_1}, $f_{m_2}, \cdots$, and f_{m_k} from I with $\phi(f_{m_i}) = m_i$, where $i = 1, 2, \cdots, k$. Thus, it is clear that $\phi(r) = \phi(f_{m_{i_1}})$ for some $1 \leq i_1 \leq k$. Since L is Euclidean, then there exist $a_{i_1}, r_1 \in L$ such that $r = a_{i_1} f_{m_{i_1}} \vee r_1$ with $r_1 = 0$ or $\phi(r_1) < \phi(f_{m_{i_1}})$. Clearly, $r_1 \in I$ by the lower set property of D in L and lower set property of I in D and the fact that

$$r_1 \leq a_{i_1} f_{m_{i_1}} \vee r_1 \in I \subseteq D.$$

Assume that $r_1 = 0$. Then $r = a_{i_1} f_{m_{i_1}}$, and by hypothesis, $\phi(r) = \phi(a_{i_1}) + \phi(f_{m_{i_1}})$ implies that $\phi(a_{i_1}) = 0$ ($\mathbb{N}$ is additively cancellative). Therefore, by Lemma 4.3 (above), $a_{i_1} \in D$. Hence, we obtain

$$f = qg \vee r = qg \vee a_{i_1} f_{m_{i_1}} \in (g, f_{m_{i_1}}).$$

Now suppose $r_1 \neq 0$ and $\phi(r_1) < \phi(f_{m_{i_1}})$. Thus, we have

$$\phi(r_1) < \phi(f_{m_{i_1}}) \leq \phi(a_{i_1}) + \phi(f_{m_{i_1}}) = \phi(a_{i_1} f_{m_{i_1}}),$$

which by hypothesis, it implies

$$\phi(a_{i_1} f_{m_{i_1}} \vee r_1) = \phi(a_{i_1} f_{m_{i_1}}).$$

Now we have

$$\phi(r) = \phi(a_{i_1} f_{m_{i_1}} \vee r_1) = \phi(a_{i_1} f_{m_{i_1}}) = \phi(a_{i_1}) + \phi(f_{m_{i_1}})$$

which implies $\phi(a_{i_1}) = 0$. Thus, again by applying Lemma 4.3, we get $a_{i_1} \in D$. In this case, it is clear that $\phi(r_1) = \phi(f_{m_{i_2}})$ for some $i_1 < i_2 \leq k$. Again, by the division algorithm, there exist $a_{i_2}, r_2 \in L$ such that $r_1 = a_{i_2} f_{m_{i_2}} \vee r_2$ with $r_2 = 0$ or $\phi(r_2) < \phi(f_{m_{i_2}})$. Again $a_2 \in I$, by the similar argument that discussed for the case $a_1 \in I$. Now, by the same argument as we showed $a_{i_1} \in D$, it can be shown that (in either case of $r_2 = 0$ or $\phi(r_2) < \phi(f_{m_{i_2}})$ that $a_{i_2} \in D$. Consequently, whenever $r_2 = 0$, we have

$$f = qg \vee r = qg \vee a_{i_1} f_{m_{i_1}} \vee r_1 = qg \vee a_{i_1} f_{m_{i_1}} \vee a_{i_2} f_{m_{i_2}}$$

which clearly belongs to the ideal $(g, f_{m_{i_1}}, f_{m_{i_2}})$. Obviously, by continuing the process of argument as above, we get the elements r, r_1, r_2, $\cdots$ of I with

$$j + t \geq \phi(r) > \phi(r_1) > \phi(r_2) > \cdots \geq j.$$

Thus, we reach an element $r_s \in I$ with $\phi(r_s) = \phi(g)$ and $r_{s+1} = 0$. Actually, by the division algorithm in L, there exist $q', r_{s+1} \in L$ such that $r_s = q'g \vee r_{s+1}$ with $r_{s+1} = 0$ or $\phi(r_{s+1}) < \phi(g)$. But the lower set property of D in L and lower set property of I in D implies that $r_{s+1} \in I$, and therefore, the minimality of $\phi(g)$ excludes the choice of $\phi(r_{s+1}) < \phi(g)$. Hence $r_s = q'g$ and $\phi(r_s) = \phi(q') + \phi(g)$, which implies $\phi(q') = 0$. Now, by Lemma, 4.3, $q' \in D$. Thus,

$$f = qg \vee a_{i_1} f_{m_{i_1}} \vee a_{i_2} f_{m_{i_2}} \vee \cdots \vee q'g$$

belongs to

$$(g, f_{m_1}, f_{m_2}, \ldots, f_{m_k}),$$

which proves that

$$I = (g, f_{m_1}, f_{m_2}, \ldots, f_{m_k}),$$

where $m_1, m_2, \cdots, m_k \in C$. This completes the proof of Part (i).

Finally, for the proof of Part (ii), assume that for each nonzero element a in L, the condition $1 \leq \phi(a) \leq t$ implies $a \notin D$, and also I contains an element h with $\phi(h) = i + \phi(g)$ for some $1 \leq i \leq t$. In this case, suppose to the contrary that I is principal in D and $I = (g)$ with $\phi(I) = \phi(g) = j$. Hence, $h = qg$ for some q in D. Therefore, $\phi(h) = i + j = \phi(q) + j$ implies $1 \leq \phi(q) = i \leq t$. Thus, by the assumption, this makes $q \notin D$, which is a contradiction. From this, we can conclude that $h \notin (g)$, i.e., I is not contained in (g). In other words, no elements of I with ϕ value equals to j can generate I in D. Now, suppose there exists an element g' in I which generates I in D. From the above argument and minimality of $j = \phi(g)$, we must have $\phi(g') > \phi(g)$. Since $g \in I$, then $g = q'g'$ for some q' in D, and hence, $\phi(g) = \phi(q') + \phi(g') \geq \phi(g')$. This is a contradiction and the proof is complete. □

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