Palestine Journal of Mathematics

Vol. 11(1)(2022) , 251–259

Symmetric and Non-symmetric Mixed Games: Games with a Combination of Chance and no Chance

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 91A46; Secondary , 91A05, 68T01.

Keywords and phrases:Mixed impartial combinatorial game, mixed nim, mixed non-combinatorial tabletop game, mixed dice-rolled (roll-and-move) board game.

This research was in part supported by grant no. 991300117 from IPM.

Abstract We introduce and study the notion of a class of games which are the combination of the elements of chance (e.g., rolling one or two dice) and no-chance (e.g., choosing one or two numbers) by the players alternatively. Explicitly, we provide and discuss a method that converts a no-chance-based game such as an impartial combinatorial game into a mixed game and a chance-based game such as backgammon into a mixed game. In a nutshell, we show how to invent many new games associated to impartial combinatorial and non-combinatorial (standard) tabletop board (dice-rolled) games that besides their recreational values for different age ranges, they open a new window and provide a solid platform for academic studies in the fields of mathematics, computer science, artificial intelligence, and game theory. Finally, we end the paper with a short philosophical speculation and exposition related to the extended mixed games and quantum mechanics.

1 Introduction and Preliminaries

The main purpose of this paper is to introduce and study the notion of *symmetric and non-symmetric mixed games* that are a class of games which are the *combination* of the elements of *chance* (e.g., rolling one or two dice) and *no-chance* (e.g., choosing one or two numbers) by the players alternatively.

$*$ In this section, we define the notion of a mixed game in general (Definition 1.2) and devote the rest of the section on a brief introduction to *Combinatorial Game Theory* with some (introductory) references; and finally end the section by recalling the definition of a *combinatorial game*. In Section 2, we introduce a *constructive method* to convert an impartial combinatorial game (which is in the class of no-chance games), such as Nim, to a new game (non-combinatorial game) by adding the element of chance to it (e.g., rolling one or two dice). Conversely, in Section 3, we *construct new games* from those *tabletop board games* that (mainly) depend on luck and chance, such as *backgammon, snakes and ladders, dice chess, and bingo* by adding the element of choice (e.g., choosing the numbers) in every other turn by the players (specially, dice-rolled (roll-and-move) tabletop board games are *good candidates* to be extended to mixed games). Thus, we can claim that mixed games of these types are (somewhat) a *generalization of (impartial) combinatorial games* such as Nim; and some tabletop non-combinatorial games or chance-based games such as backgammon, snakes and ladders, dice chess, and bingo (Remarks 2.2 and 3.3). Finally, in the third section, we end the paper with a short philosophical speculation and exposition regarding the *extended mixed games* in the real world.

Remark 1.1. For the chance part of a mixed game, which is defined on a no-chance-based game, we can use a bag of, for instance, numbered or colored marbles (e.g., small hard balls with desired numbers or colors on them) or using a (digital or mechanical) device to show the numbers randomly, or using a plain die (a die with no numbers or marks on its faces), and mark the desired numbers or colors on its faces, which depends on the structural nature and related rules, regulations, and conventions of the game. Also, different types of dice with n faces, where

$n = 4, 6, 8, 10, 12$, and 20, can be used for the chance part of a mixed game, which depends on the structure of the game (see wikipedia under “dice" for different types of dice).

Definition 1.2. A mixed game is a game whose (well-defined) playing rules and regulations depending on a combination of chance (random) and no-chance (free will) with usually two players and a well-defined clear objective (goal of the game) for the players related to winning and losing; or a draw.

We now write a brief introduction for *Combinatorial Game Theory* with some references; and end the section by recalling the notion of a combinatorial game.

• Combinatorial game theory is a fascinating and rich theory, based on a simple and intuitive recursive definition of games, which yields a very rich algebraic structure: games can be added and subtracted in a very natural way, forming an abelian theory is deeply satisfying from a theoretical point of view, and at the same time it has useful applications to specific games such as Go [3]. The theory was founded by John H. Conway in the 1970’s. Classical references are the wonderful books On Numbers and Games [4] by Conway, and Winning Ways by Berlekamp, Conway and Guy [2]. The reference [2] is a most beautiful book bursting with examples and results but with less stress on mathematical rigor and exactness of some statements. The reference [4] is still the definitive source of the theory, but rather difficult to read for novices. We recommend looking at [2], [7], or [5] to pick up the playful spirit of the theory.

• Combinatorial games are *two-person games with perfect information and no chance moves*, and with a *win-or-lose* outcome. Such a game is determined by a set of positions, including an initial position, and the player whose turn it is to move. Play moves from one position to another, with the players usually alternating moves, until a *terminal position* is reached. A *terminal position is one from which no moves are possible*. Then one of the players is declared the winner and the other the loser.

As mentioned above, there are two main references for the material on combinatorial games. One is the research book, On Numbers and Games by J. H. Conway [4]. This book introduced many of the basic ideas of the subject and led to a rapid growth of the area that continues today. The other reference, more appropriate for beginners (undergraduate students) is the book, Winning Ways for your mathematical plays by Berlekamp, Conway and Guy [2]. There are many interesting games described in this book and much of it is accessible to the undergraduate mathematics student.

This theory may be divided into two parts, *impartial games* in which the set of moves available from any given position is the same for both players, and *partizan games* in which each player has a different set of possible moves from a given position. Games like *chess or checkers* in which one player moves the white pieces and the other moves the black pieces are partizan. An elementary introduction to *impartial combinatorial games* is given in the book Fair Game by Richard K. Guy [6].

$*$ We now end the section by recalling the notion of a combinatorial game. It is a game that satisfies the following conditions.

(1) There are two players.

(2) There is a set, usually finite, of possible positions of the game.

(3) The rules of the game specify for both players and each position which moves to other positions are legal moves. If the rules make no distinction between the players, that is if both players have the same options of moving from each position, the game is called *impartial*; otherwise, the game is called *partizan*.

(4) The players alternate moving.

(5) The game ends when a position is reached from which no moves are possible for the player whose turn it is to move. Under the *normal play rule*, the last player to move wins. Under the misere play rule the last player to move loses.

If the game never ends, it is declared a draw. However, we shall nearly always add the following condition, called the ending condition. This eliminates the possibility of a draw.

(6) The game ends in a finite number of moves no matter how it is played.

It is important to note what is *omitted* in this definition. No *random moves* such as *the rolling of dice or the dealing of cards* are allowed. This *rules out* games like *backgammon and poker*. A combinatorial game is a game of *perfect information: simultaneous moves and hidden moves* are not allowed.

2 Some Examples of Mixed Impartial Combinatorial Games

This section is devoted to some *symmetric mixed games of order one* that are defined on impartial combinatorial games such as nim (e.g., subtraction, dynamic subtraction, and subtract-a-square games) and in addition, we define the *symmetric and non-symmetric mixed games of higher orders* related to this type of games (Remark 2.2). Moreover, we end the section by posing a research project related to defining new mixed combinatorial impartial and partizan games. In contrast to this, the next section is devoted to some examples of mixed games that are constructed from non-combinatorial tabletop board games (i.e., games depending on elements of chance and randomness such as backgammon). We always, by convention, *begin any mixed game with the chance part* (e.g., rolling a die in the mixed subtraction game).

Remark 2.1. For the chance part of the following examples of mixed games (in this section), we use only a *numeral regular six-sided die* $(1-6)$. Of course, it is possible to be creative and provide different settings for the chance part of any mixed impartial game rather than a regular six-sided die (see Remark 1.1). Thus, for each game of nim (e.g., subtraction game), we can define (construct) two or more different mixed games. Moreover, in the following examples of mixed games, we can also use *two regular six-sided dice* and move according to one of the numbers showing on the two dice optionally (if possible).

$*$ We next define the *symmetric mixed game of nim* of order one and assume that the reader is familiar with the rules of the game of nim. The most famous *take-away game* is the game of Nim, played as follows. There are $k \geq 2$ (a fixed integer) piles (heaps) of chips (objects) containing $x_1, x_2, \ldots, x_k$ chips (objects) respectively. Two players take turns moving. Each move consists of selecting one of the piles (heaps) and removing chips (objects) from it. You may not remove chips (objects) from more than one pile (heap) in one turn, but from the pile (heap) you selected you may remove as many chips (objects) as desired, from one chip (object) to the whole pile (heap). The winner is the player who removes the last chip(s) (object(s)).

Mixed Game of Nim of Order One: In this game two players Alice and Bob are playing the game of nim on $k \geq 2$ (a fixed integer) piles (heaps) of chips (objects) containing $x_1, x_2, \ldots, x_k$ chips (objects), respectively, with rolling one six-sided die (i.e., *chance part*) and selecting one of the piles (heaps) and removing chips (objects) from it optionally (i.e., *no-chance part*) in every other turn, respectively, as follows. Suppose that Alice is the player who should start the game. In this case, she rolls the die and plays according to the number on the die (i.e., takes some chips from the optionally selected pile according to the number showing on the die). Now, similarly, Bob in his turn rolls the die and takes some chips from the optionally selected pile according to the number showing on the die. Next it is the no-chance part for both players. In this case, Alice chooses a pile and takes any desired number of chips from that pile. After this, Bob in his turn should choose a pile and takes any desired number of chips from the chosen pile. The game continues this way by *alternating turns* between Alice and Bob until end of the game (i.e., the player who can not move legally any chip(s) from the pile(s) on his/her turn is the loser).

$*$ We next define the symmetric mixed subtraction game of order one and assume that the reader is familiar with the rules of the subtraction game. Moreover, for a general reference and rules of this game, see wikipedia under the “game of nim" and we briefly describe it as follows.

• Let us now consider a class of combinatorial games that is a special case of nim. Let $S = \{1, 2, \ldots, k\}$ be a set of positive integers with $k \geq 1$ a fixed integer. The subtraction game with subtraction set S is played as follows. From a pile (heap) with a large number, say n much larger than k, of chips (objects), two players alternate moves. A move consists of removing s chips (objects) from the pile (heap), where s belongs to S. Last player to move wins.

Mixed Subtraction Game of Order One: Let $S = \{1, 2, \ldots, k\}$ be the subtraction set of the game with $k \geq 1$ a fixed integer. In this game two players Alice and Bob are playing the (standard) subtraction game on a pile (heap) of chips (objects) of size much larger than k with rolling one six-sided die (i.e., *chance part*) and choosing one number optionally from the set S (i.e., *no-chance part*) in every other turn, respectively, as follows. Suppose that Alice is the player who should start the game. In this case, she rolls the die and plays according to the number on the die (i.e., takes some chips from the pile according to the number showing on the die). Now, similarly, Bob in his turn rolls the die and takes some chips from the pile according to the number showing on the die. Next it is the no-chance part for both players. In this case, Alice chooses and announces one number s from the set S and takes s chips from the pile. After this, Bob in his turn should choose and announce one number s from the set S and takes s chips from the pile accordingly. The game continues this way by *alternating turns* between Alice and Bob until end of the game (i.e., the player who can not move legally any chips from the pile on his/her turn is the loser).

$*$ We next define the symmetric *mixed dynamic subtraction game* of order one. One can enlarge the class of subtraction games by letting the subtraction set depend on the last move of the opponent. There is one pile of n chips. The first player to move may remove as many chips as desired, at least one chip but not the whole pile. Thereafter, the players alternate moving, each player not being allowed to remove more chips than his opponent took on the previous move.

Mixed Dynamic Subtraction Game of Order One: In this game two players Alice and Bob are playing the dynamic subtraction game on a pile (heap) of chips (objects) with rolling one six-sided die (i.e., *chance part*) and removing as many chips as desired from the pile (optionally) with some restriction, at least one chip but not the whole pile (i.e., *no-chance part*) in every other turn, respectively, as follows. Suppose that Alice is the player who should start the game. In this case, she rolls the die and plays according to the number showing on the die (i.e., takes some chips from the pile according to the number showing on the die). Now, similarly, Bob in his turn rolls the die and takes some chips from the pile according to the number showing on the die. Next it is the no-chance part for both players. In this case, Alice removing as many chips as desired from the pile optionally, at least one chip but not the whole pile. After this, Bob in his turn may remove as many chips as desired from the pile, but not being allowed to remove more chips than his opponent (i.e., Alice in this case) that took on the previous move of chips from the pile. Now after the chance part (rolling the die) for each player, Alice again removing as many chips as desired from the pile optionally, at least one chip but not the whole pile. After this, Bob again in his turn may remove as many chips as desired from the pile, but not being allowed to remove more chips than his opponent (i.e., Alice in this case) that took on the previous move of chips from the pile. The game continues this way by *alternating turns* between Alice and Bob until end of the game (i.e., the player who can not move legally any chips from the pile on his/her turn is the loser).

$*$ We next define the *mixed subtract-a-square* game. Subtract-a-square (also referred to as *take-a-square*) is a two-player mathematical subtraction game. It is played by two people with a pile of coins (or other tokens) between them. The players take turns removing coins from the pile, always removing a nonzero square number of coins. The game is usually played as a normal play game, which means that the player who removes the last coin wins.

Mixed Subtract-a-square Game of Order One: In this game two players Alice and Bob are playing the (standard) subtract-a-square game on a pile (heap) of chips (objects) with rolling one six-sided die (i.e., *chance part*) and removing a nonzero square number of chips from the pile optionally (i.e., *no-chance part*) in every other turn, respectively, as follows. Suppose that Alice is the player who should start the game. In this case, she rolls the die and plays according to the number showing on the die (i.e., takes some chips from the pile according to the number showing on the die). Now, similarly, Bob in his turn rolls the die and takes some chips from the pile according to the number showing on the die. Next it is the no-chance part for both players. In this case, Alice removing a non-zero square number of chips. After this, Bob in his turn should remove a non-zero square number of chips from the pile. The game continues this way by *alternating turns* between Alice and Bob until end of the game (i.e., the player who can not move legally any chips from the pile on his/her turn is the loser).

• Note that, in the above mixed games, the situation at the very end of the game when the number of chips (objects) are strictly less than six could be a losing point for the player who turns the die. That is, for instance, suppose that four chips are left and die shows five or six, then the player with one of these numbers is the loser since obviously he/she has no legal move at this turn. On the other hand, the player is the winner if the number on the die is 4 since he/she can remove all 4 chips from the pile.

Remark 2.2. All the above four examples are called symmetric mixed games of order one since the *frequency (number of successive repetitions) of the chance part and no-chance part* are equal to one. Generally speaking, a mixed game of order (m, n) is called symmetric [resp. non-symmetric] when $m = n$ [resp. $m \neq n$], where $m, n \geq 1$ are positive fixed integers with m the frequency of the chance part and n the frequency of the no-chance part. Thus, any impartial combinatorial game whose generalization to a mixed game is similar to one of the above four typical games, can be considered as a mixed game of order $(0, n)$ with $n \geq 1$ large enough till end of the game. In addition, symmetric and specially non-symmetric mixed games of higher orders (different from $(1, 1)$) might be challenging and having recreational and (specially) academic research values, which of course, depending on the structure, rules, and regulations of the (original) game.

We now construct a new fully chance-based game from the game of nim and call it “dice-nim".

Remark 2.3. We can convert an impartial combinatorial game such as nim to a fully chance-based game by using one regular six-sided die and move the objects from a selected pile according to the number showing on the die at each turn. We call this game “dice-nim" and note that dice-nim can be regarded as a mixed game of order $(m, 0)$ for m large enough till end of the game (see also Remarks 1.1 and 2.1 for other new constructions). This game, similar to the game of nim, will end under the normal play rule. That is, the player who takes the last legal move is the winner.

Remark 2.4. Obviously, similar to the construction of the above symmetric mixed games (e.g., mixed subtraction game) of order one from the conventional nim games (e.g., standard subtraction game), it is possible to define new symmetric and non-symmetric mixed (e.g., mixed subtraction game) games of order one or higher orders as defined in Remark 2.2.

We end this section by posing a research project related to defining some new mixed combinatorial games.

Project: Besides some recreational values, defining new symmetric and non-symmetric mixed games on some impartial combinatorial games (e.g., Graph Games: Games Played on (Directed) Graphs) and partizan combinatorial games (e.g., Partizan Poset Games) could be challenging and having (profound) academic research values (see Remarks 1.1 and 2.1).

3 Four Examples of Mixed Non-combinatorial Tabletop Games

In this section we provide some examples of symmetric mixed games of order one that are constructed from non-combinatorial tabletop board games (i.e., games depending on element of chance) such as backgammon, snakes and ladders, dice chess, and bingo; and in addition, define the *symmetric and non-symmetric mixed games of higher orders* related to this type of games (Remark 3.3). Moreover, we end the section (paper) with a short philosophical speculation and exposition regarding the *extended mixed games* in real world (e.g., quantum mechanics). We always, by convention, *begin any mixed game with the chance part* (e.g., rolling the dice in the mixed backgammon case).

∗ We start with the symmetric mixed backgammon of order one and assume that *the reader is familiar with the rules of the standard backgammon*. Moreover, for the *rules of the standard backgammon*, see, for example, http://www.bkgm.com and other backgammon-related articles and information are available at Backgammon Galore (also, see [9]). In addition, for a generalization of the standard backgammon, see the work of the author on *hyper dice backgammon of finite size* [8].

We assume that W is the player with *white checkers* and B the player having the *black checkers*. Also, the player "*who first starts the game*" depends on a *convention* between the players, which can be determined by, for instance, tossing a coin or rolling a die.

Mixed Backgammon of Order One: In this game two players are playing the standard backgammon with rolling two six-sided dice (i.e., *chance part*) and choosing two different numbers optionally from 1 to 6 (i.e., *no-chance part*) in every other turn, respectively, as follows. Suppose that W is the player who should start the game. In this case, W rolls the dice and plays according to the numbers on the dice. Now, similarly, B in his/her turn rolls the dice and moves the black checkers according to the numbers showing on the dice. Next it is the no-chance part for both players. In this case, W chooses and announces two different numbers from 1 to 6 and moves the white checkers accordingly. After this, B in his/her turn should choose and announce two different numbers from 1 to 6 and moves the black checkers accordingly. The game continues this way by *alternating turns* between W and B until end of the game according to the rules of the standard backgammon.

Remark 3.1. Obviously, similar to the above construction of the symmetric mixed backgammon of order one from the standard backgammon, it is possible to define many new mixed games by *generalizing a backgammon variant to symmetric and non-symmetric mixed backgammon variant of order one or higher orders* (see Remark 3.3 for the definition of symmetric and non-symmetric mixed games of higher orders than one). For many sources related to different *backgammon variants*, see wikipedia under the "backgammon variants".

∗ We next define the symmetric mixed snakes and ladders of order one and assume that the reader is familiar with the rules of the game of snakes and ladders. Moreover, for a general reference and rules of this game, see wikipedia under the "snakes and ladders" and we briefly describe it as follows.

• Snakes and Ladders, known originally as Moksha Patam, is an ancient Indian board game for two or more players regarded today as a worldwide classic. It is played on a *game board with numbered, gridded squares*. A number of *ladders and snakes* are pictured on the board, each connecting two specific board squares. The object of the game is to navigate one's game piece, according to die rolls, from the start (bottom square) to the finish (top square), helped by climbing ladders but hindered by falling down snakes. The game is a simple race based on sheer luck, and it is popular with young children. The historic version had its roots in morality lessons, on which a player's progression up the board represented a life journey complicated by virtues (ladders) and vices (snakes). The game is also sold under other names such as Chutes and Ladders, Bible Ups and Downs, etc., some with a morality motif.

The size of the grid varies, but is most commonly 8×8, 10×10, or 12×12 squares. Boards have snakes and ladders starting and ending on different squares; both factors affect the duration of play. Each player is represented by a *distinct game piece token*. A single die is rolled to determine random movement of a player's token in the traditional form of play.

Mixed Snakes and Ladders of Order One: In this game two players are playing snakes and ladders with rolling one six-sided die (i.e., *chance part*) and choosing one number optionally from 1 to 6 (i.e., *no-chance part*) in every other turn, respectively; and each player moves his/her token according to the random or chosen number. Note that, in contrast to the original game of snakes and ladders, the mixed version of this game shows that the phenomena in the

real life are not merely based on chance and luck, but thinking (and skills) could be a significant factor to some extend.
$*$ We next define the symmetric *mixed dice chess* of order one and assume that the reader is familiar with basic notion of *the standard (or orthodox) chess rules*. For a general reference of chess, a list of chess variants, and (specially) *dice chess*, see wikipedia under the “chess", “list of chess variants", and “dice chess", respectively.

• Dice chess can refer to a number of *chess variants* in which dice are used to alter gameplay; specifically that the moves available to each player are determined by rolling a pair of ordinary six-sided dice. There are many different variations of this form of dice chess and there is no standard set of rules for Dice Chess in general. One of them is described here and we will extend it to a mixed game of order one in the following example.

The players alternate rolling the two dice and, if possible, moving. On each of the dice, the one represents a pawn, two a knight, three a bishop, four a rook, five a queen, and six a king. The player may move either of the pieces indicated on the two dice. For example, a player rolling a one and a two may move either a pawn or a knight. A player who rolls doubles (the same number on both dice) may play any legal move.

Mixed Dice Chess of Order One: In this game two players are playing the dice chess (as defined above) with rolling two six-sided dice for their *chance part*; and moving any desired chess piece optionally according to the standard chess rules for their *no-chance part* in every other turn, respectively. Suppose that W is the player who should start the game. In this case, W rolls the dice and plays (moves a white piece) according to one of the numbers on the dice, if possible. Now, similarly, B in his/her turn rolls the dice and moves a black piece according to one of the numbers showing on the dice, if possible. Note that a player who rolls doubles (the same number on both dice) may play any legal move with any of the chess pieces, if possible. Next, it is the no-chance part for both players. In this case, W moves any desired white chess piece optionally according to the standard chess rules, if possible. After this, B in his/her turn should move any desired black chess piece optionally according to the standard chess rules, if possible. The game continues this way by *alternating turns* between W and B until end of the game according to the standard rules of chess. That is, the player should pass his/her turn to the opponent if there is no legal move for him/her according to the two different numbers on the dice and game continues this way until end of the game according to the standard rules of chess.

Remark 3.2. There are many variations (different types) of dice chess, as mentioned in the paragraph preceding the above example; therefore, we can define many new mixed dice chess games of order one or higher orders as well (see Remark 3.3 for the definition of symmetric and non-symmetric mixed games of higher orders than one).

$*$ We next define the symmetric *mixed bingo* of order one and assume that the reader is familiar with basic notion of *the game of bingo*. Bingo is a game of chance in which each player has one or more cards printed with differently numbered squares on which to place markers when the respective numbers are drawn and announced by a caller. The first player to mark a complete row of numbers is the winner.

Mixed Bingo of Order One: In this game two players are playing bingo with marking the numbered squares according to the announced called number (i.e., *chance part*) and marking a square on their cards optionally (i.e., *no-chance part*) in every other turn, respectively, until the end of the game. Note that similar to other symmetric and non-symmetric mixed games, mixed bingo can also be defined for higher orders such as 2, $(2,1)$, and $(1,2)$ as well (see Remark 3.3 for the definition of symmetric and non-symmetric mixed games of higher orders than one).

Remark 3.3. All the above four examples are called symmetric mixed games of order one since the *frequency (number of successive repetitions) of the chance part and no-chance part* are equal to one. Generally speaking, a mixed game of order (m,n) is called symmetric [resp. non-symmetric] when $m = n$ [resp. $m \neq n$], where $m, n \geq 1$ are positive fixed integers with m the frequency of the chance part and n the frequency of the no-chance part. For instance, a (symmetric [resp. non-symmetric]) mixed backgammon of order 2 [resp. $(2,1)$; or $(1,2)$] is played according to the sequence $2, 2, 2, \ldots$ (i.e., 2 times rolling the dice, 2 times choosing the numbers, ...) [resp. $2, 1, 2, 1, \ldots$ (i.e., 2 times rolling the dice, 1 time choosing the numbers, ...); or $1, 2, 1, 2, \ldots$ (i.e., 1 time rolling the dice, 2 times choosing the numbers, ...)]. Furthermore, any tabletop board game whose generalization to a mixed game is similar to one of the above four typical games, can be considered as a mixed game of order $(m, 0)$ with $m \geq 1$ large enough till end of the game. In addition, symmetric and specially non-symmetric mixed games of higher orders (different from $(1, 1)$) might be challenging and having recreational and (specially) academic research values, which of course, depending on the structure, rules, and regulations of the (original) game.

• Finally, we end the paper with a short philosophical speculation and exposition regarding the *extended mixed games* in real world. The mixed game G, as defined in 1.2, is called an extended mixed game provided that G has *two players and two opposite objectives*, where each of which belongs to one of the players. For instance, in the game of discovering the secrets and laws of the nature, man is one of the players and nature is the other player with a flexible mask on her face. In this game, one of the objective is that man is trying to remove the mask from the face of the nature as much as possible, but nature, in opposite direction, is trying to resists and keeps her face covered as much as possible. Thus, the notion of *winning and losing* in an extended mixed game is a *mixed, fuzzy, and probabilistic* concept; not absolute.

The famous word of Albert Einstein related to the denial of principles of quantum mechanics (e.g., Heisenberg's uncertainty relation) is that "God does not play dice". He is partially right and partially wrong since God does not merely play dice. Actually, God plays dice and no dice for us, but not for himself. That is, God merely plays a (simultaneous) extended mixed game in micro world and at the atomic scales for us.

Heisenberg's uncertainty principle states that measuring the momentum and position of a particle (e.g., an electron) with an arbitrarily precision is impossible and vice versa. That is, When we are measuring the momentum of a particle with a good approximation (i.e., high accuracy) and on the other hand we can not increase our accuracy for measuring the position of the same particle as much as desired; and vice versa. More precisely, pairs of non-commuting operators cannot give rise to simultaneous measurements arbitrarily precise for the associated quantities (this is usually called *Heisenberg's uncertainty principle*). Thus, this could be regarded as an extended mixed game, which is a combination of no-chance and chance (probabilistic measurement) with two players, where one is the man (observer) and the other player is the nature (e.g., electron). Indeed, the universe and real life is full of (simultaneous) extended mixed games, where (sometimes) the "chance" part or "no-chance" part is hidden or is very small that could be neglected. For a reference regarding an Introduction to Particle Physics, see [1] or any standard book in Particle Physics.

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Received: September 04, 2020.
Accepted: November 29, 2020