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# A CLASSIFICATION AND APPLICATIONS OF PRIMITIVE IDEMPOTENT ELEMENTS OF COMPLEX CLIFFORD ALGEBRAS $\mathbb{C}l(p,q)$

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**Abstract** Using its semi simplicity and its matrix representation, we will give a classification of primitive idempotent elements of the complex Clifford algebra $\mathbb{C}l(p,q)$ which we use to make some examples of Rota-Baxter operators on $\mathbb{C}l(p,q)$. A study of Rota-Baxter operators on $\mathbb{C}l(p,q)$ is given.

## 1 Introduction

The notion of Clifford algebra was invented by William Kingdon Clifford (1845-1879). The first occurrence of the result was issued in a talk in 1876, which was published posthumously in 1882. In mathematics, a Clifford algebra is an algebra generated by a vector space with a quadratic form, and it is a unital associative algebra. As $F$-algebras, they generalize the real numbers, complex numbers, quaternions and several other hypercomplex number systems [6]. The theory of Clifford algebras is intimately connected with the theory of quadratic forms and orthogonal transformations. It is well-known that Clifford algebras are defined by symmetric bilinear forms [1, 12, 5]. Given a symmetric bilinear form $B$ on a vector space $E$, one defines the Clifford algebra $Cl(E,B)$ to be the associative algebra generated by the elements of $E$ , with relations $vu + uv = 2B(u,v),\ u,v \in E$. If $B = 0$ this is just the exterior algebra $\wedge(E)$, see for example [4, 5, 14, 10], and in the general case the Clifford algebra can be regarded as a deformation of the exterior algebra. The structures of the finite dimensional Clifford algebras associated with non-degenerate quadratic forms have been well understood for a long period of time. These Clifford algebras are either full matrix algebras or the direct sums of two full matrix algebras [10, 14, 9, 16].

Clifford algebras have played an important role in a variety of fields including geometry, describing electron spin, and the fundamental representations of the orthogonal groups etc. We refer the readers to the introduction section of [18] for a discussion of the role played by Clifford algebras in quantum mechanics. Clifford algebras and spinors have been used to describe electromagnetic fields (Dirac- equation [8]), super symmetry, and celestial mechanics.

Given an algebra $A$ and a scalar $\lambda$ in a field $F$, a linear operator $R : A \longrightarrow A$ is called a Rota-Baxter operator (RB operator, shortly) on $A$ of weight $\lambda$ if the following identity

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy)$$

holds for all $x, y \in A$. The algebra $A$ is called the Rota-Baxter algebra (RB algebra). The Rota-Baxter algebras were introduced by Baxter [3], and then they were popularized by Rota and his school [2]. The linear operators with the previous property were independently introduced in the context of Lie algebras by Belavin and Drinfeld [2]. These operators were connected with the so-called R-matrices which are solutions to the classical Yang-Baxter equation. Recently, some applications of the Rota-Baxter algebras were found in such areas as the quantum field theory, the Yang-Baxter equations, the cross products, the operads, the Hopf algebras, the combinatorics, and the number theory.

The present paper is organized as follows: After introduction, section 2 aims to revisit some basic results of semi-simple algebras. In section 3, we will give a classification of primitive idempotent elements of $\mathbb{C}l(p,q)$ using its semi-simplicity. Also, examples are given. Further, in section 4, we study Rota-Baxter operators on $\mathbb{C}l(p,q)$. Some examples, using primitive idempotent elements of $\mathbb{C}l(p,q)$, are given.

## 2 Primitive idempotent elements of simple algebras

In this section, we review some algebraic preliminaries which will be needed below. We recall the most fundamental properties of semi-simple algebras and their representations, and, we give a characterization of such algebras when the basic field is algebraically closed.

Let $F$ be a field and $A$ an (associative) algebra (with identity) over $F$.

A minimal left (right) ideal of $A$ is a left (right) ideal $J \neq \{0\}$ such that $\{0\}$ and $J$ are the only left ideals contained in $J$.

An element $t$ of $A$ is said to be an idempotent if $t^2 = t$. Two idempotents $f$ and $h$ such that $hf = fh = 0$ are called orthogonal. A non-zero idempotent element of $A$ is said to be primitive if it is not a sum of two non-zero orthogonal idempotents.

The algebra $A$ is said to be simple if the only bi-ideals of $A$ are $\{0\}$ and itself. It is said to be semi-simple if it is isomorphic to a direct sum of simple algebras. Clearly every simple algebra is semi-simple.

$A$ is said to be semi-prime algebra if $\{0\}$ is the only bi-ideal $J$ of A with $J^2 = \{0\}$. It is well-known that semi-simple algebras are semi-prime [7].

The results of this section have obvious duals obtained by interchanging the roles of the right and left ideals.

**Lemma 2.1.** *Let $I$ be a left ideal of a semi-simple algebra $A$ with $I^2 = \{0\}$. Then, $I = \{0\}$.*

**Proof.** Set $K = IA$. $K$ is a bi-ideal of $A$ with $I \subset K$ and $K^2 = \{0\}$. Since $A$ is a semi-simple algebra, then $A$ is semi-prime. So, $K = \{0\}$ and so, $I = \{0\}$. □

**Proposition 2.2.** *Let $I$ be a minimal left ideal of a semi-simple algebra $A$. Then, there exists a non-zero primitive idempotent element $t$ of $A$ such that, $I = At$. Where $At = \{at/a \in A\}$.*

**Proof.** By the previous lemma, $I^2 \neq \{0\}$, since $I \neq \{0\}$. Then, there exists $x \in I$ such that $Ix \neq \{0\}$ and so, $Ax \neq \{0\}$. Since $Ix$ is a non-zero left ideal of $A$ contained in $I$, the minimality of $I$ gives $I = Ix$ and there exists $t \in I \setminus \{0\}$ with $x = tx$, also $tx = t^2x$. So, $t^2 - t \in I \cap ker(x)$, where $ker(x) = \{a \in A/ax = 0\}$. Since $I \cap ker(x)$ is a left ideal contained in $I$ and $I$ is not contained in $ker(x)$, the minimality of $I$ gives $I \cap ker(x) = \{0\}$ and so, $t^2 = t$. $At$ is a non-zero left ideal contained in $I$. Then $I = At$. on the other hand, if $t = f + h$, with $f$ and $h$ are two orthogonal idempotents of $A$, then $At = Af \oplus Ah$. By minimality of $I = At$, we have $f$ or $h$ is vanish. Hence, we get the result. □

**Proposition 2.3.** *Let $t$ be a non-zero idempotent of a semi-simple algebra $A$. Then, the left ideal $At$ is minimal if and only if $t$ is primitive.*

**Proof.** Suppose that $t$ is primitive. Let $I$ be a minimal left ideal of $A$ contained in $At$. By Proposition 2.2, there exists a primitive idempotent $t' \in A$ such that $I = At'$. Firstly, $t't = t'$, since $t' \in At$, and $tt' \neq 0$, since $\{0\} \neq At' \subset Att'$. On the other hand, $t = tt' + (t - tt')$; sum of two orthogonal idempotents. Thus, necessarily $t = tt'$, since $t$ is primitive. So, $t \in At'$ and so, $At = At'$. It follows that $At$ is minimal.
Conversely, it is easy. □

**Corollary 2.4.** *Let $t$ be a primitive idempotent element of $A$. Then, every non-zero idempotent of $At$ is primitive.*

**Proof.** Let $t'$ be a non-zero idempotent element of $At$. It is easy to see that $At = At'$. Thus, $At'$ is a minimal left ideal of $A$. By Proposition 2.3, $t'$ is primitive. □

**Proposition 2.5.** *Let $t$ be a non-zero idempotent element of a semi-simple algebra $A$. Then, $t$ is primitive if and only if $tAt$ is a division algebra; that is every non-zero element of $tAt$ has a two-sided inverse.*

**Proof.** $\Longrightarrow$ Evidently $tAt$ is an algebra with unit element $t$. Let $txt \neq 0$. Since $txt \in Atxt$, then, $\{0\} \neq Atxt \subset At$. The minimality of $At$ gives $Atxt = At$ and there exists $y \in A$ with $ytxt = t$. Thus, $(tyt)(txt) = t^2 = t$. We have proved that each non-zero element of $tAt$ has a left inverse, and therefore $tAt$ is a division algebra.
$\Longleftarrow$ Let $J$ be a left ideal of $A$ with $\{0\} \neq J \subset At$. By Lemma 2.1, we have $J^2 \neq \{0\}$, and so there exist elements $at, bt \in J$ such that $atbt \neq 0$. It follows that $tbt$ is a non-zero element of the division algebra $tAt$. Hence, there exists $c \in tAt$ such that $ctbt = t$. Thus, $At = Actbt \subset Abt \subset J$, and so, $At$ is a minimal left ideal. The Proposition 2.5 achieves the proof. $\square$

**Lemma 2.6.** *Let $A$ be an associative unit complex algebra of finite-dimensional. Then, $A$ is a division algebra if and only if $A = \mathbb{C}.1$.*

**Proof.** Suppose that $A$ is a division algebra. Let $a \in A$. The morphism $\left\{ \begin{array}{ccc} \mathbb{C}[X] & \xrightarrow{\varphi} & A \\ P & \longmapsto & P(a) \end{array} \right.$ is non injective (dimensional-reasons). It's kernel is so, a non-zero ideal of $\mathbb{C}[X]$, then there is on non-zero unitary polynomial of minimal degree $P$ in $\ker(\varphi)$. $P$ is necessary an irreducible polynomial of $\mathbb{C}[X]$, then $P = X - \alpha$ for some $\alpha \in \mathbb{C}$ and so, $a = \alpha.1$.
Conversely, if $A = \mathbb{C}.1$ then, evidently, $A$ is a division algebra. $\square$

**Remark 2.7.** The previous result is true if we substitute the field of complex numbers $\mathbb{C}$ with another algebraically closed field.

**Corollary 2.8.** *Let $t$ be a non-zero idempotent element of a semi-simple complex algebra $A$. Then, $t$ is primitive if and only if $tAt = \mathbb{C}.t$.*

**Proof.** By Proposition 2.5 and the previous lemma. $\square$

Throughout this section, $A$ denotes a semi-simple algebra of finite-dimension.
The set that we denote $\mathcal{P}$, of primitive idempotent elements of $A$ can be provided with the equivalence relation $\mathcal{R}$ given by $t\mathcal{R}t'$ if and only if $At = At'$. So, there exists $t_1, ..., t_r \in \mathcal{P}$ such that $A = \oplus_{i=1}^r At_i$, which are called representatives of the equivalence classes, with respect to $\mathcal{R}$.

**Proposition 2.9.** *There exists a pairwise orthogonal primitive idempotents $e_1, ..., e_r \in \mathcal{P}$ such that:*

*(i)* $1 = \sum_{i=1}^r e_i$.

*(ii)* $A = \oplus_{i=1}^r Ae_i$.

**Proof.** Choose $t_1, ..., t_r \in \mathcal{P}$ such that $A = \oplus_{i=1}^r At_i$. Then, there exists a unique non-zero element $(e_1, ..., e_r) \in At_1 \times ... \times At_r$ such that $1 = \sum_{i=1}^r e_i$. Moreover, for all $j \in \{1, ..., r\}$, by uniqueness of the decomposition of $e_j$, we have

$$e_j = e_j 1 = \sum_{i=1}^r e_j e_i = e_j^2 + \sum_{i \neq j} \underbrace{e_j e_i}_{=0} = e_j^2.$$

Corollary 2.4 achieves the proof. $\square$

**Remark 2.10.**

(i) The number $r$ is uniquely determined. Moreover, we have, $r = 1$ if and only if $A = F$, if and only if 1 is the only primitive idempotent element of $A$.

(ii) $A = \oplus_{i,j}^{r} e_i A e_j$.

(iii) Let $a$ and $b$ be two elements of the algebra $A$. We shall decompose them in accordance with the decomposition of 1: $a = \sum_{i,j} a_{ij}$ and $b = \sum_{i,j} b_{ij}$, where $a_{ij} = e_i a e_j$, $b_{ij} = e_i b e_j$. Then, $a + b = \sum_{i,j}(a_{ij} + b_{ij})$ and $ab = \sum_{i,k}\sum_{k,j} a_{ik}b_{kj}$. This allows the element $a$ to be written in the matrix-form $(a_{ij})$. We have just established that the addition and multiplication of these elements translates in this interpretation into the addition and multiplication of the matrices defined in the usual way.

Assume that $F$ is algebraically closed, and let $t$ be a primitive idempotent element of a simple finite dimensional algebra $A$ over $F$. Then, the natural map $\rho : A \longrightarrow End_F(At)$ defining the action of $A$ on $At$ is injective. Indeed, since the kernel is a bi-ideal, $A$ is simple.

By Burnside's lemma [15], $\rho$ is surjective, and therefore, an isomorphism which is called a spinor representation of $A$, the corresponding minimal left ideal is called a spinor space of $A$. It follows that $A$ is isomorphic to $M(n, F)$ for some integer $n \geq 1$.
Thus, we have the following result:

**Proposition 2.11.** *Assume that $A$ is simple. Let $e_1, \ldots, e_r \in A$ are the pairwise orthogonal primitive idempotents such that $1 = \sum_{i=1}^{r} e_i$. Then,*

$$r = \dim(Ae_1) = \dim(Ae_2) = \cdots = \dim(Ae_r) = \sqrt{\dim(A)}.$$

**Proof.** Since $A = \oplus_{i=1}^{r} Ae_i$ then, $\dim(A) = \sum_{i=1}^{r} \dim(Ae_i)$. On the other hand, by Burnside's lemma [15], $A \cong End(Ae_i)$, for all $i \in \{1, \ldots, r\}$ then, $\dim(A) = (\dim(Ae_i))^2$. So, we get the result. □

**Remark 2.12.** For a simple algebra $A$ over $F$ (algebraically closed) it is easy to see that:

1. $\dim A$ is, necessarily, a square integer.
2. There are $\sqrt{\dim A}$-classes (types) of primitive idempotent elements in $A$.
3. Every ideal of $A$ is of dimension $\geq \sqrt{\dim A}$.
4. Every idempotent $t \in A$ with $\dim At = \sqrt{\dim A}$ is primitive.
5. If $A$ is semi-simple, then $A \cong \oplus_{i=1}^{s} M(n_i, F)$, for some non negative integers $s, n_1, ..., n_s$.

**Example 2.13.** Let $n$ be a non-zero integer. For all $i \in \{1, ..., n\}$, $E_i$ denotes the matrix of $M(n, \mathbb{C})$ with the $ii^{th}$ entry equal to 1 and all the rest are zero. By a simple calculus, we have for all $i \in \{1, ..., n\}$, $E_i M(n, \mathbb{C}) E_i = \mathbb{C}.E_i$. Thus, by Corollary 2.8, $E_1$,..., $E_n$ are a primitive idempotent elements of $M(n, \mathbb{C})$. They are the only (with respect to $\mathcal{R}$) primitive idempotent elements of $M(n, \mathbb{C})$. Moreover, we have $\sum_{i=1}^{n} E_i = I_n$, where $I_n$ is the matrix identity of $M(n, \mathbb{C})$.

## 3 Primitive idempotent elements of $\mathbb{C}l(p,q)$

Let us recall some basic results of the complex Clifford algebra $\mathbb{C}l(p,q)$. Given $p, q$ and $n$ a non-negative integers such that $n = p + q$. We denote $Cl(p,q)$ the Clifford algebra of the quadratic space $\mathbb{R}^{p,q}$ and $\mathbb{C}l(p,q) = \mathbb{C} \otimes Cl(p,q)$ the complexifed algebra of $Cl(p,q)$. If $e_i, 1 \leq i \leq n$ is an orthonormal basis of $\mathbb{R}^{p,q}$, then $\mathbb{C}l(p,q)$ is generated by $e_i$ with relations

$$e_i e_j + e_j e_i = 2g_{ij}e, \ 1 \leq i, j \leq n, \tag{3.1}$$

where $e$ is the unitary element of $\mathbb{C}l(p,q)$, $g_{ii} = 1$ if $1 \leq i \leq p$, $g_{ii} = -1$ if $p + 1 \leq i \leq n$ and $g_{ij} = 0$ if $i \neq j$. It has basis

$$e, e_i, e_{i_1}e_{i_2}, ..., e_1...e_n, \ 1 \leq i_1 < i_2 < ... \leq n,$$

with $i, i_1, ...$ are indexes from 1 to $n$. Thus $\mathbb{C}l(p,q)$ is $2^n$-dimensional complex vector space [10]. Throughout this paper $e_1, ..., e_n$ denote an orthonormal basis of $R^{p,q}$ and $e_I$ will denote the product element $e_{i_1}\dots e_{i_k}$ of $\mathbb{C}l(p,q)$ for any $I=\{i_1,\dots,i_k\}$ with $1\le i_1<i_2\dots<i_k\le n$, and $e_\emptyset := e$. So, any Clifford algebra element $X\in\mathbb{C}l(p,q)$ can be written in the following form

$$X = xe + \sum_{I\neq\emptyset}\lambda_I e_I, \tag{3.2}$$

where $x, \lambda_I$ are complex constants.

Complex Clifford algebras $\mathbb{C}l(p,q)$ of dimension $2^n$ and different signatures $(p,q)$, $p+q=n$ are isomorphic. Clifford algebras $\mathbb{C}l(p,q)$ are isomorphic to the matrix algebras of complex matrices. In the case of even $n$, these matrices are of order $2^{\frac{n}{2}}$. In the case of odd $n$, these matrices are block diagonal of order $2^{\frac{n+1}{2}}$ with 2 blocks of order $2^{\frac{n-1}{2}}$ [16, 10]. Precisely, we have the following well-known matrix-representations of complex Clifford algebras (of minimal dimension)

$$Cl(\mathbb{C}^n)\cong\mathbb{C}l(p,q)\cong\begin{cases} Mat(2^{\frac{n}{2}},\mathbb{C}) & \text{if } n \text{ is even,}\\ Mat(2^{\frac{n-1}{2}},\mathbb{C})\oplus Mat(2^{\frac{n-1}{2}},\mathbb{C}) & \text{if } n \text{ is odd.}\end{cases}$$

In particular, these Clifford algebras $\mathbb{C}l(p,q)$ are a simple algebras (if $n$ is even) or a semi-simple algebras (if $n$ is odd). Such a property allows us to give the following results.

**Theorem 3.1.** *Let $t$ be a non-zero idempotent element of the complex Clifford algebra $\mathbb{C}l(p,q)$. Then, the following properties are equivalent.*

*(i)* *$t$ is primitive.*

*(ii)* *$\mathbb{C}l(p,q)t$ is a minimal left ideal.*

*(iii)* *$t\mathbb{C}l(p,q)t=\mathbb{C}t$.*

*(iv)* *$t\mathbb{C}l(p,q)t$ is division algebra.*

*(v)* *$t$ is the only non-zero idempotent element of $t\mathbb{C}l(p,q)t$.*

*(vi)* *(For even $n$) $\dim(\mathbb{C}l(p,q)t)=2^{\frac{n}{2}}$.*

**Proof.** From the results of the above section, it suffice to prove that $(v)\Longrightarrow(i)$.
Assume that $t$ is the only non-zero idempotent element of $t\mathbb{C}l(p,q)t$. Set $f$ and $h$ two orthogonal idempotent elements of $\mathbb{C}l(p,q)$ such that $t=f+h$. It follows that, $f$ and $h$ are two idempotent elements of $t\mathbb{C}l(p,q)t$ with $f\neq h$. So, $f$ or $h$ is zero, and so, $t$ is primitive. □

**Remark 3.2.** As the previous section, let $\mathcal{P}$ denotes the set of primitive idempotent elements of $\mathbb{C}l(p,q)$, and $\mathcal{R}$ the equivalent relation given by: $\forall t,t'\in\mathcal{P}$, $t\mathcal{R}\,t'$ if and only if $\mathbb{C}l(p,q)t=\mathbb{C}l(p,q)t'$. So, there is $2^{[\frac{n+1}{2}]}$ equivalence class representatives with respect to $\mathcal{R}$.

**Example 3.3.**
1. In $\mathbb{C}l(1,1)$, every non-zero idempotent element $t\neq 1$ is primitive. For example: $t_1=\frac{1}{2}(e-e_1)$ and $t_2=\frac{1}{2}(e+e_1)$ are two orthogonal primitive idempotent elements with $t_1+t_2=e$. They are the only (with respect to $\mathcal{R}$) primitive idempotent elements of $\mathbb{C}l(1,1)$.
2. In $Cl(1,3)$, $t_1=\frac{1}{4}(e-e_1)(e-ie_2e_3)$, $t_2=\frac{1}{4}(e-e_1)(e+ie_2e_3)$, $t_3=\frac{1}{4}(e+e_1)(e-ie_2e_3)$ and $t_4=\frac{1}{4}(e+e_1)(e+ie_2e_3)$ are four pairwise orthogonal primitive idempotent elements of $Cl(1,3)$ with $t_1+t_2+t_3+t_4=e$. They are the only (with respect to $\mathcal{R}$) primitive idempotent elements of $\mathbb{C}l(1,3)$.
3. Assume that $n\ge 3$. Set $t_0=\frac{1}{2}(e-e_1)$ and $t_k=\frac{1}{2}(e-i^{b_k}e_{2k}e_{2k+1})$ for $1\le k\le m$, where $m=[\frac{n-1}{2}]$, $b_k=0$ for $2k=p$ and $b_k=1$ for $2k\neq p$, with $e,e_1,...,e_n$ are basis generators of $\mathbb{C}l(p,q)$ satisfying (3.1).
$t_0,t_1,...,t_m$ are a pairwise commuting idempotent elements of $\mathbb{C}l(p,q)$. Thus, their product:

$$t=\prod_{k=0}^{m}t_k \tag{3.3}$$

is also an idempotent element of $\mathbb{C}l(p,q)$. Furthermore, $t$ is primitive.
Indeed: By a direct calculation, one obtains: $t_0e_1t_0 = t_ke_{2k}t_k = t_ke_{2k+1}t_k = 0$ and further, $te_1t = te_{2k}t = te_{2k+1}t = 0$, for all $k = 1,...,m$. Moreover, when $n$ is even, we have $t_0e_nt_0 = 0$, hence, $te_nt = 0$. So, for all $k = 1,...,n$, $te_kt = 0$. It follows that, $te_kt \in \mathbb{C}t$, for all $k = 1,...,n$. By a similar calculation, we can verify that for any basis element $e_I$ of $\mathbb{C}l(p,q)$, $te_It \in \mathbb{C}t$. Consequently, $t\mathbb{C}l(p,q)t = \mathbb{C}t$ and so, $t$ is primitive. (By Theorem 3.1).

**Theorem 3.4.** *For even $n$, the primitive idempotent elements of $\mathbb{C}l(p,q)$ are conjugated. That is, for all $t,t'$ two primitive idempotent elements of $\mathbb{C}l(p,q)$, there exists an invertible element $x$ of $\mathbb{C}l(p,q)$ such that $t' = xtx^{-1}$.*

**Proof.** Using the isomorphism between the Clifford algebra $\mathbb{C}l(p,q)$ and the matrix algebra $M(2^{\frac{n}{2}},\mathbb{C})$ it is enough to establish the result in $M(2^{\frac{n}{2}},\mathbb{C})$. Given a primitive idempotent matrix $T$ of $M(2^{\frac{n}{2}},\mathbb{C})$. By means of a similarly transformation, it can be transformed into its Jordan form. Consequently, the idempotency implies that the Jordan form of $T$ must be (up to basis vector transformation) of the form $E_1 = \mathrm{diag}(1,0,...,0)$, ($E_1$ is a primitive idempotent element of $M(2^{\frac{n}{2}},\mathbb{C})$, (see Example 2.13). Thus, there exists $S \in GL(2^{\frac{n}{2}},\mathbb{C})$ such that $T = SE_1S^{-1}$. So, we get the result. □

**Remark 3.5.** For even $n$, the set $\mathcal{P}$ of primitive idempotent elements of $\mathbb{C}l(p,q)$, can be given as follows

$$\mathcal{P}=\{xtx^{-1}/x \textit{ is an invertible element of } \mathbb{C}l(p,q)\},$$

for any primitive idempotent element $t$. For example $t$ is the one given by Formula (3.3).

**Theorem 3.6.** *For even $n$, any primitive idempotent element of $\mathbb{C}l(p,q)$ is necessarily of the form given by Formula (3.3), for some generators $\gamma_i$ of $\mathbb{C}l(p,q)$ (instead of the $e_i$) satisfying the relation*

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2g_{ij}e,\ 1 \le i,j \le n. \tag{3.4}$$

**Proof.** Let $t'$ be a primitive idempotent element of $\mathbb{C}l(p,q)$. By the previous theorem, there exists an invertible element $x$ of $\mathbb{C}l(p,q)$ such that $t' = x^{-1}tx$, where $t$ denotes the primitive idempotent element of $\mathbb{C}l(p,q)$ given by Formula (3.3) in the example above. The family $\gamma_i = x^{-1}e_ix$, $1 \le i \le n$ satisfies Relation (3.4). Hence, by Pauli's Theorem (see [17]), $\gamma_i$ are generator elements of $\mathbb{C}l(p,q)$. Replacing the generators $e_i$ by the $\gamma_i$, Formula (3.3) gives $t'$. □

**Proposition 3.7.** *For even $n$, $e_{2I}t, I \subset \{1,...,\frac{n}{2}\}$ form a base for the minimal left ideal $\mathbb{C}l(p,q)t$. Here, $t$ is the primitive idempotent element given by (3.3) and $2I = \{2k; k \in I\}$.*

**Proof.** Let $k \in \{1,...,\frac{n}{2}-1\}$. We have

$$\begin{aligned}
e_{2k+1}t_k &= \tfrac{1}{2}\left(e_{2k+1} + i^{b_k}e_{2k}e_{2k+1}^2\right)\\
&= \frac{i^{b_k}e_{2k+1}^2}{2}\left(e_{2k} + \frac{1}{i^{b_k}e_{2k+1}^2}e_{2k+1}\right)\\
&= i^{b_k}e_{2k+1}^2\tfrac{1}{2}\left(e_{2k} + \frac{i^{b_k}}{i^{2b_k}e_{2k}^2e_{2k+1}^2}e_{2k}^2e_{2k+1}\right)\\
&= i^{b_k}e_{2k+1}^2\tfrac{1}{2}\left(e_{2k} - i^{b_k}e_{2k}^2e_{2k+1}\right),\ \textit{since, } i^{2b_k}e_{2k}^2e_{2k+1}^2 = -1\\
&= \left(i^{b_k}e_{2k+1}^2\right)e_{2k}t_k.
\end{aligned}$$

Hence, $e_{2k+1}t = \left(i^{b_k}e_{2k+1}^2\right)e_{2k}t$. On the other hand, $e_1t = -t = -e_{\emptyset}t$. It follows that $e_{2I}t$ form a generating family of $\mathbb{C}l(p,q)t$. It is, therefore, a basis of $\mathbb{C}l(p,q)t$, (dimensional-reasons). □

**Remark 3.8.** For odd $n$, there exists two orthogonal primitive idempotent elements $t_1,t_2$ of $Cl(p,q)$ such that $\mathcal{P}= \mathcal{P}_1 \cup \mathcal{P}_2$ where $\mathcal{P}_k = \{xt_kx^{-1}; x \textit{ is an invertible element of } \mathbb{C}l(p,q)\}$, $k = 1,2$, with $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$. In particular $t_1$ and $t_2$ are not conjugated.

In the rest of this section, we will need to consider $Cl^0(p,q)$ (resp $Cl^1(p,q)$), the subspace of $Cl(p,q)$ spanned by products of even (resp. odd) number of the $e_i$. Notice that $Cl(p,q) = Cl^0(p,q) \oplus Cl^1(p,q)$ and hence, $\dim(Cl^0(p,q)) = \dim(Cl^1(p,q)) = 2^{n-1}$. The following results can be used to represent the Dirac-equation in the Minkowski space-time [8].

**Proposition 3.9.** *Let $t$ be the primitive idempotent element of $\mathbb{C}l(p,q)$ given by Formula (3.3). If $n$ is even and $q \geq 3$ then, the following two maps*

$$Cl^0(p,q) \xrightarrow{\varphi_0} \mathbb{C}l(p,q)t \text{ and } Cl^1(p,q) \xrightarrow{\varphi_1} \mathbb{C}l(p,q)t$$

*defined by multiplication by $t$ from the right are surjective $\mathbb{R}$-linear maps.*

**Proof.** By Proposition 3.7, the $\mathbb{R}$-linear space $\mathbb{C}l(p,q)t$ has basis $e_{2I}t, ie_{2I}t, I \subset \{1,...,\frac{n}{2}\}$. It easy to see that, $e_{2I}$ or $-e_{2I}e_1$ is an element of $Cl^0(p,q)$ which we denote $x$, we have $\varphi_0(x) = e_{2I}t$. On the other hand, we have $e_{n-2}e_{n-1} \in Cl^0(Cl(p,q))$ and so, $xe_{n-2}e_{n-1} \in Cl^0(p,q)$. Therefore, $e_{n-2}e_{n-1}t_{(\frac{n-2}{2})} = e_{n-2}e_{n-1}\frac{1}{2}(e - ie_{n-2}e_{n-1}) = it_{(\frac{n-2}{2})}$, (since $q \geq 3$), and so, $e_{n-2}e_{n-1}t = it$. Thus, $\varphi_0(xe_{n-2}e_{n-1}) = ie_{2I}t$. It follows that, $\varphi_0$ is a surjective map. Similarly, we show that $\varphi_1$ is a surjective map. □

**Corollary 3.10.** *As $\mathbb{R}$-vector spaces, we have*

$$Cl^0(1,3) \cong \mathbb{C}l(1,3)t \cong Cl^1(1,3) \cong \mathbb{C}^4.$$

**Proof.** Since, $\dim_{\mathbb{R}}(Cl^0(1,3)) = \dim_{\mathbb{R}}(Cl^1(1,3)) = \dim_{\mathbb{R}}(\mathbb{C}l(1,3)t) = \dim_{\mathbb{R}}(\mathbb{C}^4) = 8$. Then, Proposition 3.9 gives the result. □

## 4 Rota-Baxter operators on $\mathbb{C}l(p,q)$

Given an algebra $A$ and a scalar $\lambda$ in a field $F$, a linear operator $R : A \longrightarrow A$ is called a Rota-Baxter operator (RB operator, shortly) on $A$ of weight $\lambda$ if the following identity

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy) \tag{4.1}$$

holds for all $x, y \in A$. The algebra $A$ is called the Rota-Baxter algebra (RB algebra). The set of RB on $A$ is noted $RB(A)$.

**Example 4.1.**
1. Given an algebra $A$ of continuous functions on $\mathbb{R}$, an integration operator $R(f)(x) := \int_0^x f$ is an RB operator on $A$ of weight zero.
2. consider the algebra of sequences in a $F$-algebra, with componentwise addition and multiplication. Define an operator $R : (a_1, a_2, a_3, ..., a_n, ...) \longmapsto (0, a_1, a_1 + a_2, ..., \sum_{k<n} a_k, ...)$. $R$ is a Rota-Baxter operator of weight 1.
3. A linear map $R_a$ on the polynomial algebra $F[x]$ defined as $R(x^n) = \frac{(x^{n+1} - a^{n+1})}{n+1}$ is an RB-operator on $F[x]$ of weight zero, for any $a \in F$.

**Proposition 4.2.** *Let $A$ be an associative unital algebra.*

1. *A linear operator $R \in RB(A)$ if and only if $I - R \in RB(A)$, where $I$ is the identity map. In particular, $0, I \in RB(A)$.*
2. *Let $R$ be an RB operator on $A$ of weight $\lambda$. Then $-R - \lambda I$ is an RB operator of weight $\lambda$ and the operator $\lambda^{-1}R$ is an RB operator of weight 1, provided that $\lambda \neq 0$.*
3. *Let $R$ be an RB operator of weight $\lambda$ on $A$, and let $\psi \in Aut(A)$. Then, $R_\psi = \psi^{-1}R\psi$ is an RB operator on $A$ of weight $\lambda$.*

**Proposition 4.3.** *Assume that an algebra $A$ is split as a vector space into the direct sum of two subalgebras $A_1$ and $A_2$; $A = A_1 \oplus A_2$. Then*

*(i) An operator $R$ defined by the rule $R(x_1 + x_2) = -\lambda x_2, x_1 \in A_1, x_2 \in A_2$, is an RB operator on $A$ of weight $\lambda$.*

*(ii) If $A_1$ and $A_2$ are ideals and $R$ is an RB operator on $A$ of weight $\lambda$, then $P_iR$ is an is RB-operator of weight $\lambda$ on $A_i$, $i = 1, 2$. Here $P_i$ denotes the projection from $A$ onto $A_i$.*

*(iii) If $A_1$ and $A_2$ are ideals, then for any RB operators $R_i$ on $A_i$ (i=1,2) with a same weight, the linear map $R : (x_1, x_2) \longmapsto (R_1(x_1), R_2(x_2))$ defines a RB operator on $A$.*
*NB: It is not true that every RB on $A$ can be obtained from the above way (see [19]).*

**Proposition 4.4.** *Let $A$ be an algebra and $R : A \longrightarrow A$ be a linear isomorphism, then $R$ is a Rota-Baxter operator on $A$ if and only if $R^{-1}$ is a derivation on $A$.*

**Proof.** For any $x, y \in A$, $R$ is a Rota-Baxter operator on $A$ if and only if $R(x)R(y) = R(R(x)y+xR(y))$, which is equivalent to $R^{-1}(uv) = uR^1(v)+R^1(u)v$, where $u = R(x), v = R(y)$. Therefore, the conclusion follows. □

Throughout this section, $A$ denotes the Complex Clifford algebra $\mathbb{C}l(p,q)$, (see Section 3). Let $t_1, ..., t_r$ are orthogonal idempotent elements of $A$ such that $e = \oplus_{i=1}^{r} t_i$. Set $A_{ij} = t_iAt_j$, $A_0 = \oplus_{1\leq i\leq r} A_{ii}$, $A_- = \oplus_{1\leq i<j\leq r} A_{ij}$ and $A_+ = \oplus_{1\leq j<i\leq r} A_{ij}$. By Theorem 3.1, we have $A_0 = \oplus_{i=1}^{r} \mathbb{C}.t_i$. In the rest of this section we will give some examples and some properties of RB-operators on $A$.

**Proposition 4.5.** *(i) For any idempotent element t of $A$, the map $R_t : A \longrightarrow A : x \longmapsto xt$ is a RB operator on $A$ of weight -1.*

*(ii) If $t_1, ..., t_r$ are orthogonal idempotent elements of $A$, then for all $k = 1, ..., r$, $R_k = \sum_{i=1}^{k} R_{t_i}$ is an RB operator on $A$ of weight -1.*

*(iii) Let $t \in A$ such that $t^2 = -\lambda t$ with $\lambda \in \mathbb{C}$. The linear map $R_t : x \longmapsto xt$ is an RB operator on $A$ of weight $\lambda$.*

*(iv) Assume that $n$ is odd. For any RB-operator $R$ on $A$ of weight zero, the operator given by $R_\Gamma(x) = R(x\Gamma)$ is an RB-operator on $A$ of weight zero. Here $\Gamma = e_1...e_n$, where $e_1, ..., e_n$ are any generators of $A$ satisfying Relation (3.1).*

**Proof.**

(i) Let $t$ be an idempotent element of $A$. And let $x, y \in A$. We have $R_t(xR_t(y) + R_t(x)y - xy) = ((xR_t(y) + R_t(x)y - xy))t = (xyt + xtyt - xyt) = xtyt = R_t(x)R_t(y)$. So, $R_t$ is a RB operator on $A$ of weight -1.

(ii) Let $t_1, ..., t_r$ are orthogonal idempotent elements of $A$. Since $\sum_{i=1}^{k} t_i$ is idempotent and $\sum_{i=1}^{k} R_{t_i} = R_{\sum_{i=1}^{k} t_i}$, then, (i) gives the result.

(iii) By the same argument as (i).

(iv) It follows from Identity (4.1). □

**Proposition 4.6.** *If $R_0$ is an RB-operator of weight $\lambda$ on $A_0$, then an operator $R$ defined as*

$$R(a_- + a_0 + a_+) = R_0(a_0) - \lambda a_\pm, \; a_\pm \in A_\pm, a_0 \in A_0,$$

*is an RB-operator on $A$ of weight $\lambda$.*

**Proof.** It follows from Formula 4.1. □

**Theorem 4.7.** *A linear operator $R(t_i) = \sum_{k=1}^{r} a_{ik}t_k$, $a_{ik} \in \mathbb{C}$, is an RB-operator of weight 1 on $A_0$ if and only if the following conditions are satisfied:*

$$a_{ik}a_{jk} = a_{ji}a_{ik} + a_{ij}a_{jk} \text{ for } i \neq j, \; a_{ik}(a_{ik} - 2a_{ii} - 1) = 0 \text{ for } i = j. \tag{4.2}$$

**Proof.** For any $1 \leq i, j \leq r$,

$$R(t_i)R(t_j) = R(t_iR(t_j) + R(t_i)t_j + t_it_j)$$

$$\text{if and only if, } \sum_{k=1}^{r} a_{ik}t_k \sum_{l=1}^{r} a_{il}t_l = R\left(t_i \sum_{k=1}^{r} a_{jk}t_k + \sum_{k=1}^{r} a_{ik}t_kt_j + \delta_{ij}t_i\right)$$

$$\text{if and only if, } \sum_{k=1}^{r}\left(\sum_{l=1}^{r} a_{ik}a_{jl}t_k t_l\right) = R(a_{ji}t_i + a_{ij}t_j + \delta_{ij}t_i)$$

$$\text{if and only if, } \sum_{k=1}^{r} a_{ik}a_{jk}t_k = \sum_{k=1}^{r}(a_{ji}a_{ik} + a_{ij}a_{jk} + \delta_{ij}a_{ik})\,t_k$$

From which (4.2) follows. □

**Example 4.8.** 1. For $A = \mathbb{C}l(1,1)$, we have $A_0 = \mathbb{C}.t_1 \oplus \mathbb{C}.t_2$, (see Example 3.3 (1)). By conditions (4.2), an operator $R_0$ defined as $R_0(t_i) = \sum a_{ik}t_k$ is a Rota-Baxter operator on $A_0$ of weight 1 if ,and only if, one of the following cases is true:

a. $a_{11} = a_{22} = -1$ and ($a_{12} = -1, a_{21} = 0$ or $a_{21} = -1, a_{12} = 0$). That is

$$\begin{cases} R_0(t_1) = -t_1 - t_2 \quad \text{and} \quad R_0(t_2) = -t_2, \text{ or} \\ R_0(t_1) = -t_1 - t_2 \quad \text{and} \quad R_0(t_2) = -t_1 \end{cases}$$

b. $a_{11} = a_{22} = 0$ and ($a_{12} = 1, a_{21} = 0$, or $a_{21} = 1, a_{12} = 0$). That is

$$\begin{cases} R_0(t_1) = t_2 \quad \text{and} \quad R_0(t_2) = 0, \text{ or} \\ R_0(t_1) = 0 \quad \text{and} \quad R_0(t_2) = t_1 \end{cases}$$

On the other hand, we have $A_- = \mathbb{C}.t_3$ and $A_+ = \mathbb{C}.t_4$ where $t_3 = e_1e_2$ and $t_4 = (e_1e_2 - e_2)$. Proposition 4.5 gives RB-operators on $\mathbb{C}l(1,1)$ of weight 1. Here, $e_1, e_2$ are generators of $A$, with relation (3.1).

2. Let us consider $A = \mathbb{C}l(1,3)$. The following operator is an RB-operator of weight 1 on $A_0 = \oplus_{i=1}^{4}\mathbb{C}.t_i$.

$$R_0(t_1) = 0,\; R_0(t_2) = -t_2,\; R_0(t_3) = -t_2 - t_3,\; R_0(t_4) = -t_2 - t_3 - t_4,$$

where $t_1, t_2, t_3$ and $t_4$ are defined by example 3.3 (2).

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