Palestine Journal of Mathematics

Vol. 11(1)(2022) , 270–279 

# ORDERED REGULAR POINTED SEMIGROUPS WITH BIGGEST ASSOCIATES

G. A. Pinto

Communicated by Ayman Badawi

**MSC2020 - Mathematical Sciences Classification System** 20M17 and 06F05

Keywords and phrases: ordered semigroup; regular; naturally ordered; principally ordered; biggest associate.

**Abstract** We consider the class **BA** of ordered regular semigroups in which each element has a biggest associate $x^\dagger = \max\{y|xyx = x\}$. We investigate those ordered regular semigroups with biggest associates that are *pointed* in the sense, that the classes modulo Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$ have biggest elements which, are idempotent. Such a semigroup is necessarily a semiband. If the semigroup is also naturally ordered then, it is principally ordered. Generalisations of results in pointed principally ordered regular semigroups are obtained: (1) description of the subalgebra of $(S,^\dagger)$ generated by a pair of comparable idempotents that are $\mathcal{D}$-related; (2) those $\mathcal{D}$-classes which are subsemigroups, are ordered rectangular bands.

## 1 Introduction

In [**2**] Blyth and Almeida Santos introduced the class of ordered regular semigroups with biggest associates in the following way: when, in an ordered semigroup $S$, for each $x \in S$ there exists $x^\dagger = \max\{y \in S|xyx = x\}$ we say, that $S$ is an ordered semigroup with *biggest associates*. Note that the definition obviously implies that such a semigroup is regular.

We now list the basic properties of the unary operation $x \to x^\dagger$ in an ordered regular semigroup with biggest associates $S$ obtained in [**2**], that we shall require here.

$(P_1)$ $\Big(\forall x \in S\Big)$ $x = xx^\dagger x$

$(P_2)$ every $x \in S$ has a biggest inverse, namely $x^0 = x^\dagger x x^\dagger$

$(P_3)$ $\Big(\forall x \in S\Big)$ $x^0 \leq x^\dagger$

$(P_4)$ $\Big(\forall x \in S\Big)$ $xx^0 = xx^\dagger$ and $x^0x = x^\dagger x$

$(P_5)$ $\Big(\forall e \in E(S)\Big)$ $e \leq e^\dagger$

$(P_6)$ $\Big(\forall e \in E(S)\Big)$ $e^0 \in E(S) \iff e^\dagger \in E(S)$

$(P_7)$ $\Big(\forall x \in S\Big)$ $x \leq x^{00} \leq x^{0\dagger} = x^{\dagger\dagger} = x^{\dagger 0}$

$(P_8)$ $\Big(\forall x \in S\Big)$ $x^\dagger = x^{\dagger\dagger\dagger}$

$(P_9)$ $\Big(\forall x \in S\Big)$ $(x^\dagger x)^\dagger x^\dagger = x^\dagger = x^\dagger(xx^\dagger)^\dagger$

As it is observed in [**2**], this class is a generalisation of the class of principally ordered regular semigroups, which was introduced by Blyth and Pinto in [**3**] and whose basic properties can be found in [**1**]. An ordered regular semigroup $S$ is *principally ordered*, if for each $x \in S$ there exists $x^* = \max\{y \in S|xyx \leq x\}$.

In [**2**], it is remarked that the class of principally ordered regular semigroups **PO** is strictly contained in the class of ordered regular semigroup with biggest associates **BA**. In fact, for every element $x$ in a principally ordered regular semigroup, its biggest associate $x^\dagger$ also exists and it is equal to $x^*$.

An important property of a principally ordered regular semigroup $S$ is, that every $\mathcal{L}$-class $[x]_\mathcal{L}$ contains a biggest idempotent $x^*x$, and every $\mathcal{R}$-class $[x]_\mathcal{R}$ contains a biggest idempotent $xx^*$.

## 2 Main Theorem

In [**4**] Blyth and Pinto investigated those principally ordered regular semigroups for which the classes modulo Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$ have biggest elements, which are idempotent. In this paper, we generalise this notion and many of the results in [**4**] to the class of ordered regular semigroups with biggest associates.

**Theorem 2.1.** *Let $S$ be an ordered regular semigroup with biggest associates. The following statements are equivalent:*
(1) *every $\mathcal{L}$-class has a biggest element which is idempotent;*
(2) $(\forall x \in S)\ x^\dagger x = max[x]_{\mathcal{L}}$;
(3) *every $\mathcal{R}$-class has a biggest element which is idempotent;*
(4) $(\forall x \in S)\ xx^\dagger = max[x]_{\mathcal{R}}$;
(5) $(\forall x \in S)\ x^2 \leq x \leq x^\dagger$;
(6) $(\forall x \in S)\ x^\dagger = x^{\dagger\dagger} \in E(S)$;
(7) $(\forall x \in S)\ x^\dagger x^{\dagger\dagger} \leq x^\dagger$.
*Moreover, if $S$ satisfies any of the above conditions then,*
(8) $(\forall x \in S)\ max[x^\dagger]_{\mathcal{R}} = x^\dagger = x^{\dagger\dagger} = max[x^\dagger]_{\mathcal{L}}$
(9) *$S$ is a semiband and Green's relation $\mathcal{H}$ is equality;*
(10) *$x \in S$ is completely regular if and only if, $x \in E(S)$.*

*Proof.* (3) $\implies$ (4) : By (3) there exists an idempotent $e$ in $S$ such that
$$e = e^2 = \max[x]_{\mathcal{R}}$$
Since, $ee^\dagger$ is the biggest idempotent in its $\mathcal{R}$-class, we have using $(P_4)$ that
$$(e, ee^\dagger) \in \mathcal{R} \implies e \leq ee^\dagger = xx^\dagger$$
which means that $xx^\dagger = \max[x]_{\mathcal{R}}$.

(4) $\implies$ (5) : Since (4) holds for any element in $S$, we have in particular using $(P_8)$ that
$$(\alpha) \qquad x^\dagger \leq x^\dagger x^{\dagger\dagger} \qquad \text{and} \qquad x^{\dagger\dagger} \leq x^{\dagger\dagger}x^{\dagger\dagger\dagger} = x^{\dagger\dagger}x^\dagger$$
Then, using $(\alpha)$ repeatedly we have
$$x^\dagger = x^\dagger x^{\dagger\dagger}x^\dagger \leq x^\dagger x^{\dagger\dagger}x^\dagger x^\dagger \leq x^\dagger x^{\dagger\dagger}x^\dagger x^\dagger x^\dagger = x^\dagger x^\dagger x^\dagger \leq x^\dagger x^{\dagger\dagger}x^\dagger x^\dagger = x^\dagger x^\dagger \leq x^\dagger x^{\dagger\dagger}x^\dagger = x^\dagger$$
from which we obtain that $x^\dagger x^\dagger x^\dagger = x^\dagger$ and therefore $x^\dagger \leq x^{\dagger\dagger}$.
By (4) we have that $x \leq xx^\dagger$, and therefore on one hand
$$xx^{\dagger\dagger}x \leq xx^\dagger x^{\dagger\dagger}x \leq xx^\dagger x^{\dagger\dagger}x^\dagger x = x = xx^\dagger x \leq xx^{\dagger\dagger}x$$
from which, we obtain $x = xx^{\dagger\dagger}x$ which gives us by $(P_7)$ that $x \leq x^{\dagger\dagger} \leq x^\dagger$.
On the other hand,
$$x \leq xx^\dagger \implies xx \leq xx^\dagger x \implies x^2 \leq x$$
(5) $\implies$ (6) : For any $x \in S$ we have using (5) repeatedly that
$$x^\dagger x^{\dagger\dagger} \leq x^\dagger x^{\dagger\dagger\dagger} = x^\dagger x^\dagger \leq x^\dagger = x^\dagger x^{\dagger\dagger}x^\dagger \leq x^\dagger x^{\dagger\dagger}x^{\dagger\dagger} \leq x^\dagger x^{\dagger\dagger} \implies x^\dagger = x^\dagger x^{\dagger\dagger} \in E(S)$$
We also have that $x^\dagger \leq x^{\dagger\dagger} \leq x^{\dagger\dagger\dagger} = x^\dagger$, which implies that $x^\dagger = x^{\dagger\dagger}$.

(6) $\implies$ (3) : For any $x \in S$ we have using $(P_7)$ that
$$x^\dagger x \leq x^\dagger x^{\dagger\dagger} = x^\dagger x^\dagger = x^\dagger \implies xx^\dagger x \leq xx^\dagger \implies x \leq xx^\dagger$$
Now, considering $y \in [x]_{\mathcal{R}}$ we have that $y = yy^\dagger y \leq yy^\dagger y^{\dagger\dagger} = yy^\dagger y^\dagger = yy^\dagger = xx^\dagger$ which gives us that $xx^\dagger = \max[x]_{\mathcal{R}}$.

(1) $\implies$ (2) $\implies$ (5) $\implies$ (6) $\implies$ (1) : Are proved similarly.

(7) $\implies$ (6) : From (7) we have that
$$x^\dagger x^{\dagger\dagger} \leq x^\dagger \implies x^\dagger x^{\dagger\dagger}x^\dagger \leq x^\dagger x^\dagger \implies x^\dagger \leq x^\dagger x^\dagger$$
and
$$x^\dagger x^{\dagger\dagger} \leq x^\dagger \implies x^{\dagger\dagger}x^\dagger x^{\dagger\dagger}x^{\dagger\dagger} \leq x^{\dagger\dagger}x^\dagger x^{\dagger\dagger} \implies x^{\dagger\dagger}x^{\dagger\dagger} \leq x^{\dagger\dagger}$$
Replacing $x$ by $x^\dagger$ in the last inequality, we obtain that $x^{\dagger\dagger\dagger}x^{\dagger\dagger\dagger} \leq x^{\dagger\dagger\dagger}$ that is $x^\dagger x^\dagger \leq x^\dagger$ and we conclude that $x^\dagger$ is an idempotent. From this fact, we obtain $x^\dagger x^\dagger x^\dagger = x^\dagger$ and therefore $x^\dagger \leq x^{\dagger\dagger}$. Replacing $x$ by $x^\dagger$ gives $x^{\dagger\dagger} \leq x^{\dagger\dagger\dagger} = x^\dagger$, and therefore $x^\dagger = x^{\dagger\dagger}$ which proves (6).
(6) $\implies$ (7) : We have that
$$xx^\dagger x^{\dagger\dagger}x = xx^\dagger x^\dagger x = xx^\dagger x = x \implies x^\dagger x^{\dagger\dagger} \leq x^\dagger$$

Suppose now that the above conditions are satisfied.

$(8)$ : We have by (6) that $x^\dagger = x^{\dagger\dagger} \in E(S)$ which implies $x^\dagger = x^{\dagger\dagger}x^\dagger$ and therefore by (4)

$$x^\dagger = x^{\dagger\dagger}x^{\dagger\dagger\dagger} = \max[x^{\dagger\dagger}]_{\mathcal{R}} = \max[x^\dagger]_{\mathcal{R}}$$

Dually, we obtain that $x^\dagger = \max[x^\dagger]_{\mathcal{L}}$

$(9)$ : For every $x \in S$ we have by (6), that $x^\dagger$ is an idempotent and $x = xx^\dagger x = xx^\dagger \cdot x^\dagger x$, which means that every element of $S$ is a product of two idempotents, whence $S$ is a semiband. Moreover, if $x\mathcal{H}y$ then

$$x = xx^\dagger x = xx^\dagger \cdot x^\dagger x = yy^\dagger \cdot y^\dagger y = yy^\dagger y = y$$

whence $\mathcal{H}$ reduces to equality.

$(10)$ : If $x \in S$ is completely regular, then there exists $x' \in V(x)$ such that $xx' = x'x$. Then, by (5) $x' = x'xx' = x'x'x = (x')^2x \leq x'x$ from which it follows that

$$x = xx'x \leq xx'xx = x^2$$

and consequently $x \in E(S)$. The converse is clear. □

**Definition 2.2.** We shall say that an ordered regular semigroup with biggest associates is *pointed*, whenever it satisfies any of the seven equivalent properties of Theorem 2.1.

Let us present now some basic properties that hold in such a semigroup.

**Theorem 2.3.** *If $S$ is an ordered regular pointed semigroup with biggest associates then,*

(1) $(\forall x \in S) \quad (x^\dagger x)^\dagger = x^\dagger = (xx^\dagger)^\dagger$

(2) $(\forall x \in S) \quad x^0 = x^{00} \in E(S)$

(3) $(\forall x \in S) \quad x^0 = max[xx^\dagger]_{\mathcal{L}} = max[x^\dagger x]_{\mathcal{R}}$

*Proof.* Consider a general element $x$ in $S$.

$(1)$ : We have that

$$xx^\dagger \cdot x^\dagger \cdot xx^\dagger = x(x^\dagger x^\dagger) \cdot xx^\dagger = xx^\dagger xx^\dagger = xx^\dagger \implies x^\dagger \leq (xx^\dagger)^\dagger$$

Now, using $(P_9)$

$$x \cdot (xx^\dagger)^\dagger \cdot x = xx^\dagger x(xx^\dagger)^\dagger x \leq xx^\dagger x^\dagger (xx^\dagger)^\dagger x = xx^\dagger (x^\dagger (xx^\dagger)^\dagger)x = xx^\dagger x^\dagger x = x$$

and

$$x = xx^\dagger x \leq x(xx^\dagger)^\dagger x$$

imply that $x \cdot (xx^\dagger)^\dagger \cdot x = x$ which means that $(xx^\dagger)^\dagger \leq x^\dagger$ and, therefore $(xx^\dagger)^\dagger = x^\dagger$. Similarly, we can see that $(x^\dagger x)^\dagger = x^\dagger$

$(2)$ and $(3)$ : We have by (1), that

$$x^0 = x^\dagger xx^\dagger = x^\dagger x(x^\dagger x)^\dagger = \max[x^\dagger x]_{\mathcal{R}}$$

dually $x^0 = \max[xx^\dagger]_{\mathcal{L}}$. Finally, from $(P_7)$ we obtain

$$x^{00} = x^{0\dagger}x^0x^{0\dagger} = x^{\dagger\dagger}x^\dagger xx^\dagger x^{\dagger\dagger} = x^\dagger x^\dagger xx^\dagger x^\dagger = x^\dagger xx^\dagger = x^0$$ □

It follows from the definition that in a ordered regular pointed semigroup with biggest associates, the classes modulo Green's relation $\mathcal{R}$ and $\mathcal{L}$ have biggest elements which, are idempotent. In the next Theorem we show that the same is true for Green's relation $\mathcal{D}$.

**Theorem 2.4.** *Let $S$ be an ordered regular pointed semigroup with biggest associates. Green's relation $\mathcal{D}$ is given by*

$$(x, y) \in \mathcal{D} \iff x^0 = y^0$$

*and $x^0 = max[x]_{\mathcal{D}}$.*

*Proof.* Let $x$ and $y$ be $\mathcal{D}$-related elements of $S$. Since $\mathcal{D}$ is the composition of $\mathcal{R}$ and $\mathcal{L}$, there exists $z \in S$ such that $x\mathcal{L}z\mathcal{R}y$. Then, $x^\dagger x = z^\dagger z$ and $zz^\dagger = yy^\dagger$ which imply by Theorem 2.3(3) that,

$$x^0 = \max[x^\dagger x]_{\mathcal{R}} = \max[z^\dagger z]_{\mathcal{R}} = z^0$$

and similarly

$$y^0 = \max[yy^\dagger]_{\mathcal{L}} = \max[zz^\dagger]_{\mathcal{L}} = z^0$$

from which we conclude that, $x^0 = y^0$. Conversely, if $x, y \in S$ are such that $x^0 = y^0$, then $x\mathcal{L}x^\dagger x\mathcal{R}x^0 = y^0\mathcal{L}yy^\dagger\mathcal{R}y$, that is $x\mathcal{D}y$. Consequently, $(x, y) \in \mathcal{D}$ if and only if, $x^0 = y^0$.

Finally, by Theorem 2.1(2,4) $x \leq xx^\dagger \leq x^\dagger xx^\dagger = x^0$ and $x^0 = \max[x]_{\mathcal{D}}$. □

## 3 Natural Order

The next result is a source of examples of ordered regular pointed semigroups with biggest associates. With that in mind, we recall that the *natural order* $\leq_n$ on the idempotents of a regular semigroup is defined by

$$e \leq_n f \quad \Longleftrightarrow \quad e = ef = fe$$

and that an ordered regular semigroup $(T; \leq)$ is said to be *naturally ordered* if, the order $\leq$ extends the natural order in the sense that if $e \leq_n f$ then, $e \leq f$. In this case, a fundamental property (see, for example [**1**, Theorem 13.11]) is that if $e \leq f$ then, $e = efe$. In particular, if $T$ has a biggest idempotent $\xi$, then $e\xi e = e$, for all $e \in E(S)$.

**Theorem 3.1.** *If $T$ is a naturally ordered regular semigroup with a biggest idempotent $\xi$ then, the semiband $\langle E(T)\rangle$ is an ordered regular pointed semigroup with biggest associates.*

*Proof.* We first observe that if $\xi$ is the biggest idempotent of $T$, it is necessarily the biggest idempotent of $\langle E(T)\rangle$, in fact its biggest element.
Let $\overline{e} = e_1e_2\cdots e_n \in \langle E(T)\rangle$, with $e_1, e_2, ..., e_n \in E(T)$. From the observation previous to this theorem's statement we can say that, in particular that $e_1\xi e_1 = e_1$ and consequently,

$$\overline{e}\xi\overline{e} = e_1e_2\cdots e_n\xi e_1e_2\cdots e_n \leq e_1\xi\cdots\xi\xi e_1e_2\cdots e_n = e_1\xi e_1e_2\cdots e_n = e_1e_2\cdots e_n = \overline{e}$$

Fitz-Gerald in [**5**] proved that $\langle E(T)\rangle$ is a regular semigroup. Thus, there exists

$$y = z_1z_2...z_m \in \langle E(T)\rangle,$$

with $z_1, z_2, \cdots, z_m \in E(T)$ such that

$$\overline{e} = \overline{e}y\overline{e} = e_1e_2\cdots e_nz_1z_2\cdots z_me_1e_2\cdots e_n \leq e_1e_2\cdots e_n\xi\cdots\xi e_1e_2\cdots e_n = \overline{e}\xi\overline{e}$$

and we conclude that, $\overline{e}\xi\overline{e} = \overline{e}$ that is, each element of $\langle E(T)\rangle$ has a biggest associate more specifically

$$(\forall\overline{e} \in \langle E(S)\rangle) \qquad \overline{e}^\dagger = \xi$$

which implies in particular that $\overline{e} \leq \overline{e}^\dagger$.
Furthermore, since $\xi$ is the biggest element of $\langle E(T)\rangle)$, we have that $e_1 \leq \xi$ and

$$\overline{e}^2 = e_1e_2\cdots e_ne_1e_2\cdots e_n = e_1e_2\cdots e_ne_1e_1e_2\cdots e_n = \overline{e}e_1\overline{e} \leq \overline{e}\xi\overline{e} = \overline{e}$$

Thus, it follows by Theorem 2.1(5) that $\langle E(T)\rangle$ is an ordered regular pointed semigroup with biggest associates. □

In the next results we are going to analyse the existence and the effect of the presence of maximal idempotents or maximal elements, and in particular of a biggest idempotent. We start with a result in a general ordered regular semigroup with biggest associates.

**Lemma 3.2.** *An ordered regular semigroup with biggest associates has at most, one idempotent that is a maximal element.*

*Proof.* Let us consider the idempotents $e$ and $f$ and assume, that they are maximal elements of $S$. We have that

$$f(ef)^0e \cdot e \cdot f(ef)^0e = f(ef)^0e \implies e \leq (f(ef)^0e)^\dagger$$

and since $e$ is maximal in $S$, we can conclude that $e = (f(ef)^0e)^\dagger$. Similarly, we obtain that $f = (f(ef)^0e)^\dagger$.
Therefore, $S$ has at most one idempotent that is a maximal element. □

**Theorem 3.3.** *Let $S$ be an ordered regular pointed semigroup with biggest associates.*
(1) *An element $x \in S$ is a maximal idempotent if and only if, it is a maximal element.*
(2) *$S$ contains at most a maximal element.*

*Proof.* (1) : Let us assume first, that $e$ is a maximal idempotent of $S$. If $x \in S$ is such that $e \leq x$ then, by Theorem 2.1(5 and 6) we have that $e \leq x \leq x^\dagger \in E(S)$, whence by the hypothesis that $e$ is maximal in $E(S)$ gives $e = x = x^\dagger$ which means, that $e$ is a maximal element of $S$. Conversely, if $x \in S$ is a maximal element of $S$ then, $x \leq x^\dagger$ gives $x = x^\dagger \in E(S)$.
(2) : By (1) any maximal element of $S$ has to be idempotent. Thus from Lemma 3.2, we can conclude that such an element if it exists, has to be unique. □

By [**2**, Theorem 2] an ordered regular semigroup with biggest associates $S$ is naturally ordered if and only if, the assignment $x \to x^\dagger$ is weakly antitone that is, if for any $e, f \in E(S)$ such that $e \leq f$ we obtain that $f^\dagger \leq e^\dagger$. In this case, it is proved in [**2**, Theorem 4(2)] that $(xx^\dagger)^\dagger$ and $(x^\dagger x)^\dagger$ are maximal idempotents for all $x \in S$. Using this fact in the case where $S$ is pointed, we obtain the following characterisation.

**Theorem 3.4.** *Let $S$ be an ordered regular pointed semigroup with biggest associates. The following statements are equivalent:*
(1) *$S$ is naturally ordered;*
(2) *$S$ has a biggest idempotent $\xi$ and $x^\dagger = \xi$ for every $x \in S$.*

*Proof.* (1) $\implies$ (2) : By [**2**, Theorem 4] we have for all $x \in S$, that $(xx^\dagger)^\dagger$ and $(x^\dagger x)^\dagger$ are maximal idempotents. Therefore, by Theorem 2.3(1) we have that $x^\dagger$ is a maximal idempotent for all $x \in S$. By [**2**, Theorem 6], there exists in $S$ a biggest idempotent $\xi$ and we can conclude that, $\xi = x^\dagger$ for all $x \in S$.
(2) $\implies$ (1) : Let $e, f \in E(S)$ be such that $e \leq_n f$. By (2) we have that $e^\dagger = \xi = f^\dagger$ and we can conclude that
$$e = ef = fef \leq fe^\dagger f = ff^\dagger f = f$$
which means that $S$ is naturally ordered. □

**Corollary 3.5.** *Let $S$ be an ordered regular pointed semigroup with biggest associates. If $S$ is naturally ordered then Green's relations $\mathcal{D}$ and $\mathcal{J}$ coincide.*

*Proof.* By Theorem 3.4 $(x^2)^\dagger = \xi = x^\dagger$. Then, $x^2 = x^2(x^2)^\dagger x^2 = x^2 x^\dagger x^2 = x^3$ and, consequently $x^2 \in E(S)$. Thus, $S$ is a group bound and it follows by [**6**, Theorem 1.2.20] that $\mathcal{D}$ and $\mathcal{J}$ coincide. □

Let us now see in the next Theorem the effect of assuming that an ordered regular pointed semigroup with biggest associates is also, naturally ordered.

In order to obtain our result, we need to recall that a *strong Dubreil-Jacotin semigroup* (see for example [**1**]) is an ordered semigroup $S$, for which there exists an ordered group $G$ and an epimorphism $f : S \to G$ that is *residuated*, in the sense that the pre-image under $f$ of every principal order ideal of $G$ is a principal order ideal of $S$. In particular, the pre-image of the negative cone $N(G) = \{x \in G | x \leq 1\}$ is a principal order ideal $\xi^\downarrow = \{x \in S | x \leq \xi\}$ of $S$, the so-called *bimaximum element* $\xi$ being *equiresidual* in the sense that, for every $x \in S$ the order ideals $\{y \in S | xy \leq \xi\}$ and $\{y \in S | yx \leq \xi\}$ coincide, and have a greatest element denoted by $\xi : x$. When $S$ is regular, the bimaximum element $\xi$ is the biggest idempotent of $S$ and if, $e$ is an idempotent then, $\xi : e = \xi$.

Let us recall that **PO** is a strict subclass of **BA**. In the pointed situation next Theorem proves, that these classes coincide if we impose the additional hypothesis that $S$ is naturally ordered.

**Theorem 3.6.** *Let $S$ be an ordered regular pointed semigroup with biggest associates. If $S$ is naturally ordered then $S$ is principally ordered, in which $x^* = x^\dagger$ for all $x \in S$.*

*Proof.* From Theorem 2.1(6) we have for every $x \in S$, that $x^\dagger \in E(S)$. We can conclude from [**2**, Theorem 6] that $S$, has a biggest idempotent $\xi$. From Theorem 3.4 we have that $x \leq x^\dagger = \xi$, which allows us to conclude that $\xi$ is the biggest element of $S$.
Now, for every $x \in S$, we have that
$$\xi x \leq \xi\xi = \xi \quad \implies \quad \xi \in \{y \in S | yx \leq \xi\}$$
and
$$x\xi \leq \xi\xi = \xi \quad \implies \quad \xi \in \{y \in S | xy \leq \xi\}$$
from which we obtain that $\xi$ is equiresidual with $\xi : x = \xi$. By [**1**, Corollary to Theorem 13.4], $S$ is a strong Dubreil-Jacotin semigroup and therefore by [**1**, Theorem 13.28] we can conclude that $S$ is principally ordered.
For any $x \in S$ we have that
$$xx^*x = x \implies x^* \in \{y \in S | xyx = x\} \implies x^* \leq x^\dagger$$
and
$$xx^\dagger x = x \implies x^\dagger \in \{y \in S | xyx \leq x\} \implies x^\dagger \leq x^*$$
from which we conclude that $x^* = x^\dagger$. □

Since in an ordered regular semigroup with biggest associates $S$, the condition of being naturally ordered implies that $S$ is principally ordered, there is not much more to investigate under this assumption, since as it can be seen in [**1**], naturally ordered and principally ordered regular semigroups are already well known and studied.

## 4 Compacteness and Identity Element

Let us consider now the subset

$$S^\dagger = \{x^\dagger | x \in S\}.$$

This set is related to the subset $S^0 = \{x^0 | x \in S\}$ and to the set $C = \{x \in S | x^\dagger = x^0\}$ of *compact elements* as follows.

**Theorem 4.1.** *If $S$ is an ordered regular pointed semigroup with biggest associates then,*

$$S^\dagger = C \cap S^0.$$

*Proof.* We have using the identity $x^{\dagger\dagger} = x^{\dagger 0}$ that, $S^\dagger \subseteq C$. Similarly, $x^\dagger = x^{\dagger\dagger\dagger} = x^{\dagger\dagger 0}$ show that, $S^\dagger \subseteq S^0$.
Conversely, if $x \in C \cap S^0$ then, $x^\dagger = x^0$ and $x = x^{00}$ from which we obtain that

$$x = x^{00} = x^{\dagger 0} = x^{\dagger\dagger} \in S^\dagger.$$ □

We now recall an example that can be found in [**4**, Example 1] which illustrates that $S^\dagger$ is not in general a subsemigroup of $S$.

**Example 4.2.** Let $L$ be a lattice and consider the cartesian ordered set

$$L^{[2]} = \{(x, y) \in L \times L | y \leq x\}$$

With respect to the multiplication defined by

$$(x, y)(a, b) = (x \vee a, y \wedge b)$$

it is clear that $L^{[2]}$ is an ordered band. Since,

$$(x, y)(a, b)(x, y) = (x, y) \iff (x \vee a, y \wedge b) = (x, y)$$

we can conclude that $a \leq x$ and $y \leq b$, and therefore $(x, y)^\dagger = (x, x)$. Then, each element of $L^{[2]}$ has a biggest associate and we can write

$$(L^{[2]})^\dagger = \{(x, x) | x \in L\}$$

The fact that $L^{[2]}$ is a band implies in particular, that it is pointed.
Now,

$$(x, y)^\dagger (a, b)^\dagger = (x, x)(a, a) = (x \vee a, x \wedge a)$$

which is an element of $(L^{[2]})^\dagger$ if and only if, $x = a$. Thus, $(L^{[2]})^\dagger$ is not a subsemigroup of $L^{[2]}$.

In the presence of an identity element 1 in an ordered regular pointed semigroup with biggest associates $S$, it is possible to prove that the subset $S^\dagger$ is a subsemigroup of $S$ and has an interesting description.

**Theorem 4.3.** *Let $S$ be an ordered regular pointed monoid with biggest associates and identity element 1. Then,*

$$S^\dagger = \{x \in S | 1 \leq x\}$$

*and is a join semilattice in which $x \vee y = xy$.*

*Proof.* Let $x \in S$ be such that $1 \leq x$. Then,

$$x^\dagger = \begin{cases} 1x^\dagger \leq xx^\dagger \leq x^\dagger x^\dagger = x^\dagger \implies x^\dagger = xx^\dagger \mathcal{R} x \\ x^\dagger 1 \leq x^\dagger x \leq x^\dagger x^\dagger = x^\dagger \implies x^\dagger = x^\dagger x \mathcal{L} x \end{cases}$$

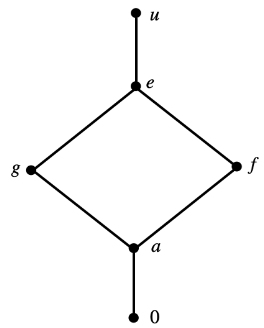
and therefore $(x^\dagger, x) \in \mathcal{H}$, which by Theorem 2.1(9) gives us, since $\mathcal{H}$ is the equality relation, that $x = x^\dagger \in S^\dagger$. Thus, $\{x \in S | 1 \leq x\} \subseteq S^\dagger$. Conversely, if $x \in S^\dagger$ we have by Theorem 2.1(6) that $x^\dagger = x^{\dagger\dagger}$ is an idempotent. Then,

$$x^\dagger \cdot 1 \cdot x^\dagger = x^\dagger x^\dagger = x^\dagger \quad \implies \quad 1 \leq x^{\dagger\dagger} = x^\dagger$$

which means that $S^\dagger \subseteq \{x \in S | 1 \leq x\}$ and therefore $S^\dagger = \{x \in S | 1 \leq x\}$. For any elements of $S$ greater or equal than 1, it is obvious that their product is also greater or equal than 1, which means that $S^\dagger$ is a sub-band of $S$.

Now, for any $x, y \in S^\dagger$, we immediately have that $x = x \cdot 1 \leq xy$ and $y = 1 \cdot y \leq xy$, which means that $xy$ is an upper bound of $x$ and $y$. If $z \in S$ is an upper bound of $\{x, y\}$ then, $1 \leq x \leq z$ implies that $z \in S^\dagger$. Thus, $xy \leq zz = z$ which implies that $xy$ is the join of $\{x, y\}$. Consequently, $S^\dagger$ is a join semilattice in which $x \vee y = xy$. □

**Example 4.4.** In [**4**, Example 2] Blyth and Pinto presented a semigroup constructed using the isotone mappings from a three element chain into itself, preserving the bottom element. It is denoted by Res **3** and it can be defined by the following Hasse diagram and Cayley table:




| | $u$ | $e$ | $f$ | $g$ | $a$ | 0 |
|---|---|---|---|---|---|---|
| $u$ | $u$ | $u$ | $u$ | $g$ | $g$ | 0 |
| $e$ | $u$ | $e$ | $f$ | $g$ | $a$ | 0 |
| $f$ | $f$ | $f$ | $f$ | $a$ | $a$ | 0 |
| $g$ | $u$ | $g$ | 0 | $g$ | 0 | 0 |
| $a$ | $f$ | $a$ | 0 | $a$ | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

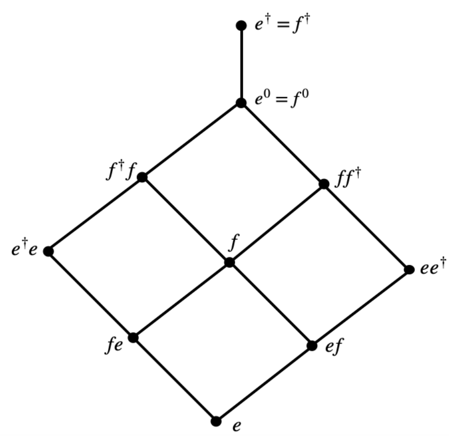
In this semiband, where $e$ is the identity element, every element has a biggest associate namely, $0^\dagger = a^\dagger = f^\dagger = g^\dagger = u^\dagger = u$ and $e^\dagger = e$. Clearly, Res **3** is pointed with $(Res\mathbf{3})^\dagger = \{e, u\}$ as a subsemigroup.

## 5 $\mathcal{D}$ classes

Let us now look more closely to the $\mathcal{D}$ classes of an ordered regular pointed semigroup with biggest associates, $S$. We first focus on the subalgebra of $(S,^\dagger)$ generated by $\{e, f\}$ where $e, f$ are idempotents such that $e \leq f$ and $(e, f) \in \mathcal{D}$.

The next two Theorems were obtained in [**4**, Theorems 9 and 10] in a pointed principally ordered regular semigroup and are now generalised to ordered regular pointed semigroup with biggest associates.

**Theorem 5.1.** *Let $S$ be an ordered regular pointed semigroup with biggest associates and $e, f$ idempotents of $S$, such that $e \leq f$ and $e\mathcal{D}f$. If $T$ is the subalgebra of $(S,^\dagger)$ generated by $\{e, f\}$ then $T$ is a band having at most 10 elements. In the case where $T$ has precisely* 10 *elements it is represented by the Hasse diagram*




*in which elements joined by lines of positive slope are $\mathcal{R}$-related, those joined by lines of negative slope are $\mathcal{L}$-related, and the vertical line also indicates $\leq_n$.*

*Proof.* Since $e\mathcal{D}f$ it follows from Theorem 2.4 that $e^0 = f^0$ whence using $(P_7)$,

$$e^\dagger = e^{\dagger\dagger\dagger} = e^{0\dagger\dagger} = f^{0\dagger\dagger} = f^{\dagger\dagger\dagger} = f^\dagger$$

The elements of $T$ are the finite products of the elements $e$, $f$ and $e^\dagger$ (which is equal to $f^\dagger$). Moreover, since $e \le f$ we have that every $x \in T$ is such that $e \le x \le e^\dagger$. By [**7**, Theorem 7] we can conclude that $e$ and $f$ are mutually inverses so, we obtain

$$f = fef \le fxf \le fe^\dagger f = ff^\dagger f = f \quad \Longrightarrow \quad fTf = f$$
$$e = eee \le exe \le ee^\dagger e = e \quad \Longrightarrow \quad eTe = e$$
$$ee^\dagger = eee^\dagger \le exe^\dagger \le ee^\dagger e^\dagger = ee^\dagger \quad \Longrightarrow \quad eTe^\dagger = ee^\dagger$$
$$ff^\dagger = feff^\dagger \le fxf^\dagger f^\dagger \le fe^\dagger f^\dagger f^\dagger = ff^\dagger f^\dagger f^\dagger = ff^\dagger \quad \Longrightarrow \quad fTf^\dagger = ff^\dagger$$
$$ef = eef \le exf = eexf \le efe^\dagger f = eff^\dagger f = ef \quad \Longrightarrow \quad eTf = ef$$

Similarly, we can obtain that $e^\dagger Te = e^\dagger e$, $f^\dagger Tf = f^\dagger f$ and $fTe = fe$.
Now, in order to conclude what $e^\dagger Te^\dagger$ is equal to, let us consider an element $y$ in $e^\dagger Te^\dagger$ which is of the form $y = e^\dagger k_1 k_2 \cdots k_n e^\dagger$, where $k_1, k_2, ..., k_n \in \{e, f, e^\dagger\}$.
We need to consider several cases. If $e \in \{k_1, k_2, ..., k_n\}$ we have

$$e^0 = e^\dagger ee^\dagger = e^\dagger ee \cdots ee^\dagger \le e^\dagger k_1 k_2 \cdots k_n e^\dagger \le e^\dagger ee^\dagger = e^0 \quad \Longrightarrow \quad y = e^0$$

If on the contrary, $e \notin \{k_1, k_2, ..., k_n\}$ and also $f \notin \{k_1, k_2, ..., k_n\}$ then we have that $k_1, k_2, ..., k_n$ are all equal to $e^\dagger$ and we conclude that $y = e^\dagger k_1 k_2 \cdots k_n e^\dagger = e^\dagger$.
Finally if $e \notin \{k_1, k_2, ..., k_n\}$, but $f \in \{k_1, k_2, ..., k_n\}$ then

$$y = e^\dagger k_1 k_2 \cdots k_n e^\dagger \le f^\dagger ff^\dagger = f^0 = e^0 = e^\dagger ee^\dagger \le e^\dagger k_1 k_2 \cdots k_n e^\dagger = y$$

Therefore, we can conclude that $e^\dagger Te^\dagger = \{e^0, e^\dagger\}$.
Now, it follows from the above that $T$ is a band that consist of at most 10 elements, with two $\mathcal{D}$ classes $D_e = T \setminus \{e^\dagger\}$ and $D_{e^\dagger} = \{e^\dagger\}$ and is described by the above Hasse diagram. □

**Example 5.2.** Consider the ordered semigroup $\mathbf{B}_2$ of $2 \times 2$ matrices with entries in a boolean algebra $\mathbf{B}$, where the notation for the basic operations in $\mathbf{B}$ is $a + b$ (for $a \vee b$) and $ab$ (for $a \wedge b$). With matrix multiplication it is shown in [**1**] that $\mathbf{B}_2$ is a regular semigroup. In [**4**, Example 3] it is proved that

$$\langle E(\mathbf{B}_2)\rangle = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \quad bc \le ad \right\}$$

is a pointed principally ordered regular semigroup with

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} 1 & a' + d' + b \\ a' + d' + c & 1 \end{bmatrix} \in E(\mathbf{B}_2)$$

and therefore, we can say that it is an ordered regular pointed semigroup with biggest associates with

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} 1 & a' + d' + b \\ a' + d' + c & 1 \end{bmatrix}$$

In $\mathbf{B}_2$ the elements

$$e = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} a & b \\ b & b \end{bmatrix}$$

where $0 < b < a < 1$ are $\mathcal{D}$-related idempotents for which $e < f$.
It can be verified like in [**4**, Example 4], that if $|\mathbf{B}| \ge 8$ then, the subalgebra of $\langle E(\mathbf{B}_2)\rangle$ generated by $\{e, f\}$ has the Hasse diagram presented in Theorem 5.1.

In the next Theorem, we describe the structure of the $\mathcal{D}$ classes that are subsemigroups of an ordered regular pointed semigroup with biggest associates. Note that in the previous Theorem, the $\mathcal{D}$ class of $e$, is a subsemigroup of $T$.

**Theorem 5.3.** *Let $S$ be an ordered regular pointed semigroup with biggest associates. Given $e \in E(S)$ suppose that $D_e$ is a subsemigroup of $S$. Then $L_{e^0}$ is a left zero semigroup, $R_{e^0}$ is a right zero semigroup, and $D_e$ is isomorphic to the ordered rectangular band $L_{e^0} \times R_{e^0}$.*

*Proof.* We first note that from Theorem 2.4, two elements of $S$ are $\mathcal{D}$ related if and only if, they have the same biggest inverse and therefore by Theorem 2.1(6), the same biggest associate. Considering $e \in E(S)$ we have that $e\mathcal{D}e^0$ since $e^0 = e^{00}$. Let us take any element $x$ in $\mathcal{D}_e$. We have by $(P_7)$ that

$$x \leq x^{00} = (x^0)^0 = (e^0)^0 = e^{00} = e^0$$

which means that $\mathcal{D}_e$ has a biggest element $e^0$, in particular it is its biggest idempotent.
If $e, f$ are idempotents in $D_e$ such that $e \leq_n f$, we have since $e\mathcal{D}f$ that $e^\dagger = f^\dagger$, which implies

$$e = fe = fef \leq fe^\dagger f = ff^\dagger f = f$$

and therefore we can conclude that $D_e$ is a naturally ordered regular semigroup with a biggest idempotent $e^0$.
For any $x, y \in D_e$, we have that $(xy)^0 = e^0 = e^0e^0 = y^0x^0$, from which we conclude by [**1**, Theorem 13.18] that $e^0$ is a middle unit of $D_e$, that is, $xe^0y = xy$. Now, let us consider an arbitrary element $x$ in $\mathcal{D}_e$. Note that

$$x \in L_{e^0} \iff x^0x = e^0 \iff x = xx^0 \in D_e$$

and

$$x \in R_{e^0} \iff xx^0 = e^0 \iff x = x^0x \in D_e$$

from which we obtain for $x, y \in L_{e^0}$ that

$$xy = xx^0y = xe^0y = xy^0y = xe^0 = xx^0 = x$$

and consequently $L_{e^0}$ is a left zero semigroup. Similarly, $R_{e^0}$ is a right zero semigroup, and therefore we can consider the rectangular band

$$L_{e^0} \times R_{e^0} = \{(xe^0, e^0y) | x, y \in D_e\}$$

We can then consider the mapping $\vartheta : D_e \to L_{e^0} \times R_{e^0}$ defined by $\vartheta(x) = (xe^0, e^0x)$, which is obviously isotone.
For $(a, b) \in L_{e^0} \times R_{e^0}$ we have that $ab \in D_e$ and therefore

$$\vartheta(ab) = (abe^0, e^0ab) = (abb^0, a^0ab) = (ae^0, e^0b) = (aa^0, b^0b) = (a, b)$$

proves that $\vartheta$ is surjective. Moreover, since

$$\vartheta(x) \leq \vartheta(y) \iff xe^0 \leq ye^0, \quad e^0x \leq e^0y \iff x = xe^0x \leq ye^0y = y$$

we can conclude that $\vartheta$ is an order isomorphism.
Finally, using the fact that $e^0$ is a middle unit of $D_{e^0}$ we obtain that $\vartheta$ is a morphism:

$$\vartheta(x)\vartheta(y) = (xe^0, e^0x)(ye^0, e^0y) = (xe^0ye^0, e^0xe^0y) = (xye^0, e^0xy) = \vartheta(xy)$$

Thus, $\vartheta$ defines an ordered semigroup isomorphism $D_e \simeq L_{e^0} \times R_{e^0}$. □

## References

[1] T. S. Blyth; *Lattices and Ordered Algebraic Structures*. Springer, 2005.
doi:10.1017/b139095

[2] T. S. Blyth and M. H. Almeida Santos; *Ordered regular semigroups with biggest associates*. Discussiones Mathematicae, General Algebra and Applications, 2019, **39**, 5-21.
doi:10.7151/dmgaa.1299

[3] T. S. Blyth and G. A. Pinto; *Principally ordered regular semigroups*. Glasgow Math. J., 1990, **32**, 349-364.
doi:10.1017/S0017089500009435

[4] T. S. Blyth and G. A. Pinto; *Pointed principally ordered regular semigroups*. Discussiones Mathematicae, General Algebra and Applications, **36**, 2016, 101-111.
doi:10.7151/dmgaa.1243

[5] D. G. Fitz-Gerald; *On inverses of products of idempotents in regular semigroups*. J. of the Australian Math. Soc., **13**, 1972, 335-337.
doi:10.1017/S1446788700013756

[6] P. M. Higgins; *Techniques of Semigroup Theory*. Oxford Science Publications, 1992.
doi:10.1007/BF02573500

[7] G. A. Pinto; *Eventually pointed principally ordered regular semigroups*. SQU Journal for Science, **24(2)**, 2019, 139-146.
doi:10.24200/squjs.vol24iss2pp139-146

### Author information

G. A. Pinto, Department of Mathematics, College of Science, Sultan Qaboos University, Muscat, Oman.
E-mail: gcapinto@gmail.com

Received: November 7, 2020.
Accepted: January 21, 2021.