Palestine Journal of Mathematics

Vol. 11(1)(2022) , 280–284

UNIQUENESS THEOREM FOR MEROMORPHIC FUNCTIONS AND DIFFERENTIAL POLYNOMIALS SHARE ONE VALUE WITH FINITE WEIGHT

Rajeshwari S., Husna V., and Nagarjun V.

Communicated by H. M. Srivastava

MSC 2010 Classifications: Primary 30D35

Keywords and Phrases: Uniqueness, Meromorphic functions, Differential Polynomials.

Abstract In this present research article, we investigate the uniqueness problems of meromorphic functions concerning differential polynomials sharing a value with finite weight and give some results which improves and generalizes the several earlier results of Jin-Dong Li [10].

1 Introduction

Throughout of this article, we shall use the following standard notations of Nevanlinna's Value Distribution Theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$ etc...see Haymann [1], Yang [2], and Yi and Yang [3]. A meromorphic function $g(z)$ is said to be rational if and only if $T(r,g) = O(log\, r)$, otherwise, $g(z)$ is called a transcendental meromorphic function. Let $f(z)$ be transcendental meromorphic function, defined in the complex plane $\mathbb{C}$. We denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = o\{T(r,f)\}, \;\; as \;\; r \longrightarrow \infty, \;\; r \notin E$$

where E is a subset of positive real numbers of finite linear measure, not necessarily the same at each occurence.

For any constant a we define,

$$\Theta(a,f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r,f)}.$$

where $\overline{N}\left(r, \frac{1}{f-a}\right)$ is the reduced counting function which counts zeros of $f(z) - a$ in $|z| \leq r$, counted only once. Let $f(z)$ and $g(z)$ be non-constant two meromorphic functions. Let a be any finite complex number. If $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities then we say that $f(z)$ and $g(z)$ share the value a CM (Counting Multiplicities) and we say that $f(z)$ and $g(z)$ share the value a IM (Ignoring Multiplicity) if we do not consider the multiplicities.

Definition 1.1. (see [12]) A meromorphic function $b(z)$ $(\not\equiv 0, \infty)$ defined in $\mathbb{C}$ is called a "small function" with respect to $f(z)$ if $T(r, b(z)) = S(r,f)$.

Definition 1.2. (see [12]) Let k be a positive integer, for any constant a in the complex plane $\mathbb{C}$. We denote
(i) by $N_{k)}\left(r, \frac{1}{f-a}\right)$ the counting function of a-points of $f(z)$ with multiplicity $\leq k$.
(ii) by $N_{(k}\left(r, \frac{1}{f-a}\right)$ the counting function of a-points of $f(z)$ with multiplicity $\geq k$.
Similarly, the reduced counting functions $\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)$ and $\overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$ are defined.

Recently, Jin-Dong Li [10] proved the following theorems.

Theorem 1.3. *(see [10]) Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $n,\ k$ be two positive integers with $n > 3k + 11$. If $\Theta(\infty, f) > \frac{2}{n}$, $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g-1)]^{(k)}$ share 1 (1,2) then $f(z) \equiv g(z)$ or $[f^n(z)(f(z)-1)]^{(k)}[g^n(z)(g(z)-1)]^{(k)} \equiv 1$.*

Theorem 1.4. *(see [10]) Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $n,\ k$ be two positive integers with $n > 5k + 14$. If $\Theta(\infty, f) > \frac{2}{n}$, $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share 1 (1,1) then $f(z) \equiv g(z)$ or $[f^n(z)(f(z)-1)]^{(k)}[g^n(z)(g(z)-1)]^{(k)} \equiv 1$.*

Now, we generalize the above results and obtained the following theorems.

Theorem 1.5. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, $P(f)$ and $P(g)$ be a polynomials of degree m and let $n,\ k$ be two positive integers with $t(n+m) > 3k+8$. If $\Theta(\infty, f) > \frac{2}{n+m}$, $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share 1 (1,2), then either $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv 1$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g) = 0$ where*

$$R(w_1, w_2) = w_1^m(a_mw_1^m + a_{m-1}w_1^{m-1} + ... + a_0) - w_2^m(a_mw_2^m + a_{m-1}w_2^{m-1} + \cdots + a_0).$$

Theorem 1.6. *Let $f(z)$ and $g(z)$ be a non-constant meromorphic functions, $P(f)$ and $P(g)$ be a polynomials of degree m and let $n,\ k$ be two positive integers with $t(n+m) > 5k+10$. If $\Theta(\infty, f) > \frac{2}{n+m}$, $[f^nP(f)]^{(k)}$ and $[f^nP(f)]^{(k)}$ share 1 (1,1) then $f(z) \equiv g(z)$ or $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv 1$.*

2 Some Lemmas

To prove our result we need the following Lemmas.

Lemma 2.1. *(see [1]) Let $f(z)$ be a non-constant meromorphic function, and let $a_0,\ a_1, \ldots, a_n$ be finite complex numbers such that $a_n \neq 0$. Then*

$$T(r, a_nf^n + a_{n-1}f^{n-1} + ... + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2. *(see [1]) Let $f(z)$ be a non-constant meromorphic function and k be a positive integer and c a non-zero finite complex number. Then*

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, f) + N\Big(r, \frac{1}{f}\Big) + N\Big(r, \frac{1}{f^{(k)} - c}\Big) - N\Big(r, \frac{1}{f^{(k+1)}}\Big) + S(r, f) \\ &\leq \overline{N}(r, f) + N_{k+1}\Big(r, \frac{1}{f}\Big) + \overline{N}\Big(r, \frac{1}{f^{(k)} - c}\Big) - N_0\Big(r, \frac{1}{f^{(k+1)}}\Big) + S(r, f). \end{aligned}$$

where $N_0\Big(r, \frac{1}{f^{(k+1)}}\Big)$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but note that $f(f^{(k)} - c) \neq 0$.

Lemma 2.3. *(see [8]) Let $f(z)$ be a non-constant meromorphic function, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N\Big(r, \frac{1}{f^{(k)}}\Big) \leq N\Big(r, \frac{1}{f}\Big) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.4. *(see [11]) Let $f(z)$ be a non-constant meromorphic function and $s,\ k$ be any two positive integers. Then*

$$N_s\Big(r, \frac{1}{f^{(k)}}\Big) \leq k\overline{N}(r, f) + N_{s+k}\Big(r, \frac{1}{f}\Big) + S(r, f).$$

Clearly, $\overline{N}\Big(r, \frac{1}{f^{(k)}}\Big) = N_1\Big(r, \frac{1}{f^{(k)}}\Big)$.

Lemma 2.5. *(see [1]) Let $f(z)$ be a transcendental meromorphic function, and let $a_1(z),\ a_2(z)$ be two meromorphic functions such that $T(r,a_i)=S(r,f)$, $i=1,\ 2\ldots,n$. Then*

$$T(r,f)\le \overline{N}(r,f)+\overline{N}\Big(r,\frac{1}{f-a_1}\Big)+\overline{N}\Big(r,\frac{1}{f-a_2}\Big)+S(r,f).$$

Lemma 2.6. *(see [10]) Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $k\ge 1$, $l\ge 1$ be two positive integers. Suppose that $f^{(k)}$ and $g^{(k)}$ share $(1,l)$,*
(i) If $l=2$ and

$$\Delta_1=2\Theta(\infty,f)+(k+2)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)>k+7.$$

then either $f^{(k)}g^{(k)}\equiv 1$ or $f(z)\equiv g(z)$.
(ii) If $l=1$ and

$$\Delta_2=(k+3)\Theta(\infty,f)+(k+2)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+2\delta_{k+1}(0,f)+\delta_{k+1}(0,g)>2k+9.$$

then either $f^{(k)}g^{(k)}\equiv 1$ or $f(z)\equiv g(z)$.

3 Proof of Theorems

Proof of Theorem 1.5.

Proof. Let $F(z)=f^nP(f)$ and $G(z)=g^nP(g)$. We have from Lemma 2.6,

$$\Delta_1=2\Theta(\infty,f)+(k+2)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+\delta_{k+1}(0,f)+\delta_{k+1}(0,g). \tag{3.1}$$

$$\overline{N}\Big(r,\frac{1}{F}\Big)=\overline{N}\Big(r,\frac{1}{f^nP(f)}\Big)\le\frac{1}{t(n+m)}N(r,\frac{1}{F})\le\frac{1}{t(n+m)}(T(r,F)+O(1)). \tag{3.2}$$

Therefore,

$$\Theta(0,f)=1-\overline{\lim_{r\to\infty}}\frac{N(r,\frac{1}{f})}{T(r,f)}\ge 1-\frac{1}{t(n+m)}. \tag{3.3}$$

Similarly,

$$\delta_{k+1}(0,g)\ge 1-\frac{k+1}{t(n+m)}. \tag{3.4}$$

$$\delta_{k+1}(0,f)\ge 1-\frac{k+1}{t(n+m)}. \tag{3.5}$$

$$\Theta(\infty,f)\ge 1-\frac{1}{t(n+m)}. \tag{3.6}$$

$$\Theta(\infty,g)\ge 1-\frac{1}{t(n+m)}. \tag{3.7}$$

From the inequalities (3.3)-(3.7), we get,

$$\Delta_1\ge 2\Big(1-\frac{1}{t(n+m)}\Big)+(k+2)\Big(1-\frac{1}{t(n+m)}\Big)+\Big(1-\frac{1}{t(n+m)}\Big)+\Big(1-\frac{1}{t(n+m)}\Big)$$
$$+2\Big(1-\frac{k+1}{t(n+m)}\Big). \tag{3.8}$$

Since $t(n+m)>3k+8$, we get $\Delta_1>k+7$. Considering that $F^{(k)}$ and $G^{(k)}$ share $(1,2)$, then by Lemma 2.6 we deduce that either $F^{(k)}G^{(k)}\equiv 1$ or $F\equiv G$.

Next, we consider the following two cases.

Case 1. $F^{(k)}G^{(k)} \equiv 1$, that is

$$[f^nP(f)]^{(k)}[g^nP(f)]^{(k)} \equiv 1. \tag{3.9}$$

Case 2. $F \equiv G$, that is

$$f^nP(f) = g^nP(g). \tag{3.10}$$

Suppose that $f \not\equiv g$, then we consider following two cases:

(i) Let $h = \frac{f}{g}$ be a constant. Then from (3.10) it follows that $h^n \neq 1,\ h^{n+m} \neq 1,\ h^{n+m-1} \neq 1$ and $a_m g^{n+m}(1-h^{n+m}) + \ldots + a_0 g^n(1-h^n) = 0$ which implies $h^d = 1$ where $d = (n+m,\ n+m-i\ldots n)$, $a_{m-i} \neq 0$, for some $i = 0,\ 1\ldots m$.

(ii) If h is not constant, then we know by (3.10) that f and g satisfy the algebraic equation $R(f,g) = 0$, where

$$R(w_1,w_2) = w_1^m(a_m w_1^m + a_{m-1}w_1^{m-1} + \ldots + a_0) - w_2^m(a_m w_2^m + a_{m-1}w_2^{m-1} + \ldots + a_0).$$

It follows that,

$$T(r,f) = T(r,gh) = (n+m)T(r,h) + S(r,f).$$

On the other hand, by the second fundamental theorem of Nevanlinna we get,

$$\overline{N}(r,f) = \sum_{j=1}^{N}\overline{N}(r,\frac{1}{h-a_j}) \geq (n+m-2)T(r,h) + S(r,f).$$

where $a_j(\neq 1)$ $(j = 1,\ 2\ldots,n)$ are the distinct roots of the algebraic equation $h^{n+m} = 1$. So, we have

$$\begin{aligned}\Theta(\infty,f) &= 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{T(r,f)}\\ &\leq 1 - \overline{\lim_{r\to\infty}}\frac{(n+m-2)T(r,g)+S(r,g)}{(n+m)T(r,h)}\\ &\leq 1 - \frac{n+m-2}{n+m}\\ &\leq \frac{2}{n+m}.\end{aligned}$$

which contradicts to the assumption that $\Theta(\infty,f) > \frac{2}{n+m}$. Thus $F \equiv G$. Hence the proof of Theorem 1.5. □

Proof of Theorem 1.6.

Proof. From (3.3)-(3.7) and by Lemma 2.6, we get,

$$\Delta_2 \geq (k+3)\Big(1-\frac{1}{t(n+m)}\Big) + (k+2)\Big(1-\frac{1}{t(n+m)}\Big) + \Big(1-\frac{1}{t(n+m)}\Big) + \Big(1-\frac{1}{t(n+m)}\Big) + 2\Big(1-\frac{k+1}{t(n+m)}\Big) + \Big(1-\frac{k+1}{t(n+m)}\Big).$$

On simplyfying, the above expression, we get,

$$\Delta_2 \geq 5k + 10 - \frac{5k+10}{t(n+m)}.$$

Since $t(n+m) > 5k+10$, we get $\Delta_1 > 2k+9$. Considering that $F^{(k)}$ and $G^{(k)}$ share (1,1) then by Lemma 2.6, we deduce that either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$. Next, by proceeding as in Theorem 1.5, we obtain the conclusion of Theorem 1.6. Here we omit the details. □

References

[1] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[2] L. Yang, *Value distribution theory*, translated and revised from the 1982 Chinese original, Springer-Verlag, Berlin, 1993.

[3] H. X. Yi and C. C. Yang, Uniqueness theory of Meromorphic functions, Science press Beijing, China, 1995.

[4] C. Y. Fang and M. L. Fang, Uniqueness of meromorphic functions and differential polynomials, Comput. Math. Appl. **44** (2002), no. 5-6, 607–617.

[5] M. L. Fang, Uniqueness and value-sharing of entire functions, Comput. Math. Appl. **44** (2002), no. 5-6, 823–831.

[6] L. Liu, Uniqueness of meromorphic functions and differential polynomials, Comput. Math. Appl. **56** (2008), no. 12, 3236–3245.

[7] W. C. Lin and H. X. Yi, Uniqueness theorems for meromorphic function, Indian J. Pure Appl. Math. **35** (2004), no. 2, 121–132.

[8] S. S. Bhoosnurmath and R. S. Dyavanal, Uniqueness and value-sharing of meromorphic functions, Comput. Math. Appl. **53** (2007), no. 8, 1191–1205.

[9] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, Nagoya mathematical journal vol. 161, pp 193-206, 2001.

[10] J. D. Li, Uniqueness of meromorphic functions and differential polynomials, Int. J. Math. Math. Sci. **2011**, Art. ID 514218, 12 pp.

[11] T. Zhang and W. Lü, Uniqueness theorems on meromorphic functions sharing one value, Comput. Math. Appl. **55** (2008), no. 12, 2981–2992.

[12] S. S. Bhoosnurmath, B. Chakraborty and H. M. Srivastava, A note on the value distribution of differential polynomials, Commun. Korean Math. Soc. **34** (2019), no. 4, pp.1145–1155.

[13] V. Husna, S. Rajeshwari and S. H. Naveen Kumar, A note on uniqueness of transcendental entire functions concerning differential-difference polynomials of finite order, Electron. J. Math. Anal. Appl. **9** (2021), no. 1, 248–260.

[14] V. Husna, Results on difference polynomials of an entire function and its kth derivative shares a small function J. Phys.: Conf. Ser. 1597 (2020), 012026.

[15] P.W. Harina, V. Husna, Results on uniqueness of product of certain type of difference polynomials, Adv. Stud. Contemp. Math. 31 (2021), 67 - 74.

[16] Husna V. and Veena, Results on meromorphic and entire functions sharing CM and IM with their difference operators, J.Math.Comput. Sci.11(2021),**No.4**, 5012-5030.

[17] Husna V., Some results on uniqueness of meromorphic functions concerning differential polynomials, The Journal of Analysis, Vol.**29** April(2021).

[18] V. Husna, S. Rajeshwari and S.H. Naveen Kumar, Results on Uniqueness of product of certain type of shift polynomials, Poincare J. Anal. Appl. **7** (2020), 197-210.

Author information

Rajeshwari S., Husna V., and Nagarjun V.,
Corresponding author: Husna V.
Department of Mathematics, School of Engineering, Presidency University, Itgalpura, Rajanukunte, Yelahanka, Bengaluru-560 064, INDIA.
E-mail: `rajeshwaripreetham@gmail.com, rajeshwari.s@presidencyuniversity.in`

`husnav43@gmail.com, husna@presidencyuniversity.in`

`phalguniarjun95@gmail.com`

Received: July 23, 2020.
Accepted: June 3, 2021