UNIQUENESS THEOREM FOR MEROMORPHIC FUNCTIONS AND DIFFERENTIAL POLYNOMIALS SHARE ONE VALUE WITH FINITE WEIGHT

Rajeshwari S., Husna V., and Nagarjun V.

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Abstract In this present research article, we investigate the uniqueness problems of meromorphic functions concerning differential polynomials sharing a value with finite weight and give some results which improves and generalizes the several earlier results of Jin-Dong Li [10].

1 Introduction

Throughout of this article, we shall use the following standard notations of Nevanlinna’s Value Distribution Theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $N(r, f)$ etc... see Haymann [1], Yang [2], and Yi and Yang [3]. A meromorphic function $g(z)$ is said to be rational if and only if $T(r, g) = O(\log r)$, otherwise, $g(z)$ is called a transcendental meromorphic function. Let $f(z)$ be transcendental meromorphic function, defined in the complex plane $\mathbb{C}$. We denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o\{T(r, f)\}, \quad as \quad r \to \infty, \quad r \notin E$$

where $E$ is a subset of positive real numbers of finite linear measure, not necessarily the same at each occurrence.

For any constant $a$ we define,

$$\Theta(a, f) = 1 - \lim_{r \to \infty} \frac{N_r(1/f - a)}{T(r, f)}.$$ 

where $N(r, 1/f - a)$ is the reduced counting function which counts zeros of $f(z) - a$ in $|z| \leq r$, counted only once. Let $f(z)$ and $g(z)$ be non-constant two meromorphic functions. Let $a$ be any finite complex number. If $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities then we say that $f(z)$ and $g(z)$ share the value $a$ CM (Counting Multiplicities) and we say that $f(z)$ and $g(z)$ share the value $a$ IM (Ignoring Multiplicity) if we do not consider the multiplicities.

**Definition 1.1.** (see [12]) A meromorphic function $b(z)$ ($\neq 0, \infty$) defined in $\mathbb{C}$ is called a “small function” with respect to $f(z)$ if $T(r, b(z)) = S(r, f)$.

**Definition 1.2.** (see [12]) Let $k$ be a positive integer, for any constant $a$ in the complex plane $\mathbb{C}$. We denote

(i) by $N_k(r, 1/f - a)$ the counting function of $a$-points of $f(z)$ with multiplicity $\leq k$.

(ii) by $N_k(r, 1/f - a)$ the counting function of $a$-points of $f(z)$ with multiplicity $\geq k$.

Similarly, the reduced counting functions $N_k(r, 1/f - a)$ and $N_k(r, 1/f - a)$ are defined.
Recently, Jin-Dong Li [10] proved the following theorems.

**Theorem 1.3. (see [10])** Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions, and let \( n, k \) be two positive integers with \( n > 3k + 11 \). If \( \Theta(\infty, f) > \frac{2}{n+m}, [f^n(z)(f(z) - 1)]^{[k]} \) and \([g^n(z)(g(z) - 1)]^{[k]} \) share \( I(1,2) \) then \( f(z) \equiv g(z) \) or \([f^n(z)(f(z) - 1)]^{[k]}[g^n(z)(g(z) - 1)]^{[k]} \equiv 1 \).

**Theorem 1.4. (see [10])** Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions, and let \( n, k \) be two positive integers with \( n > 5k + 14 \). If \( \Theta(\infty, f) > \frac{2}{n+m}, [f^n(z)(f(z) - 1)]^{[k]} \) and \([g^n(z)(g(z) - 1)]^{[k]} \) share \( I(1,1) \) then \( f(z) \equiv g(z) \) or \([f^n(z)(f(z) - 1)]^{[k]}[g^n(z)(g(z) - 1)]^{[k]} \equiv 1 \).

Now, we generalize the above results and obtained the following theorems.

**Theorem 1.5.** Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions, \( P(f) \) and \( P(g) \) be polynomials of degree \( m \) and let \( n, l \) be two positive integers with \( \ell(n + m) > 3k + 8 \). If \( \Theta(\infty, f) > \frac{2}{n+m}, [f^n P(f)]^{[k]} \) and \([g^n P(g)]^{[k]} \) share \( I(1,2) \), then either \([f^n P(f)]^{[k]}[g^n P(g)]^{[k]} \equiv 1 \) or \( f(z) \) and \( g(z) \) satisfy the algebraic equation \( R(w_1, w_2) = 0 \).

\[
R(w_1, w_2) = w_1^n (a_1 w_1^n + a_2 w_2^{n-1} + \ldots + a_0) - w_1^n (a_3 w_1^n + a_4 w_2^{n-1} + \cdots + a_0).
\]

**Theorem 1.6.** Let \( f(z) \) and \( g(z) \) be a non-constant meromorphic functions, \( P(f) \) and \( P(g) \) are polynomials of degree \( m \) and let \( n, l \) be two positive integers with \( \ell(n + m) > 5k + 10 \). If \( \Theta(\infty, f) > \frac{2}{n+m}, [f^n P(f)]^{[k]} \) and \([f^n P(f)]^{[k]} \) share \( I(1,1) \) then \( f(z) \equiv g(z) \) or \([f^n P(f)]^{[k]}[g^n P(g)]^{[k]} \equiv 1 \).

## 2 Some Lemmas

To prove our result we need the following Lemmas.

**Lemma 2.1. (see [1])** Let \( f(z) \) be a non-constant meromorphic function, and let \( a_0, a_1, \ldots, a_n \) be finite complex numbers such that \( a_n \neq 0 \). Then

\[
T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_0) = nT(r, f) + S(r, f).
\]

**Lemma 2.2. (see [1])** Let \( f(z) \) be a non-constant meromorphic function and \( k \) be a positive integer and \( c \) a non-zero finite complex number. Then

\[
T(r, f) \leq \mathcal{N}(r, f) + N\left(r, \frac{1}{f^{(k)}}\right) + N\left(r, \frac{1}{f^{(k-1)}}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f)
\]

\[
\leq \mathcal{N}(r, f) + N_{k+1}\left(r, \frac{1}{f^{(k)}}\right) + \mathcal{N}\left(r, \frac{1}{f^{(k-1)}}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f).
\]

where \( N_0\left(r, \frac{1}{f^{(k)}}\right) \) is the counting function which only counts those points such that \( f^{(k+1)} = 0 \) but note that \( f^{(k+1)} - c \neq 0 \).

**Lemma 2.3. (see [8])** Let \( f(z) \) be a non-constant meromorphic function, and let \( k \) be a positive integer. Suppose that \( f^{(k)} \neq 0 \), then

\[
N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\mathcal{N}(r, f) + S(r, f).
\]

**Lemma 2.4. (see [11])** Let \( f(z) \) be a non-constant meromorphic function and \( s, k \) be any two positive integers. Then

\[
N_s\left(r, \frac{1}{f^{(k)}}\right) \leq k\mathcal{N}(r, f) + N_{s+k}\left(r, \frac{1}{f}\right) + S(r, f).
\]

Clearly, \( \mathcal{N}\left(r, \frac{1}{f^{(k)}}\right) = N_1\left(r, \frac{1}{f^{(k)}}\right) \).
Lemma 2.5. (see [1]) Let \( f(z) \) be a transcendental meromorphic function, and let \( a_1(z), a_2(z) \) be two meromorphic functions such that \( T(r, a_i) = S(r, f), i = 1, 2, \ldots, n \). Then

\[
T(r, f) \leq N(r, f) + \mathcal{N}\left(r, \frac{1}{f - a_1}\right) + \mathcal{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f).
\]

Lemma 2.6. (see [10]) Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions, and let \( k \geq 1, l \geq 1 \) be two positive integers. Suppose that \( f^{(k)} \) and \( g^{(k)} \) share \((1, l)\),

(i) If \( l = 2 \) and

\[
\Delta_1 = 2\Theta(\infty, f) + (k + 2)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > k + 7.
\]

then either \( f^{(k)} g^{(k)} \equiv 1 \) or \( f(z) \equiv g(z) \).

(ii) If \( l = 1 \) and

\[
\Delta_2 = (k + 3)\Theta(\infty, f) + (k + 2)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 2\delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 2k + 9.
\]

then either \( f^{(k)} g^{(k)} \equiv 1 \) or \( f(z) \equiv g(z) \).

3 Proof of Theorems

Proof of Theorem 1.5.

Proof. Let \( F(z) = f^n P(f) \) and \( G(z) = g^n P(g) \). We have from Lemma 2.6,

\[
\Delta_1 = 2\Theta(\infty, f) + (k + 2)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g).
\]

(3.1)

\[
\mathcal{N}\left(r, \frac{1}{F}\right) = \mathcal{N}\left(r, \frac{1}{f^n P(f)}\right) \leq \frac{1}{t(n + m)} N\left(r, \frac{1}{F}\right) \leq \frac{1}{t(n + m)} (T(r, F) + O(1)).
\]

(3.2)

Therefore,

\[
\Theta(0, f) = 1 - \lim_{r \to \infty} \frac{N(r, \frac{1}{F})}{T(r, f)} \geq 1 - \frac{1}{t(n + m)}.
\]

(3.3)

Similarly,

\[
\delta_{k+1}(0, g) \geq 1 - \frac{k + 1}{t(n + m)}.
\]

(3.4)

\[
\delta_{k+1}(0, f) \geq 1 - \frac{k + 1}{t(n + m)}.
\]

(3.5)

\[
\Theta(\infty, f) \geq 1 - \frac{1}{t(n + m)}.
\]

(3.6)

\[
\Theta(\infty, g) \geq 1 - \frac{1}{t(n + m)}.
\]

(3.7)

From the inequalities (3.3)-(3.7), we get,

\[
\Delta_1 \geq 2\left(1 - \frac{1}{t(n + m)}\right) + (k + 2)\left(1 - \frac{1}{t(n + m)}\right) + \left(1 - \frac{1}{t(n + m)}\right) + \left(1 - \frac{1}{t(n + m)}\right)
\]

\[
+ 2\left(1 - \frac{k + 1}{t(n + m)}\right).
\]

(3.8)

Since \( t(n + m) > 3k + 8 \), we get \( \Delta_1 > k + 7 \). Considering that \( F^{(k)} \) and \( G^{(k)} \) share \((1, 2)\), then by Lemma 2.6 we deduce that either \( F^{(k)} G^{(k)} \equiv 1 \) or \( F \equiv G \).
Next, we consider the following two cases.

Case 1. $F^{(k)} G^{(k)} \equiv 1$, that is

$$[f^n P(f)]^{(k)} [g^n P(f)]^{(k)} \equiv 1. \quad (3.9)$$

Case 2. $F \equiv G$, that is

$$f^n P(f) = g^n P(g). \quad (3.10)$$

Suppose that $f \neq g$, then we consider following two cases:

(i) Let $h = \frac{f}{g}$ be a constant. Then from (3.10) it follows that $h^n \neq 1$, $h^{n+m} \neq 1$, $h^{n+m-1} \neq 1$ and $a_m g^{n+m}(1-h^{n+m}) + \ldots + a_0 g^n(1-h^n) = 0$ which implies $h^d = 1$ where $d = (n+m, n+m-1, \ldots, n)$, $a_{m-i} \neq 0$, for some $i = 0, 1, \ldots m$.

(ii) If $h$ is not constant, then we know by (3.10) that $f$ and $g$ satisfy the algebraic equation

$$R(w_1, w_2) = w_1^m (a_m w_1^m + a_{m-1} w_1^{m-1} + \ldots + a_0) - w_2^m (a_m w_2^m + a_{m-1} w_2^{m-1} + \ldots + a_0).$$

It follows that

$$T(r, f) = T(r, g) = (n+m)T(r, h) + S(r, f).$$

On the other hand, by the second fundamental theorem of Nevanlinna we get

$$\overline{N}(r, f) = \sum_{j=1}^{N} \overline{N}(r, \frac{1}{h-a_j}) \geq (n + m - 2)T(r, h) + S(r, f).$$

where $a_j \neq 1$ ($j = 1, 2, \ldots, n$) are the distinct roots of the algebraic equation $h^{n+m} = 1$. So, we have

$$\Theta(\infty, f) = 1 - \lim_{r \to \infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 1 - \lim_{r \to \infty} \frac{(n + m - 2)T(r, g) + S(r, g)}{(n+m)T(r, h)} \leq 1 - \frac{n + m - 2}{n+m} \leq \frac{2}{n+m},$$

which contradicts to the assumption that $\Theta(\infty, f) > \frac{2}{n+m}$. Thus $F \equiv G$. Hence the proof of Theorem 1.5.

Proof of Theorem 1.6.

Proof. From (3.3)-(3.7) and by Lemma 2.6, we get,

$$\Delta_2 \geq (k+3) \left(1 - \frac{1}{t(n+m)} \right) + (k+2) \left(1 - \frac{1}{t(n+m)} \right) + \left(1 - \frac{1}{t(n+m)} \right) + \left(1 - \frac{1}{t(n+m)} \right) + 2 \left(1 - \frac{k+1}{t(n+m)} \right) + \left(1 - \frac{k+1}{t(n+m)} \right).$$

On simplifying the above expression, we get

$$\Delta_2 \geq 5k + 10 - \frac{5k+10}{t(n+m)}.$$
References


Author information
Rajeshwari S., Husna V., and Nagarjun V.,
Corresponding author: Husna V.
Department of Mathematics, School of Engineering, Presidency University, Itgalpura, Rajanukunte, Yelahanka, Bengaluru-560 064, INDIA.
E-mail: rajeshwaripreetham@gmail.com, rajeshwari.s@presidencyuniversity.in
husnav43@gmail.com, husna@presidencyuniversity.in
phalguniarjun95@gmail.com

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