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MATRIX SEQUENCE OF THE BINOMIAL FORM OF THE COMPLEX COMBINED JACOBSTHAL-AKIN SEQUENCE

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Abstract In this note, we consider a complex sequence $\langle U_n \rangle_{n\in\mathbb{N}}$ which have a similar structure with Jacobsthal sequence and we nominate this sequence as a complex combined Jacobsthal-Akin sequence or simply combined Jacobsthal-Akin sequence. After that we study a binomial form $\langle X_n \rangle_{n\in\mathbb{N}}$ of $\langle U_n \rangle_{n\in\mathbb{N}}$ and we call $\langle X_n \rangle_{n\in\mathbb{N}}$ as binomial sequence. Finally we delineate a matrix sequence $\langle Z_n \rangle_{n\in\mathbb{N}}$ of the binomial sequence $\langle X_n \rangle_{n\in\mathbb{N}}$.

1 Introduction

Many more authors worked on the generalizations of Fibonacci sequences [1] by various angles or patterns. Especially some of them employed matrix methods as well as introduced complex plane concept for the study of generalizations of Fibonacci numbers.

Horadam [2] in 1963 introduced the concept of complex Fibonacci numbers. Jordan [3] in 1965 considered a Gaussian Fibonacci sequence $\langle GF_n \rangle$ and established some results between Gaussian Fibonacci sequence and classical Fibonacci sequence. Gaussian Fibonacci numbers are recursively defined by

$$GF_n = GF_{n-1} + GF_{n-2}, \;\; n \geq 2 \;\; and \;\; GF_0 = i, \; GF_1 = 1 \tag{1.1}$$

Later on Berzsenyi [4], Harman [5] and Pethe [6] used different approaches of extensions of Fibonacci numbers on the complex plane.

Now we give some literature where the authors studied the generalizations of Jacobsthal numbers and Jacobsthal-Lucas numbers (see [7]). Asci and Gurel [7] delineated and studied Gaussian Jacobsthal and Gaussian Jacobsthal-Lucas numbers. These numbers are given, respectively, as

$$GJ_{n+1} = GJ_n + 2GJ_{n-1}, \;\; n \geq 1 \;\; and \;\; GJ_0 = \frac{i}{2}, \; GJ_1 = 1 \tag{1.2}$$

$$Gj_{n+1} = Gj_n + 2Gj_{n-1}, \;\; n \geq 1 \;\; and \;\; Gj_0 = 2 - \frac{i}{2}, \; Gj_1 = 1 + 2i. \tag{1.3}$$

Again Asci ans Gurel [8] examined Gaussian Jacobsthal and Gaussian Jacobsthal-Lucas polynomials.

In [10] defined a sequence $\langle b_n \rangle_{n\in\mathbb{Z}_0}$ ($\mathbb{Z}_0$ is the set of non-negative numbers) as the binomial transform of the sequence $\langle a_n \rangle_{n\in\mathbb{Z}_0}$ if

$$b_n = \sum_{k=0}^{n} a_k \tag{1.4}$$

and Wani et al. [11] obtained the binomial form of Fibonacci-Like sequence.. A trend has been going on from several past years that many authors added parameters s and t in the classical Fibonacci, Jacobsthal sequences etc in the recurrence relation system of these sequences and then designate these sequences as (s,t)-type sequences. Uygun [9] presented (s,t)-Jacobsthal sequence $\langle \hat{j}_n(s,t) \rangle$ and (s,t)-Jacobsthal-Lucas sequence $\langle \hat{c}_n(s,t) \rangle$ such that

$$\hat{j}_n(s,t) = s\hat{j}_{n-1}(s,t) + 2t\hat{j}_{n-2}(s,t), \;\; n \geq 2 \;\; and \;\; \hat{j}_0(s,t) = 0, \; \hat{j}_1(s,t) = 1 \tag{1.5}$$

$$\hat{c}_n(s,t) = s\hat{c}_{n-1}(s,t) + 2t\hat{c}_{n-2}(s,t), \quad n \geq 2 \;\; and \;\; \hat{c}_0(s,t) = 2, \; \hat{c}_1(s,t) = s \tag{1.6}$$

where $s > 0, t \neq 0$ and $s^2 + 8t > 0$.

Since 2008 several authors explored the recurrence relations of Fibonacci sequences, generalized Fibonacci sequences and other second order sequences into the sequences known as matrix sequences that is, the sequences in which the terms of the sequences are in the form of matrices and the elements of matrices are the terms of general sequences. In 2008 Civciv and and Turkmen [12] presented (s,t)-Fibonacci sequence $\langle F_n(s,t)\rangle$ and (s,t)-Fibonacci matrix sequence $\langle \mathcal{F}_n(s,t)\rangle$ and obtained various properties for these sequences. These sequences $\langle F_n(s,t)\rangle$ and $\langle \mathcal{F}_n(s,t)\rangle$ are delineated by

$$F_{n+1}(s,t) = sF_n(s,t) + tF_{n-1}(s,t) \quad n \geq 1 \;\; and \;\; F_0(s,t) = 0, \; F_1(s,t) = 1 \tag{1.7}$$

$$\mathcal{F}_{n+1}(s,t) = s\mathcal{F}_n(s,t) + t\mathcal{F}_{n-1}(s,t), \quad n \geq 1 \tag{1.8}$$

with $\mathcal{F}_0(s,t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathcal{F}_1(s,t) = \begin{bmatrix} s & 1 \\ t & 0 \end{bmatrix}$ and $s > 0, \; t \neq 0, \; s^2 + 4t > 0$.

Again Civciv and Turkmen [13] delineated (s,t)-Lucas matrix sequence which is defined as follows:

$$F_{n+1}(s,t) = sF_n(s,t) + tF_{n-1}(s,t) \quad n \geq 1 \;\; and \;\; F_0(s,t) = 0, \; F_1(s,t) = 1 \tag{1.9}$$

$$\mathcal{L}_{n+1}(s,t) = s\mathcal{L}_n(s,t) + t\mathcal{L}_{n-1}(s,t), \quad n \geq 1 \tag{1.10}$$

with $\mathcal{L}_0(s,t) = \begin{bmatrix} s & 2 \\ 2t & -s \end{bmatrix}$, $\mathcal{L}_1(s,t) = \begin{bmatrix} s^2+2t & s \\ st & 2t \end{bmatrix}$ and $s > 0, \; t \neq 0, \; s^2 + 4t > 0$.

The main motive of this article to obtain the matrix sequence of the binomial form of the Combined Jacobsthal-Akin sequence.

2 Combined Jacobsthal-Akin Sequence

Definition 2.1. [7] *The Jacobsthal and Jacobsthal-Lucas sequences $\langle J_n\rangle$ and $\langle j_n\rangle$ are respectively given by the following recurrence relations*:

$$J_n = J_{n-1} + 2J_{n-2}, \quad n \geq 2 \;\; and \;\; J_0 = 0, \; J_1 = 1 \tag{2.1}$$

$$j_n = j_{n-1} + 2j_{n-2}, \quad n \geq 2 \;\; and \;\; J_0 = 2, \; J_1 = 1 \tag{2.2}$$

The n^{th} terms of both the sequences are mentioned by the ensuing relations:

$$J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2.3}$$

$$j_n = \alpha^n + \beta^n \tag{2.4}$$

where $\alpha = 2$ and $\beta = -1$.

Definition 2.2. *For $s,t \in \mathbb{Z}^+$ and $i \left(= \sqrt{-1}\right)$, the combined Jacobsthal-Akin sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ is recurrently defined by*

$$U_n = iU_{n-1} + 2U_{n-2}, \quad n \geq 2 \tag{2.5}$$

with seeds $U_0 = s - 2t$ and $U_1 = i(s-t)$

The first few terms of the the combined Jacobsthal-Akin sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ are given by

$$U_0 = s - 2t, \; U_1 = i(s-t), \; U_2 = s - 3t, \; U_3 = i(3s - 5t), \; U_4 = -(s+t)$$

and so on.

Let θ and ϑ be the two complex roots of the characteristic equation $u^2 - iu - 2 = 0$ of $\langle U_n\rangle$. The values of θ and ϑ are determined by

$$\theta = \frac{\sqrt{7} + i}{2} \quad \text{and} \quad \vartheta = \frac{-\left(\sqrt{7} - i\right)}{2} \tag{2.6}$$

Theorem 2.3. *For $n \in \mathbb{Z}_0$, the n^{th} term of the combined Jacobsthal-Akin sequence is delineated as*

$$U_n = s\frac{\theta^{n+1} - \vartheta^{n+1}}{\theta - \vartheta} - t\left(\theta^n + \vartheta^n\right) \tag{2.7}$$

Proof. Its proof can be easily seen by using induction method. □

Now let

$$\widehat{U}_n = \frac{\theta^{n+1} - \vartheta^{n+1}}{\theta - \vartheta} \tag{2.8}$$

is called Jacobsthal-Like sequence.
and

$$\overline{U}_n = \theta^n + \vartheta^n \tag{2.9}$$

is called Jacobsthal-Lucas-Like sequence.
Clearly from the equations (2.8) and (2.9) Jacobsthal-Akin sequence $\langle U_n \rangle$ is the combination of two sequences such as jacobsthal-Like sequence (2.8) and Jacobsthal-Lucas-Like sequence (2.9) and so the sequence $\langle U_n \rangle$ is called combined Jacobsthal-Akin sequence.

3 Binomial Form of Combined Jacobsthal-Akin sequence $\langle W_n \rangle$

In the present section first of all we express combined Jacobsthal-Akin sequence $\langle U_n \rangle$ in terms of binomial form $\langle X_n \rangle$ and we call $\langle X_n \rangle$ as binomial sequence. After that we obtain a recurrence relation for $\langle X_n \rangle$. Furthermore we obtain binomial forms or binomial sequences of the Jacobsthal-Like and Jacobsthal-Lucas-Like sequences.

Definition 3.1. *For $n \in \mathbb{Z}_0$, the binomial form of the combined Jacobsthal-Like sequence $\langle U_n \rangle$ is defined by*

$$X_n = \sum_{l=0}^{n} \binom{n}{l} U_l \tag{3.1}$$

Lemma 3.2. *For $n \in \mathbb{Z}_0$, the following property holds for $\langle X_n \rangle$:*

$$X_{n+1} = \sum_{l=0}^{n} \binom{n}{l} \left(U_l + U_{l+1}\right) \tag{3.2}$$

Proof. Its proof can be easily obtained by using the relation $\binom{n+1}{l} = \binom{n}{l} + \binom{n}{l-1}$ □

Theorem 3.3. (Recurrence relation for $\langle U_n \rangle$) *For $s, t \in \mathbb{Z}^+$ and $i \left(= \sqrt{-1}\right)$, the binomial recurrence relation $\langle X_n \rangle$ of the combined Jacobsthal-Akin sequence $\langle U_n \rangle$ is given by*

$$X_{n+1} = (2+i)\, X_n + (1-di)\, X_{n-1}, \quad n \geq 1 \tag{3.3}$$

with $X_0 = s - 2t$ and $X_1 = (s-2t) + i\,(s-t)$

Proof. Since

$$\begin{aligned} X_{n+1} &= \sum_{l=0}^{n} \binom{n}{l} \left(U_l + U_{l+1}\right) \\ &= U_0 + U_1 + \sum_{l=1}^{n} \binom{n}{l} \left(U_l + U_{l+1}\right) \end{aligned}$$

$$= U_0 + U_1 + \sum_{l=1}^{n} \binom{n}{l} \left(U_l + iU_l + 2U_{l-1}\right) \qquad \text{By Eqn. (2.5)}$$

$$= U_0 + U_1 + \sum_{l=1}^{n} \binom{n}{l} \left[(1+i)\, U_l + 2U_{l-1}\right]$$

$$= (1+i) \sum_{l=1}^{n} \binom{n}{l} U_l + 2 \sum_{l=1}^{n} \binom{n}{l} U_{l-1} + U_0 + U_1$$

$$= (1+i) \sum_{l=1}^{n} \binom{n}{l} U_l + (1+i)\, U_0 + 2 \sum_{l=1}^{n} \binom{n}{l} U_{l-1} - (1+i)\, U_0 + U_0 + U_1$$

$$= (1+i) \sum_{l=0}^{n} \binom{n}{l} U_l + 2 \sum_{l=1}^{n} \binom{n}{l} U_{l-1} - iU_0 + U_1$$

$$= (1+i)\, X_n + 2 \sum_{l=1}^{n} \binom{n}{l} U_{l-1} - iU_0 + U_1 \qquad \text{By Eqn. (3.1)} \tag{3.4}$$

By replacing n by $n-1$, we get

$$X_n = (1+i)\, X_{n-1} + 2 \sum_{l=1}^{n-1} \binom{n-1}{l} U_{l-1} - iU_0 + U_1$$

$$= iX_{n-1} + \sum_{l=0}^{n-1} \binom{n-1}{l} U_l + 2 \sum_{l=1}^{n-1} \binom{n-1}{l} U_{l-1} - iU_0 + U_1$$

$$= iX_{n-1} + \sum_{l=1}^{n} \binom{n-1}{l-1} U_{l-1} + 2 \left[\binom{n-1}{1} U_0 + \binom{n-1}{2} U_1 + \binom{n-1}{3} U_2 + \cdots + \binom{n-1}{n-1} U_{n-2} + \binom{n-1}{n} U_{n-1} \right] - iU_0 + U_1$$

After using the fact $\binom{n-1}{n} = 0$, we have

$$X_n = iX_{n-1} + \sum_{l=1}^{n} \binom{n-1}{l-1} U_{l-1} + 2 \sum_{l=1}^{n} \binom{n-1}{l} U_{l-1} - iU_0 + U_1$$

$$X_n = iX_{n-1} + \sum_{l=1}^{n} \left[\binom{n-1}{l-1} + 2 \binom{n-1}{l} \right] U_{l-1} - iU_0 + U_1$$

$$= iX_{n-1} + \sum_{l=1}^{n} \left[\binom{n-1}{l-1} + 2 \binom{n-1}{l} + 2 \binom{n-1}{l-1} - 2 \binom{n-1}{l-1} \right] U_{l-1} - iU_0 + U_1$$

$$= iX_{n-1} + \sum_{l=1}^{n} \left[(1-2) \binom{n-1}{l-1} + 2 \binom{n}{l} \right] U_{l-1} - iU_0 + U_1$$

$$= iX_{n-1} - \sum_{l=1}^{n} \binom{n-1}{l-1} U_{l-1} + 2 \sum_{l=1}^{n} \binom{n}{l} U_{l-1} - iU_0 + U_1$$

$$= iX_{n-1} - \sum_{l=0}^{n-1} \binom{n-1}{l} U_l + 2\sum_{l=1}^{n} \binom{n}{l} U_{l-1} - iU_0 + U_1$$

$$= iX_{n-1} - X_{n-1} + 2\sum_{l=1}^{n} \binom{n}{l} U_{l-1} - iU_0 + U_1 \qquad \text{By Eqn. (3.1)}$$

$$= (i-1)\, X_{n-1} + 2\sum_{l=1}^{n} \binom{n}{l} U_{l-1} - iU_0 + U_1$$

Thus

$$X_n - (i-1)\, X_{n-1} = 2\sum_{l=1}^{n} \binom{n}{l} U_{l-1} - iU_0 + U_1$$

Hence from the equation (3.4), we get

$$\begin{aligned} X_{n+1} &= (1+i)\, X_n + X_n - (i-1)\, X_{n-1} \\ &= (2+i)\, X_n + (1-i)\, X_{n-1} \end{aligned}$$

as required. □

First few terms of the binomial sequence $\langle X_n \rangle$ defined in equation (3.3) are as under $X_0 = (s-2t)\,,\ X_1 = (s-2t) + i\,(s-t)\,,\ X_2 = (2s-6t) + i\,(2s-2t)\,,\ X_3 = (4s-13t) + i\,(6s-9t)\,,\ X_4 = (6s-25t) + i\,(16s-27t)$ and so on.
Clearly $v^2 - (2+i)\,v - (1-i) = 0$ is the characteristic equation of $\langle X_n \rangle$. Suppose that γ and δ be its two roots and are given as

$$\begin{aligned} \gamma &= \frac{(2+i) + \sqrt{(2+i)^2 + 4\,(1-i)}}{2} \\ &= \frac{\sqrt{7} + i + 2}{2} \\ &= \frac{\sqrt{7} + i}{2} + 1 \\ &= \theta + 1 \end{aligned} \tag{3.5}$$

Similarly

$$\delta = \vartheta + 1 \tag{3.6}$$

Some noticeable points about γ and δ are

$$\gamma + \delta = 2 + i, \quad \gamma\delta = i - 1 = -\,(1-i) \quad \text{and} \quad \gamma - \delta = \sqrt{7} \tag{3.7}$$

Now to obtain the binomial forms or binomial sequences of the Jacobsthal-Like $\langle \widehat{U}_n \rangle$ and Jacobsthal-Lucas-Like $\langle \overline{U}_n \rangle$ sequences we should prove the following result:

Theorem 3.4. *For $n \in \mathbb{Z}_0$, the n^{th} term of $\langle X_n \rangle$ is given by*

$$X_n = s\left(\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta} - \frac{\gamma^n - \delta^n}{\gamma - \delta}\right) - t\left(\gamma^n + \delta^n\right) \tag{3.8}$$

Proof. Let us consider a square matrix $X = \begin{bmatrix} 2+i & 1-i \\ 1 & 0 \end{bmatrix}$ and u be the eigenvalue of X. Then by Cayley Hamilton theorem the characteristic equation of X is given by the equation:

$$\left| X - uI \right| = 0$$

$$\begin{vmatrix} 2+i-u & 1-i \\ 1 & u \end{vmatrix} = 0$$

$$u^2 - (2+i)\,u - (1-i) = 0$$

Let γ and δ be the characteristic roots as well as eigenvalues of the matrix X. The eigenvectors corresponding to γ and δ are $\begin{bmatrix} \gamma \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \delta \\ 1 \end{bmatrix}$ respectively. Let $V_1 = \begin{bmatrix} \gamma & \delta \\ 1 & 1 \end{bmatrix}$ be the matrix of the eigenvectors. Since your matrix $V_1 = \begin{bmatrix} \gamma & \delta \\ 1 & 1 \end{bmatrix}$, the its inverse is $\big(\gamma - \delta\big)^{-1} \begin{bmatrix} 1 & -\delta \\ -1 & \gamma \end{bmatrix}$ and $V_2 = \begin{bmatrix} \gamma & 0 \\ 0 & \delta \end{bmatrix}$ is the diagonal matrix. Then by the process of diagonalization of matrices, we achieve

$$\begin{aligned} X^n &= V_1 V_2^n V_1^{-1} \\ &= (\gamma-\delta)^{-1} \begin{bmatrix} \gamma & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma^n & 0 \\ 0 & \delta^n \end{bmatrix} \begin{bmatrix} 1 & -\gamma \\ -1 & \delta \end{bmatrix} \\ &= (\gamma-\delta)^{-1} \begin{bmatrix} \gamma^{n+1} - \delta^{n+1} & -\delta\gamma^{n+1} + \gamma\delta^{n+1} \\ \gamma^n - \delta^n & -\delta\gamma^n + \gamma\delta^n \end{bmatrix} \end{aligned}$$

Since $\begin{bmatrix} X_{n+1} \\ X_n \end{bmatrix} = X^n \begin{bmatrix} X_1 \\ X_0 \end{bmatrix}$, we have

$$\begin{aligned} \begin{bmatrix} X_{n+1} \\ X_n \end{bmatrix} &= (\gamma-\delta)^{-1} \begin{bmatrix} \gamma^{n+1} - \delta^{n+1} & -\delta\gamma^{n+1} + \gamma\delta^{n+1} \\ \gamma^n - \delta^n & -\delta\gamma^n + \gamma\delta^n \end{bmatrix} \begin{bmatrix} X_1 \\ X_0 \end{bmatrix} \\ &= (\gamma-\delta)^{-1} \begin{bmatrix} X_1\gamma^{n+1} - X_1\delta^{n+1} - X_0\delta\gamma^{n+1} + X_0\gamma\delta^{n+1} \\ X_1\gamma^n - X_1\delta^n - X_0\delta\gamma^n + X_0\gamma\delta^n \end{bmatrix} \end{aligned}$$

Thus

$$\begin{aligned} X_n &= \frac{X_1\gamma^n - X_1\delta^n - X_0\delta\gamma^n + X_0\gamma\delta^n}{\gamma-\delta} \\ &= \frac{1}{\gamma-\delta}\Big[\big(X_1 - \delta X_0\big)\gamma^n + \big(\gamma X_0 - X_1\big)\delta^n\Big] \end{aligned}$$

Let

$$X_n = \frac{1}{\gamma-\delta}\left(V_3 + V_4\right)$$

where

$$\begin{aligned} V_3 &= \big(X_1 - \delta X_0\big)\gamma^n \\ &= \Big[s\,(1+i) - t\,(2+i) - \delta\,(s-2t)\Big] \\ &= \big(is + s - it - 2t - \delta s + 2\delta t\big)\gamma^n \\ &= is\gamma^n + s\gamma^n - s\delta\gamma^n - it\gamma^n - 2t\gamma^n + 2\delta t\gamma^n \\ &= is\gamma^n + s\gamma^n - s\,(2+i-\gamma)\,\gamma^n - it\gamma^n - 2t\gamma^n + 2t\,(2+i-\gamma)\,\gamma^n \end{aligned}$$

By Eqn. (3.7)

$$
\begin{aligned}
&= is\gamma^n + s\gamma^n - 2s\gamma^n - is\gamma^n + s\gamma^{n+1} - it\gamma^n - 2t\gamma^n + 4t\gamma^n + i2t\gamma^n - 2t\gamma^{n+1} \\
&= -s\gamma^n + s\gamma^{n+1} + it\gamma^n + 2t\gamma^n - 2t\gamma^{n+1} \\
&= s\gamma^{n+1} - s\gamma^n + t\gamma^n\big(2 + i - 2\gamma\big) \\
&= s\gamma^{n+1} - s\gamma^n + t\gamma^n\big(\gamma + \delta - 2\gamma\big) \qquad \text{By Eqn. (3.7)} \\
&= s\gamma^{n+1} - s\gamma^n - t\gamma^n\left(\gamma - \delta\right)
\end{aligned}
$$

Similarly

$$V_4 = -s\delta^{n+1} + s\delta^n - t\delta^n\left(\gamma - \delta\right)$$

Therefore

$$
\begin{aligned}
X_n &= \frac{1}{\gamma - \delta}\Big[s\gamma^{n+1} - s\gamma^n - t\gamma^n\left(\gamma - \delta\right) - s\delta^{n+1} + s\delta^n - t\delta^n\left(\gamma - \delta\right)\Big] \\
&= \frac{1}{\gamma - \delta}\Big[s\gamma^{n+1} - s\delta^{n+1} - s\gamma^n + s\delta^n - t\gamma^n\left(\gamma - \delta\right) - t\delta^n\left(\gamma - \delta\right)\Big] \\
&= s\left(\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta} - \frac{\gamma^n - \delta^n}{\gamma - \delta}\right) - t\Big(\gamma^n + \delta^n\Big)
\end{aligned}
$$

Hence the result. □

Again we know that the recurrence relation for $\langle X_n \rangle$ is a second order homogeneous linear recurrence relation. Then the general solution or n^{th} term of the binomial sequence $\langle X_n \rangle$ is also given according to

$$X_n = A\gamma^n + B\delta^n \tag{3.9}$$

where A and B are constants and the values of A and B are as

$$A = \frac{X_1 - \delta X_0}{\gamma - \delta} \text{ and } B = \frac{\gamma X_0 - X_1}{\gamma - \delta} \Rightarrow AB = \frac{d}{\left(\gamma - \delta\right)^2} \tag{3.10}$$

where d is the fixed quantity dependent only on X_0 and X_1.
Now we express the binomial sequence $\langle X_n \rangle$ in terms of two sequences $\langle M_n \rangle$ and $\langle N_n \rangle$, where

$$
\begin{aligned}
M_n &= \frac{\gamma^n - \delta^n}{\gamma - \delta} \\
&= A_1\gamma^n + B_1\delta^n, \quad A_1 = \frac{M_1 - \delta M_0}{\gamma - \delta}, \; A_2 = \frac{\gamma M_0 - M_1}{\gamma - \delta} \\
&\Rightarrow A_1 A_2 = \frac{-1}{\left(\gamma - \delta\right)^2}
\end{aligned} \tag{3.11}
$$

and

$$
\begin{aligned}
N_n &= \gamma^n + \delta^n \\
&= B_1\gamma^n + B_2\delta^n, \quad B_1 = \frac{N_1 - \delta N_0}{\gamma - \delta}, \; B_2 = \frac{\gamma N_0 - N_1}{\gamma - \delta} \Rightarrow B_1 B_2 = 1
\end{aligned} \tag{3.12}
$$

Clearly $M_{n+1} - M_n$ is the binomial form or binomial sequence of the Jacobsthal-Like sequence $\left\langle \widehat{U}_n \right\rangle$ and $\langle N_n \rangle$ is the binomial form or binomial sequence of the Jacobsthal-Lucas-Like sequence $\langle \overline{U}_n \rangle$.

4 Matrix Sequence of the Binomial Sequence $\langle X_n \rangle$

In this section we define a matrix sequence $\langle Z_n \rangle$ by using binomial sequence $\langle X_n \rangle$ and so called $\langle Z_n \rangle$ as binomial matrix sequence. In addition to this we give some results related to sequences $\langle X_n \rangle$, $\langle M_n \rangle$, $\langle N_n \rangle$ and $\langle Z_n \rangle$.

Definition 4.1. *For* $i\left(=\sqrt{-1}\right)$*, the binomial matrix sequence* $\langle Z_n\rangle_{n\in\mathbb{N}}$ *is defined by the following equation:*

$$Z_{n+1} = (2+i)\,Z_n + (1-i)\,Z_{n-1}, \quad n \geq 1 \tag{4.1}$$

with $Z_0 = \begin{bmatrix} 4+3i & 3-i \\ 2+i & 1-i \end{bmatrix}$ and $Z_1 = \begin{bmatrix} 8+9i & 7-i \\ 4+3i & 3-i \end{bmatrix}$

Some few initial few terms of the the binomial matrix sequence $\langle Z_n\rangle_{n\in\mathbb{N}}$ are given by

$$Z_0 = \begin{bmatrix} 4+3i & 3-i \\ 2+i & 1-i \end{bmatrix}, \; Z_1 = \begin{bmatrix} 8+9i & 7-i \\ 4+3i & 3-i \end{bmatrix}, \; Z_2 = \begin{bmatrix} 14+25i & 17+i \\ 8+9i & 7-i \end{bmatrix}$$

$$Z_3 = \begin{bmatrix} 20+65i & 39+11i \\ 14+25i & 17+i \end{bmatrix}$$

and so on.

As we know that the elements of the binomial matrix sequence $\langle Z_n\rangle$ are in the form of of matrices and the entries of these matrices are the elements of binomial sequence $\langle X_n\rangle$. Now in the next theorem we give the n^{th} term of the binomial matrix sequence $\langle Z_n\rangle$ in terms of the binomial sequence $\langle X_n\rangle$.

Theorem 4.2. *For* $n \in \mathbb{Z}_0$*, the* n^{th} *term of the matrix sequence* $\langle Z_n\rangle$ *is given by*

$$Z_n = d^{-1}\begin{bmatrix} X_0X_{n+4} - X_1X_{n+3} & (1-i)\left(X_0X_{n+3} - X_1X_{n+2}\right) \\ X_0X_{n+3} - X_1X_{n+2} & (1-i)\left(X_0X_{n+2} - X_1X_{n+1}\right) \end{bmatrix} \tag{4.2}$$

Proof. Let $Z = \begin{bmatrix} 2+i & 1-i \\ 1 & 0 \end{bmatrix}$ be a square matrix correspond to the binomial matrix sequence $\langle Z_n\rangle$ and assuredly $\begin{bmatrix} Z_{n+1} \\ Z_n \end{bmatrix} = Z^n \begin{bmatrix} Z_1 \\ Z_0 \end{bmatrix}$. Then by similar manner from the proof of the Theorem (3.4), we write

$$\begin{aligned}
Z_n &= \frac{Z_1\gamma^n - Z_1\delta^n - Z_0\delta\gamma^n + Z_0\gamma\delta^n}{\gamma-\delta} \\
&= \frac{1}{\gamma-\delta}\left[\left(Z_1 - \delta Z_0\right)\gamma^n + \left(\gamma Z_0 - Z_1\right)\delta^n\right] \\
&= \frac{1}{\gamma-\delta}\left[\begin{pmatrix} 8+9i & 7-i \\ 4+3i & 3-i \end{pmatrix}\gamma^n - \begin{pmatrix} 4+3i & 3-i \\ 2+i & 1-i \end{pmatrix}\delta\gamma^n + \begin{pmatrix} 4+3i & 3-i \\ 2+i & 1-i \end{pmatrix}\gamma\delta^n \right. \\
&\qquad \left. - \begin{pmatrix} 8+9i & 7-i \\ 4+3i & 3-i \end{pmatrix}\delta^n\right] \qquad (4.3) \\
&= \frac{AB\left(\gamma-\delta\right)}{d}\left[\begin{pmatrix} 8+9i & 7-i \\ 4+3i & 3-i \end{pmatrix}\gamma^n - \begin{pmatrix} 4+3i & 3-i \\ 2+i & 1-i \end{pmatrix}\delta\gamma^n + \begin{pmatrix} 4+3i & 3-i \\ 2+i & 1-i \end{pmatrix}\gamma\delta^n \right. \\
&\qquad \left. - \begin{pmatrix} 8+9i & 7-i \\ 4+3i & 3-i \end{pmatrix}\delta^n\right] \qquad \text{By Eqn. (3.10)} \\
&= \frac{AB\left(\gamma-\delta\right)}{d}\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}
\end{aligned}$$

Here

$$\begin{aligned} a_1 &= (8+9i)\gamma^n - (4+3i)\delta\gamma^n + (4+3i)\gamma\delta^n - (8+9i)\delta^n \\ &= 8\gamma^n + 9\gamma^n i - 4\gamma^n(2+i-\gamma) - 3\gamma^n i(2+i-\gamma) + 4\delta^n(2+i-\gamma) \\ &\quad\quad + 3\delta^n i(2+i-\gamma) - 8\delta^n - 9\delta^n i \\ &= 3\gamma^n - \gamma^n i + 4\gamma^{n+1} + 3\gamma^{n+1}i - 3\delta^n + \delta^n i - 4\delta^{n+1} - 3\delta^n - 3\delta^{n+1}i \\ &= \gamma^n\Big[(4+3i)\gamma + (3-i)\Big] - \delta^n\Big[(4+3i)\delta + (3-i)\Big] \end{aligned}$$

Since

$$\begin{aligned} 4+3i &= (2+i)^2 + (1-i) = (\gamma+\delta)^2 + (\gamma\delta) \quad \text{and} \\ 3-i &= (2+i)(1-i) = -(\gamma+\delta)(\gamma\delta) \end{aligned}$$

Hence, we get

$$\begin{aligned} a_1 &= \gamma^n\Big[(\gamma+\delta)^2\gamma - (\gamma\delta)\gamma - (\gamma+\delta)(\gamma\delta)\Big] - \delta^n\Big[(\gamma+\delta)^2\delta - (\gamma\delta)\delta - (\gamma+\delta)(\gamma\delta)\Big] \\ &= \gamma^{n+3} - \delta^{n+3} \end{aligned}$$

Therefore

$$\begin{aligned} \frac{AB(\gamma-\delta)}{d}a_1 &= \frac{AB(\gamma-\delta)\gamma^{n+3} - AB(\gamma-\delta)\delta^{n+3}}{d} \\ &= \frac{A(\gamma X_0 - X_1)\gamma^{n+3} - B(\gamma X_1 - \delta X_0)\delta^{n+3}}{d} \qquad \text{By Eqn. (3.10)} \\ &= \frac{X_0(A\gamma^{n+4} + B\delta^{n+4}) - X_1(A\gamma^{n+3} + B\delta^{n+3})}{d} \\ &= \frac{X_0X_{n+4} - X_1X_{n+3}}{d} \end{aligned}$$

Now

$$\begin{aligned} a_2 &= (7-i)\gamma^n - (3-i)\delta\gamma^n + (3-i)\gamma\delta^n - (7-i)\delta^n \\ &= 7\gamma^n - i\gamma^n - 3\gamma^n(2+i-\gamma) + \gamma^n i(2+i-\gamma) + 3\delta^n(2+i-\delta) - \delta^n i(2+i-\delta) \\ &\quad - 7\delta^n + \delta^n i \\ &= \gamma^n[-2i + \gamma(3-i)] + \delta^n[2i - \delta(3-i)] \end{aligned}$$

Since

$$\begin{aligned} -2i &= (1-i)(1-i) = (\gamma\delta)(\gamma\delta) \quad \text{and} \\ 3-i &= (2+i)(1-i) = -(\gamma+\delta)(\gamma\delta) \end{aligned}$$

This implies that

$$\begin{aligned} a_2 &= (1-i)\gamma^n\Big[-\gamma\delta + \gamma(\gamma+\delta)\Big] + (1-i)\delta^n\Big[\gamma\delta - \delta(\gamma+\delta)\Big] \\ &= (1-i)\Big(\gamma^{n+2} - \delta^{n+2}\Big) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{AB(\gamma-\delta)}{d}a_2 &= (1-i)\frac{AB(\gamma-\delta)\gamma^{n+2} - AB(\gamma-\delta)\delta^{n+2}}{d} \\ &= (1-i)\frac{A(\gamma X_0 - X_1)\gamma^{n+2} - B(\gamma X_1 - \delta X_0)\delta^{n+2}}{d} \end{aligned}$$

By Eqn. (3.10)

$$= (1-i)\frac{X_0\left(A\gamma^{n+3}+B\delta^{n+3}\right)-X_1\left(A\gamma^{n+2}+B\delta^{n+2}\right)}{d}$$
$$= (1-i)\frac{X_0X_{n+3}-X_1X_{n+2}}{d}$$

Similarly

$$\frac{AB\left(\gamma-\delta\right)}{d}a_3 = \frac{X_0X_{n+3}-X_1X_{n+2}}{d} \quad \text{and}$$
$$\frac{AB\left(\gamma-\delta\right)}{d}a_4 = (1-i)\frac{X_0X_{n+2}-X_1X_{n+1}}{d}$$

Thus, we get

$$Z_n = d^{-1}\begin{bmatrix} X_0X_{n+4}-X_1X_{n+3} & (1-i)\left(X_0X_{n+3}-X_1X_{n+2}\right) \\ X_0X_{n+3}-X_1X_{n+2} & (1-i)\left(X_0X_{n+2}-X_1X_{n+1}\right) \end{bmatrix}$$

□

Lemma 4.3. *For $n \in \mathbb{Z}_0$, we have*

$$M_n = \frac{X_0X_{n+1}-X_1X_n}{d} \tag{4.4}$$

Theorem 4.4. *For $n \in \mathbb{Z}_0$, the following result holds*

$$Z_n = \begin{bmatrix} M_{n+3} & (1-i)M_{n+2} \\ M_{n+2} & (1-i)M_{n+1} \end{bmatrix} \tag{4.5}$$

Proof. The proof of this theorem is clearly visible from the equations (4.2) and (4.4). □

Theorem 4.5. *For $n \in \mathbb{Z}_0$, we have*

$$Z_n = \left(\gamma-\delta\right)^{-2}\begin{bmatrix} N_0N_{n+4}-N_1X_{n+3} & (1-i)\left(N_0N_{n+3}-N_1N_{n+2}\right) \\ N_0N_{n+3}-N_1N_{n+2} & (1-i)\left(N_0N_{n+2}-N_1N_{n+1}\right) \end{bmatrix} \tag{4.6}$$

Proof. By using Equation (4.3) from the proof of Theorem (4.2), we have

$$Z_n = \frac{1}{\gamma-\delta}\left[\begin{pmatrix} 8+9i & 7-i \\ 4+3i & 3-i \end{pmatrix}\gamma^n - \begin{pmatrix} 4+3i & 3-i \\ 2+i & 1-i \end{pmatrix}\delta\gamma^n + \begin{pmatrix} 4+3i & 3-i \\ 2+i & 1-i \end{pmatrix}\gamma\delta^n \right.$$
$$\left. - \begin{pmatrix} 8+9i & 7-i \\ 4+3i & 3-i \end{pmatrix}\delta^n\right]$$
$$= \frac{1}{\gamma-\delta}\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

Since

$$a_1 = \gamma^{n+3} - \delta^{n+3}$$

We have

$$\frac{a_1}{\gamma-\delta} = \frac{\left(\gamma-\delta\right)\left(\gamma^{n+3}-\delta^{n+3}\right)}{\left(\gamma-\delta\right)^2}$$

$$= \frac{B_1B_2\big(\gamma-\delta\big)\big(\gamma^{n+3}-\delta^{n+3}\big)}{\big(\gamma-\delta\big)^2} \qquad \text{By Eqn. (3.12)}$$

$$= \frac{B_1B_2\big(\gamma-\delta\big)\gamma^{n+3}-B_1B_2\big(\gamma-\delta\big)\delta^{n+3}}{\big(\gamma-\delta\big)^2}$$

$$= \frac{B_1\big(\gamma N_0-N_1\big)\gamma^{n+3}-B_2\big(N_1-\delta N_0\big)\delta^{n+3}}{\big(\gamma-\delta\big)^2} \qquad \text{By Eqn. (3.12)}$$

$$= \frac{N_0\Big(B_1\gamma^{n+4}+B_2\delta^{n+4}\Big)-N_1\Big(B_1\gamma^{n+3}+B_2\delta^{n+3}\Big)}{\big(\gamma-\delta\big)^2}$$

$$= \frac{N_0N_{n+4}-N_1N_{n+3}}{\big(\gamma-\delta\big)^2}$$

Again

$$\frac{a_2}{\gamma-\delta} = \big(1-i\big)\frac{\big(\gamma-\delta\big)\big(\gamma^{n+2}-\delta^{n+2}\big)}{\big(\gamma-\delta\big)^2}$$

$$= \big(1-i\big)\frac{B_1B_2\big(\gamma-\delta\big)\big(\gamma^{n+2}-\delta^{n+2}\big)}{\big(\gamma-\delta\big)^2} \qquad \text{By Eqn. (3.12)}$$

$$= \big(1-i\big)\frac{B_1B_2\big(\gamma-\delta\big)\gamma^{n+2}-B_1B_2\big(\gamma-\delta\big)\delta^{n+2}}{\big(\gamma-\delta\big)^2}$$

$$= \big(1-i\big)\frac{B_1\big(\gamma N_0-N_1\big)\gamma^{n+2}-B_2\big(N_1-\delta N_0\big)\delta^{n+2}}{\big(\gamma-\delta\big)^2} \qquad \text{By Eqn. (3.12)}$$

$$= \big(1-i\big)\frac{N_0\Big(B_1\gamma^{n+3}+B_2\delta^{n+3}\Big)-N_1\Big(B_1\gamma^{n+2}+B_2\delta^{n+2}\Big)}{\big(\gamma-\delta\big)^2}$$

$$= \big(1-i\big)\frac{N_0N_{n+3}-N_1N_{n+2}}{\big(\gamma-\delta\big)^2}$$

Equivalently

$$\frac{a_3}{\gamma-\delta} = \frac{N_0N_{n+3}-N_1N_{n+2}}{\big(\gamma-\delta\big)^2} \quad \text{and}$$

$$\frac{a_4}{\gamma-\delta} = \big(1-i\big)\frac{N_0N_{n+2}-N_1N_{n+1}}{\big(\gamma-\delta\big)^2}$$

Hence, we achieve

$$Z_n = \big(\gamma-\delta\big)^{-2}\begin{bmatrix} N_0N_{n+4}-N_1X_{n+3} & \big(1-i\big)\big(N_0N_{n+3}-N_1N_{n+2}\big) \\ N_0N_{n+3}-N_1N_{n+2} & \big(1-i\big)\big(N_0N_{n+2}-N_1N_{n+1}\big) \end{bmatrix}$$

□

Corollary 4.6. *For $n \in \mathbb{Z}_0$, the ensuing results hold*

$$X_0X_{n+2}-X_1X_{n+1} = dM_{n+1} \tag{4.7}$$

$$N_0N_{n+2}-N_1N_{n+1} = \big(\gamma-\delta\big)^2M_{n+1} \tag{4.8}$$

Corollary 4.7. *Let $n \geq 0$, the following properties hold*

$$X_0\big(X_{n+4}+X_{n+2}\big)-X_1\big(X_{n+3}+X_{n+1}\big)=dN_{n+2} \tag{4.9}$$

$$N_0\big(N_{n+4}+N_{n+2}\big)-N_1\big(N_{n+3}+N_{n+1}\big)=\big(\gamma-\delta\big)^2N_{n+2} \tag{4.10}$$

Proof. By equating corresponding terms of matrices from the Equations (4.2) and (4.5), we have

$$X_0X_{n+4}-X_1X_{n+3}=dM_{n+3}$$
$$X_0X_{n+2}-X_1X_{n+1}=dM_{n+1}$$

Adding together both the equations, we get

$$\begin{aligned}
&X_0\big(X_{n+4}+X_{n+2}\big)-X_1\big(X_{n+3}+X_{n+1}\big)\\
&=d\big(M_{n+3}+M_{n+1}\big)\\
&=\frac{d}{\gamma-\delta}\Big(\gamma^{n+3}-\delta^{n+3}+\gamma^{n+1}-\delta^{n+1}\Big) && \text{By Eqn. (3.11)}\\
&=\frac{d}{\gamma-\delta}\Big[\gamma^{n+1}\big(\gamma^2+1\big)-\delta^{n+1}\big(\delta^2+1\big)\Big]\\
&=\frac{d}{\gamma-\delta}\Big[\gamma^{n+2}\big(\gamma-\delta\big)+\delta^{n+2}\big(\gamma-\delta\big)\Big] && \text{By Eqn. (3.7)}\\
&=dN_{n+2} && \text{By Eqn. (3.12)}
\end{aligned}$$

Hence the result. □

Conclusion

In this paper we studied the matrix sequence of the binomial form of second order Jacobsthal-Like sequence. In addition to this we obtained some basic results about the said matrix sequence. As an extension of this article, future work will examine the matrix sequence of the binomial form of other second order sequences or higher order sequences.

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