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ANALYTICAL STUDY OF NONLINEAR ROLL MOTION OF SHIPS: A HOMOTOPY PERTURBATION APPROACH

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Abstract In this article, a mathematical model for a nonlinear roll motion of a ship is discussed and analytically solved. The model is a second-order nonlinear differential equation containing nonlinear terms of damping and restoring moments. Under normal or irregular waves, analytical expressions of roll angle, roll velocity, and moments of damping and restoring are derived using a modified version of the homotopy perturbation method. The obtained analytical expressions are shown to be in satisfactory agreements with MATLAB generated numerical simulations.

1 Introduction

For every ship, protection from capsizing is of great importance. One of the leading causes of a ship capsizing in waves is the loss of stability due to nonlinear roll motion. Over the past few decades, a significant effort has been made to develop a basic ship safety evaluation methodology that is applicable during the design process and ship operation.

The roll motion of a ship is described by a second order differential equation, where damping coefficients can be linear or nonlinear. Typically, the linear model is used for low rolling amplitudes, where the spectral analysis describes the problem in a frequency domain. However, a large-amplitude ship movement is expected to result in a strongly nonlinear motion or even a chaotic behavior, and in this case, the nonlinear rolling formula, which could be set for regular or irregular waves, aims at foreseeing the ship's nonlinear response. The ship-rolling problem can be investigated in the frequency or time domains. However, the analysis in the time domain is more feasible than that in the frequency domain, because in the latter case, there is a need for conducting a significant number of realizations to determine the probability of capsizing. But with the advancements in numerical simulations and probabilistic analysis, the development of instructions for a ship handling based on risk has become possible [1].

Various viscous roll damping prediction methods have been applied by researchers over the past few decades. Numerical methods for the simulation of ship motions are built in parallel with the growing computing power. Mathematical techniques vary in complexity and ability to compensate for different phenomena of flow, such as the two-phase flow, turbulence, and ship motion [2]. Flazarano et al. [3] presented an overview of roll damping and viscous roll damping prediction methods. Graham et al. [4] discussed the viscous damping prediction of large floating bodies in waves. Seah et al. [5] have coupled a vortex method for simulating the roll decline of a floating cylinder with a rigid body design. Several scientists have also studied the effects of a moving ship's nonlinear recovery moments. For example, Koskinen [6] discussed the numerical simulation of ship motion due to waves and maneuvering. Ibrahim [7] presented the modelling of a ship roll dynamics and its coupling with a heave and pitch. The oscillation of the roll as one of the most important movements that can result in the boat being capsized has also been investigated by researchers. Ibrahim [7] articulated the ships' response through a linear formula for small angles of roll motions. Unneland et al. [8] carried out theoretical studies on low-order potential damping models for surface vessels. Perez et al. [9] presented a detailed simulation model of the naval coastal patrol vessel. Wu and Sheu [10] obtained an exact solution for the design of a ship hull's heave and pitch movements, and Thuhad [11] investigated the coupled heave and pitch motions of a non-uniform hull on a still surface.

In this paper, a nonlinear model of roll motion of a ship in a time domain is discussed and analytically solved by using a modified version of the homotopy perturbation method. In addition, approximate analytical expressions for roll angle and velocity and the moments of damping and restoring are also derived as they are essential parameters of ship dynamics. Numerical simulations are performed to investigate the characteristics of the movements of ships and to validate the derived analytical results.

2 Mathematical formulation of the problem

Using Newton's law, the roll motion of a ship can be described by the following second-order ordinary differential equation

$$A\ddot{x} + B\dot{x} + Cx + f(t) = 0, \tag{2.1}$$

where A, B and C represent the coefficients of inertia, damping and restoring terms, respectively. The first term represents the force of inertia, and the second and third terms describe the moments of damping and regeneration. Eq. (2.1) can be reduced to a single differential equation with one degree of freedom as follows:

$$I\ddot{\theta} + B\dot{\theta} + C\theta = M_\theta(t), \tag{2.2}$$

where the roll angle θ is the independent variable, I is the moment of inertia and B and C are nonlinear functions representing the roll damping and restoring moments of the roll angle. Figure 1 shows a general roll damping prediction model [2].

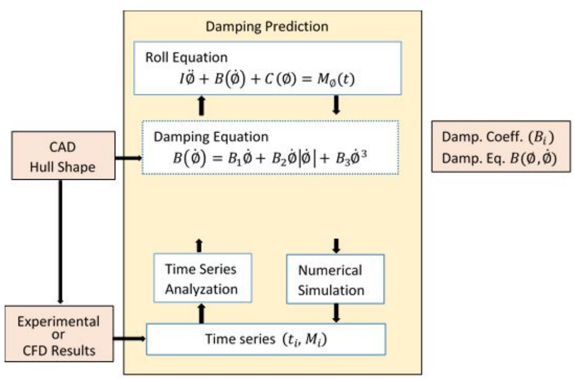


Figure 1: Design of the roll damping prediction system development.

3 Analytical expression of roll angle

As shown in Section 2, the governing equation of the roll motion is a nonlinear boundary value problem for which no exact solution is known. There are many numerical schemes that can be employed to find an approximate solution of the equation of roll motion such as, finite difference methods [12], spline collocation methods [13], and wavelet-based methods [14]. But although numerical solutions can be efficiently obtained, there are some pitfalls that make analytical solutions more desirable by researchers. Of the most serious pitfalls that come with numerical solutions, we mention instability and the difficulty of adjusting parameters to match the numerical data. In addition, understanding the nonlinear behavior vessel rolling analysis can not be achieved without analytical methods. Of the analytical methods that have proved to be effective in solving nonlinear differential systems, we mention the variational iteration method [15, 16], homotopy analysis method [17], Green's function based method [18, 19], and differential transformation method [20].

The homotopy perturbation method (HPM), which was first introduced by He [21] in 1999, is a powerful and efficient method for finding analytical solutions of nonlinear differential equations. Of the many advantages of the HPM, we emphasize that HPM does not involve discretization of the variables involved, and hence it is free from rounding off errors. In addition, the HPM is applicable to various kinds of complicated nonlinear, stiff, delay, and fractional-order differential equations. Accurate results can be obtained with just a few iterations, although more iterations may be required in cases like strong oscillation models, which may be regarded as a possible limitation of the method. For the past two decades, HPM has undergone various modifications and has been successfully employed for finding reliable solutions for many nonlinear equations arising in physical, chemical, and engineering sciences [22–28]. In the next two subsections, we show how a modified version of the HPM is employed to derive approximate analytical expressions for roll angle in the case of linear damping and restoring moment with and without wave exciting moment.

3.1 Linear damping and nonlinear restoring moments with wave exciting moment

The ship roll in a regular wave is described by the second order nonlinear equation [11]

$$\ddot{\theta}(t) + d_1\dot{\theta}(t) + k_1\theta(t) + k_3\theta^3(t) + k_5\theta^5(t) - a\cos(\omega t) = 0, \tag{3.1}$$

where d_1 is a relative damping coefficient, k_1, k_3 and k_5 are relative restoring coefficients, a is the excitation amplitude and ω is the frequency. The initial conditions are given by

$$\theta|_{t=0} = l,\ \dot{\theta}|_{t=0} = 0. \tag{3.2}$$

We construct the homotopy for Eq. (3.1) as follows (see Appendix A)

$$\begin{aligned} &(1-p)[\ddot{\theta}(t) + d_1\dot{\theta}(t) + k_1\theta(t)] \\ &+p[\ddot{\theta}(t) + d_1\dot{\theta}(t) + k_1\theta(t) + k_3\theta^3(t) + k_5\theta^5(t) - a\cos(\omega t)] = 0 \end{aligned} \tag{3.3}$$

where $p \in [0,1]$ is an embedding parameter. The approximate homotopy solution is expressed in the series form

$$\theta = \theta_0 + p\theta_1 + p^2\theta_2 + \cdots \tag{3.4}$$

Substituting (3.4) into (3.3) and equating like powers of p leads to the linear system

$$p^0 \quad : \ddot{\theta}_0 + d_1\dot{\theta}_0 + k_1\theta_0(t) = 0 \tag{3.5}$$

$$p^1 \quad : \ddot{\theta}_1 + d_1\dot{\theta}_1 + k_1\theta_1(t) + k_3\theta_0^3(t=0) + k_5\theta_0^5(t=0) - a\cos(\omega t) = 0 \tag{3.6}$$

$$\vdots$$

Subject to initial boundary conditions

$$\theta_0(0) \quad = l, \dot{\theta}_0(0) = 0, \tag{3.7}$$

$$\theta_1(0) \quad = l, \dot{\theta}_1(0) = 0, \tag{3.8}$$

$$\vdots$$

Solving system (3.5)-(3.8) gives

$$\theta_0(t) \quad = \quad e^{-\frac{d_1 t}{2}}\left\{l\cos\left(\frac{\gamma t}{2}\right) + \frac{ld_1}{\gamma}\sin\left(\frac{\gamma t}{2}\right\}, \tag{3.9}$$

$$\begin{aligned} \theta_1(t) \quad &= \quad e^{-\frac{d_1 t}{2}}\left\{(\alpha - \beta(k_1-\omega^2))\cos\left(\frac{\gamma t}{2}\right) + \frac{d_1}{\gamma}(\alpha - \beta(k_1+\omega^2))\sin\left(\frac{\gamma t}{2}\right)\right\} \\ &-\alpha + \beta(d_1\omega\sin(\omega t) + (k_1-\omega^2)\cos(\omega t)), \end{aligned} \tag{3.10}$$

$$\vdots$$

The two-term HPM solution for the roll angle is now obtained from Eq. (3.4) by taking the limit as $p \to 1$. That is,

$$\begin{aligned} \theta(t) =&e^{-\frac{d_1 t}{2}}\left\{(l+\alpha-\beta(k_1-\omega^2)\cos\left(\frac{\gamma t}{2}\right) + \frac{d_1}{\gamma}(l+\alpha-\beta(k_1+\omega^2))\sin\left(\frac{\gamma t}{2}\right)\right\} \\ &-\alpha + \beta(d_1\omega\sin(\omega t) + (k_1-\omega^2)\cos(\omega t)). \end{aligned} \tag{3.11}$$

By direct differentiation with respect to time, we obtain the explicit velocity expression

$$\begin{aligned}\frac{d\theta}{dt} =& e^{\frac{d_1 t}{2}}\left\{-\Big((l+\alpha-\beta(k_1+\omega^2))\frac{d_1^2}{2\gamma}+(l+\alpha-\beta(k_1-\omega^2))\frac{\gamma}{2}\Big)\sin\Big(\frac{\gamma t}{2}\Big)\right\}\\ &-\beta d_1\omega^2 e^{\frac{d_1 t}{2}}\cos\Big(\frac{\gamma t}{2}\Big)+\beta(d_1\omega^2\cos(\omega t)-\omega(k_1-\omega^2)\sin(\omega t)),\end{aligned} \tag{3.12}$$

and the explicit acceleration expression

$$\begin{aligned}\frac{d^2\theta}{dt^2} =& e^{\frac{d_1 t}{2}}\left\{\Big(-\frac{\beta d_1^2\omega^2}{2}-((l+\alpha-\beta(k_1+\omega^2))\Big(\frac{d_1^2}{4}\Big)-((l+\alpha-\beta(k_1-\omega^2))\Big(\frac{\gamma^2}{4}\Big)\Big)\cos\Big(\frac{\gamma t}{2}\Big)\right.\\ &\left.+\Big(\frac{\beta d_1\omega^2\gamma}{2}+((l+\alpha-\beta(k_1+\omega^2))\Big(\frac{d_1^3}{4\gamma}\Big)+((l+\alpha-\beta(k_1\omega^2))\Big(\frac{d_1\gamma}{4}\Big)\Big)\sin\Big(\frac{\gamma t}{2}\Big)\right\}\\ &-\beta\omega^2(d_1\omega\sin(\omega t)+(k_1-\omega^2)\cos(\omega t)),\end{aligned} \tag{3.13}$$

where

$$\alpha=\frac{k_3 l^3+k_5 l^5}{k_1},\ \beta=\frac{a}{d_1^2\omega^2+(k_1-\omega^2)^2},\ \gamma=\sqrt{4k_1-d_1^2}. \tag{3.14}$$

Now the analytical expressions for the restoring and damping moments are, respectively, given by

$$\text{Restoring moment} = c\theta = k_1\theta(t)+k_3\theta^3(t)+k_5\theta^5(t), \tag{3.15}$$

and

$$\text{Damping moment} = B\dot{\theta}(t) = d_1\dot{\theta}(t), \tag{3.16}$$

where $\theta(t)$ and $\dot{\theta}(t)$ are given by Eqs. (3.11) and (3.12).

3.2 Linear damping and nonlinear restoring moments without wave exciting moment

The nonlinear equation representing a ship without wave exciting moment is given by

$$\ddot{\theta}(t)+d_1\dot{\theta}(t)+k_1\theta(t)+k_3\theta^3(t)+k_5\theta^5(t)=0, \tag{3.17}$$

where d_1, k_1, k_3 and k_5 are as given in Eq. (3.1) subject to initial conditions (3.2). Following same procedure as in section 3.1, the homotopy perturbation method gives the following analytical expression for the roll angle

$$\theta(t)=e^{-\frac{d_1 t}{2}}\left\{(l+\alpha)\cos\Big(\frac{\gamma t}{2}\Big)+\frac{d_1}{\gamma}(l+\alpha)\sin\Big(\frac{\gamma t}{2}\Big)\right\}-\alpha, \tag{3.18}$$

where α and γ are as given in Eq.(3.14). The roll velocity and acceleration are, respectively given by

$$\frac{d\theta}{dt}=e^{-\frac{d_1 t}{2}}\left\{(l+\alpha)\Big(-\frac{d_1^2}{2\gamma}-\frac{\gamma}{2}\Big)\sin\Big(\frac{\gamma t}{2}\Big)\right\}, \tag{3.19}$$

$$\frac{d^2\theta}{dt^2}=e^{-\frac{d_1 t}{2}}\left\{(l+\alpha)\Big(\frac{-d_1^2}{2\gamma}-\frac{\gamma}{2}\Big)\Big(-\frac{d_1}{2}\sin\Big(\frac{\gamma t}{2}\Big)+\frac{\gamma}{2}\cos\Big(\frac{\gamma t}{2}\Big)\Big)\right\}. \tag{3.20}$$

4 Results and discussion

The following experimental values of parameters [1] will be used to test the accuracy of the derived analytical expressions for the roll angle, velocity and acceleration with and without wave exciting moments.

$$\begin{aligned}&d_1=0.0126513,\ k_1=0.67199703,\ k_3=-0.5392039,\ k_5=-0.086792,\ a=0.1,\\ &\omega=0.1,\ l=0.3.\end{aligned} \tag{4.1}$$

Using these parameters, the simplified roll angle, velocity and acceleration in the presence of wave exciting moment are, respectively, given by

$$\begin{aligned}\theta(t) &= e^{-0.00633t}(0.127\cos(0.81973t) + 0.00094\sin(0.81973t)) \\ &\quad + 0.00029\sin(0.1t) + 0.1511\cos(0.1t) + 0.02198,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\frac{d\theta}{dt} &= e^{-0.00633t}(-0.00029\cos(0.81973t) - 0.10408\sin(0.81973t)) \\ &\quad + 0.000029\cos(0.1t) - 0.01511\sin(0.1t),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\frac{d^2\theta(t)}{dt^2} &= (-0.08532\cos(0.81973t) + 0.000682\sin(0.81973t))e^{-0.00633t} \\ &\quad - 0.001511\cos(0.1t) - 2.88683\times 10^{-6}\sin(0.1t).\end{aligned} \tag{4.4}$$

In the absence of wave exciting moments, the roll angle, velocity and acceleration are, respectively

$$\theta(t) = 0.021979 + (0.002145\sin(0.81973t) + 0.27802\cos(0.81973t))e^{-0.0063t} \tag{4.5}$$

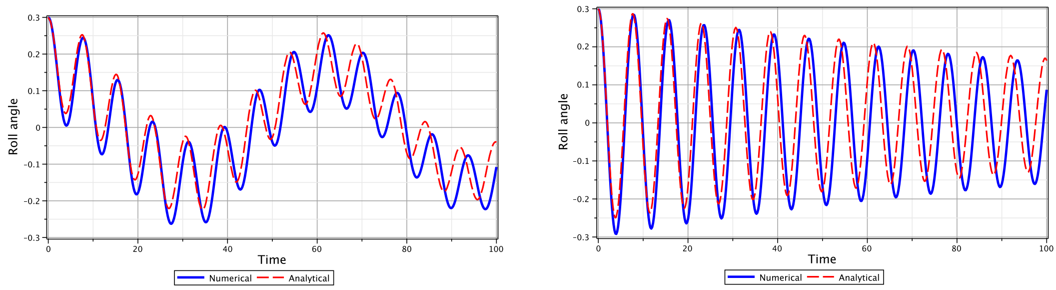
$$\frac{d\theta}{dt} = -0.22792e^{-0.00632565t}\sin(0.81973t) \tag{4.6}$$

$$\frac{d^2\theta(t)}{dt^2} = (0.001442\sin(081973t8062t) - 0.18683\cos(0.81973t))e^{-0.006326t} \tag{4.7}$$

Tables 1 and 2 show that the analytical and numerical solutions for the roll angle and the velocity are in a strong agreement when t is sufficiently close to the initial value. Figures 2-5 show that although as time increases the analytical curves for the roll angle and velocity deviate from the numerical curves but they are still satisfactorily close to them. It is also noted that all analytical curves maintain stability over a very large time domain.

Table 1: Comparison between numerical and analytical results for roll angle and velocity with wave exciting moment

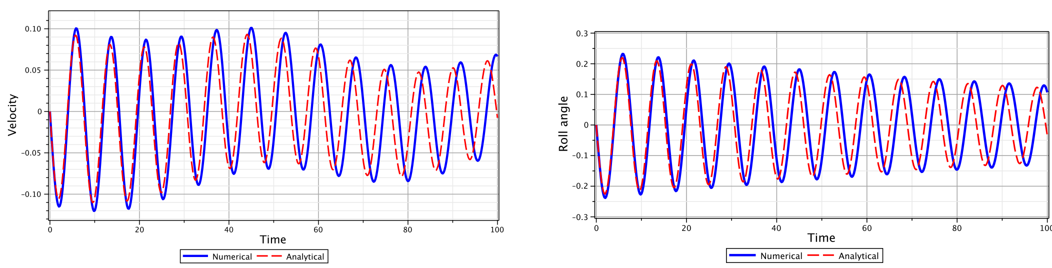
	Roll angle θ (rad)			Velocity $\dot{\theta}$ (cm/s)		
Time (s)	analytical	numerical	abs. error	analytical	numerical	abs. error
0.0	0.30000	0.30000	0	0	0	0
0.1	0.29957	0.29957	0	-0.00867	-0.00867	0
0.2	0.29827	0.29827	0	-0.01727	-0.01729	0.00002
0.3	0.29612	0.29611	0	-0.02574	-0.02580	0.00006
0.4	0.29312	0.29311	0.00001	-0.03404	-0.03417	0.00013
0.5	0.28932	0.28928	0.00003	-0.04210	-0.04236	0.00026
0.6	0.28472	0.28465	0.00006	-0.04987	-0.05031	0.00044
0.7	0.27935	0.27923	0.00012	-0.05730	-0.05799	0.00069
0.8	0.27327	0.27306	0.00021	-0.06435	-0.06535	0.00100
0.9	0.26650	0.26617	0.00033	-0.07096	-0.07235	0.00139
1.0	0.25909	0.25861	0.00049	-0.07710	-0.07895	0.00185



(a) With wave exciting moment (Eq. (4.2))

(b) Without wave exciting moment (Eq. (4.5))

Figure 2: Analytical and numerical curves of roll angle in the case of linear damping.



(a) With wave exciting moment (Eq. (4.3))

(b) Without wave exciting moment (Eq. (4.6))

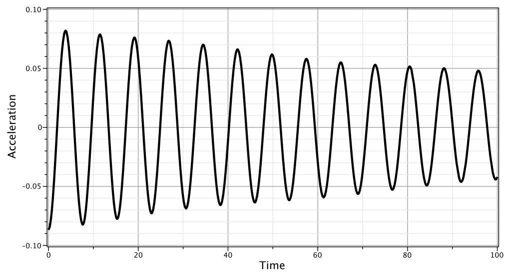
Figure 3: Analytical and numerical curves of velocity in the case of linear damping.

Table 2: Comparison between numerical and analytical results for roll angle and velocity without wave exciting moment

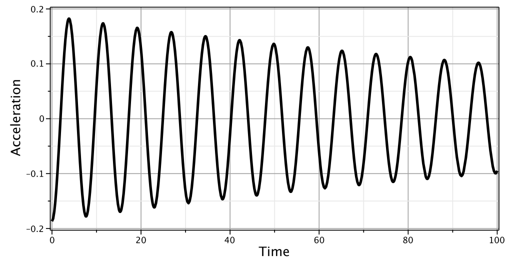
	Roll angle θ (rad)			Velocity $\dot{\theta}$ (cm/s)		
Time (s)	analytical	numerical	abs. error	analytical	numerical	abs. error
0.0	0.30000	0.30000	0	0	0	0
0.1	0.29907	0.29907	0	-0.01865	-0.01865	0
0.2	0.29627	0.29627	0	-0.03715	-0.03719	0.00004
0.3	0.29165	0.29163	0.00002	-0.05538	-0.05550	0.00012
0.4	0.28521	0.28518	0.00003	-0.07321	-0.07350	0.00028
0.5	0.27702	0.27695	0.00007	-0.09053	-0.09108	0.00054
0.6	0.26713	0.26698	0.00014	-0.10722	-0.10814	0.00091
0.7	0.25560	0.25534	0.00026	-0.12317	-0.12458	0.00140
0.8	0.24252	0.24209	0.00043	-0.13828	-0.14029	0.00201
0.9	0.22798	0.22731	0.00066	-0.15243	-0.15518	0.00275
1.0	0.21207	0.21109	0.00098	-0.16555	-0.16914	0.00359

Figure 2 shows the roll decay of a ship motion from an initial angle together with the main properties of damped roll motion. As a ship performs harmonic roll oscillation from rest with an initial roll angle and an initial velocity, then the roll damping can be measured from the decrement of each successive roll angle maxima [2].

Figure 3 depicts the analytical curves of velocity with and without wave exciting moments. It is inferred from this figure that the amplitude velocity of linear damping is smaller in the presence of exciting wave moments. Figure 4, which portrayed the acceleration curves versus time, shows that the amplitudes of both curves decrease as time increases.

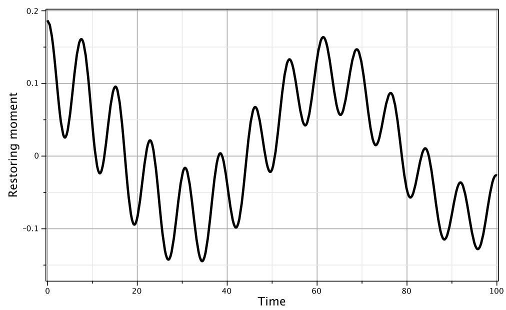


(a) With wave exciting moment (Eq. (4.4))

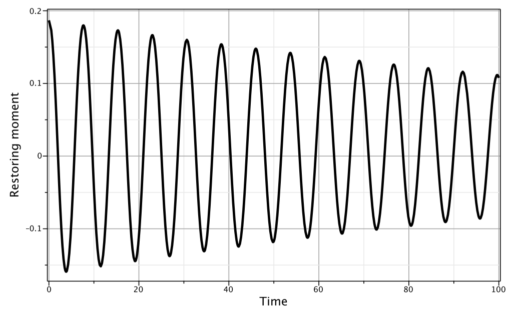


(b) Without wave exciting moment (Eq. (4.7))

Figure 4: Analytical acceleration curve in the case of linear damping.

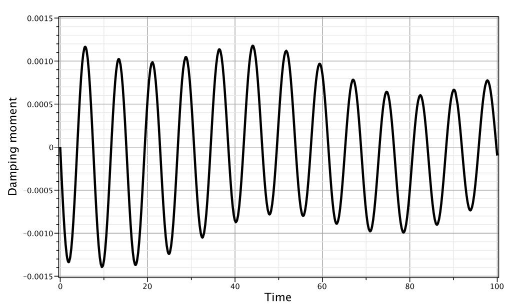


(a) With wave exciting moment

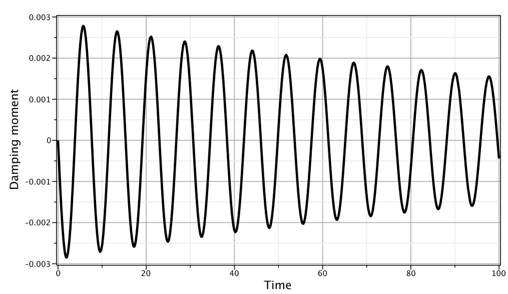


(b) Without wave exciting moment

Figure 5: Analytical curve of restoring moment in the case of linear damping (Eq. (3.15)).



(a) With wave exciting moment



(b) Without wave exciting moment

Figure 6: Analytical curve of damping moment in the case of linear damping (Eq. (3.16)).

The analytical curves of the restoring and damping moments, which are used to measure the bilge keel height, are given in Figures 5 and 6, respectively. From these figures, it is noted that the period of oscillation is not constant but is slowly increasing with time. Using the envelope function $f = \pm\theta(t=0)e^{-\frac{d_1 t}{2}}$, the damping coefficient can be obtained.

5 Conclusions

In this paper, a theoretical model describing the nonlinear roll motion of a ship was discussed. A modified version of the homotopy perturbation method was employed to derive an analytical solution of the model that is represented by a nonlinear time-dependent differential equation. Approximate analytical expressions of the roll angle, velocity, acceleration, and resorting and damping moments for all possible values of parameters were obtained. The derived analytical results were shown to maintain a satisfactory agreement with numerical simulation throughout the time domain. These analytical results can play an essential role in validating the experimental and numerical results, such as finding a roll decay test and improve the level of design and safety of a ship. The methodology presented in this paper can be intrinsically extended to solve other nonlinear systems in various fields of science and engineering.

Appendix A The basic idea of the homotopy perturbation methd

Consider the nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{Appendix A.1}$$

with the boundary condition

$$B\Big(u, \frac{du}{dr}\Big) = 0, \quad r \in \Gamma, \tag{Appendix A.2}$$

where $A, B, f(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω , respectively. Expressing $A(u)$ as the sum of linear (L) and nonlinear (N) parts, Eq. (Appendix A.1) becomes

$$L(u) + N(u) - f(r) = 0. \tag{Appendix A.3}$$

The homotopy technique begins by defining $v(r,p) : \Omega \times [0,1] \to R$, such that

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{Appendix A.4}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. (Appendix A.1) that satisfies boundary conditions (Appendix A.2). Evidently, Eq.(Appendix A.4) implies that

$$H(v,0) \;=\; L(v) - L(u_0) = 0, \tag{Appendix A.5}$$
$$H(v,1) \;=\; A(v) - f(r) = 0. \tag{Appendix A.6}$$

As p changes from 0 to 1, $v(r,p)$ changes from u_0 to u_r, a process known as a homotopy. The solution of Eq. (Appendix A.4) may be expressed in terms of a power series in the form:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots . \tag{Appendix A.7}$$

With $p = 1$, an approximate solution to Eq. (Appendix A.4) is given by:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{Appendix A.8}$$

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