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SOME GROWTH ANALYSIS OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS IN THE LIGHT OF THEIR GENERALIZED ORDER (α, β) AND GENERALIZED TYPE (α, β)

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Abstract In this paper we wish to prove some results relating to the growth rates of composite entire and meromorphic functions with their corresponding left and right factors on the basis of their generalized order (α, β) and generalized type (α, β), wher α, β are continuous non-negative functions defined on $(-\infty, +\infty)$.

1 Introduction, Definitions and Notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [3, 5, 11]. We also use the standard notations and definitions of the theory of entire functions which are available in [9] and therefore we do not explain those in details. Let f be an entire function and $M_f(r) = \max\{|f(z)| : |z| = r\}$. When f is meromorphic, the Nevanlinna's characteristic function $T_f(r)$ (see [3, p.4]) plays the same role as $M_f(r)$, which is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r,a)(\overline{N}_f(r,a))$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as follows:

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a) \log r$$

$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(0,a)}{t} dt + \overline{n}_f(0,a) \log r\right),$$

in addition we represent by $n_f(r,a)(\overline{n}_f(r,a))$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f is defined as:

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \quad \text{where}$$

$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0 .$$

Also we may employ $m(r, \frac{1}{f-a})$ by $m_f(r, a)$.

For an entire function f, the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r).$$

For $x \in [0, \infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers, we define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$, with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$, and $\exp^{[-1]} x = \log x$. Further we assume that p and q always denote positive integers. Now considering this, let us recall that Juneja et al. [4] defined the (p, q)-th order and (p, q)-th lower order of an entire function as follows:

Definition 1.1. [4] Let $p \geq q$. The (p, q)-th order $\varrho^{(p,q)}(f)$ and (p, q)-th lower order $\lambda^{(p,q)}(f)$ of an entire function f are defined as:

$$\varrho^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}.$$

If f is a meromorphic function, then

$$\varrho^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}.$$

For any entire function f, using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ $\{cf.$ [3]$\}$, one can easily verify that

$$\begin{aligned} \varrho^{(p,q)}(f) &= \limsup_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} = \limsup_{r\to+\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \\ \text{and } \lambda^{(p,q)}(f) &= \liminf_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} = \liminf_{r\to+\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}, \end{aligned}$$

when $p \geq 2$.

Extending the notion of (p, q)-th order, recently Shen et al. [6] introduced the new concept of $[p, q]$-φ order of entire and meromorphic function where $p \geq q$. Later on, combining the definition of (p, q)-order and $[p, q]$-φ order, Biswas (see, e.g., [2]) redefined the (p, q)-order of an entire and meromorphic function without restriction $p \geq q$.

However the above definition is very useful for measuring the growth of entire and meromorphic functions. If $p = l$ and $q = 1$ then we write $\varrho^{(l,1)}(f) = \varrho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\varrho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire or meromorphic function f. For details about generalized order one may see [8]. Moreover when $p = 3$ and $q = 1$ then we write $\varrho^{(3,1)}(f) = \overline{\varrho}(f)$ and $\lambda^{(3,1)}(f) = \overline{\lambda}(f)$ where $\overline{\varrho}(f)$ and $\overline{\lambda}(f)$ are respectively known as hyper order and hyper lower order of entire or meromorphic function f (see [10]). Also for $p = 2$ and $q = 1$, we respectively denote $\varrho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\varrho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire or meromorphic function f.

Now let L be a class of continuous non-negative on $(-\infty, +\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. For any $\alpha \in L$, we say that $\alpha \in L_1^0$, if $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ and $\alpha \in L_2^0$, if $\alpha(\exp((1 + o(1))x)) = (1 + o(1))\alpha(\exp(x))$ as $x \to +\infty$. Finally for any $\alpha \in L$, we also say that $\alpha \in L_1$, if $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$ and $\alpha \in L_2$, if $\alpha(\exp(cx)) = (1 + o(1))\alpha(\exp(x))$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$. Clearly, $L_1 \subset L_1^0$, $L_2 \subset L_2^0$ and $L_2 \subset L_1$.Throughout the present paper we assume that $\beta, \beta_1, \beta_2 \in L_1$ and $\alpha_1, \alpha_2 \in L_2$, unless otherwise specifically stated.

Considering this, the value

$$\varrho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} \ (\alpha \in L, \beta \in L)$$

is called [7] generalized order (α, β) of an entire function f. For details about generalized order (α, β) one may see [7]. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order (α, β) in some different direction. For the purpose of further applications, Biswas et al. [1] introduced the definition of the generalized order (α, β) of entire and meromorphic function in the following way after giving a minor modification to the original definition (e.g. see, [7]) which are as follows:

Definition 1.2. [1] The generalized order (α, β) denoted by $\varrho_{(\alpha,\beta)}[f]$ and generalized lower order (α, β) denoted by $\lambda_{(\alpha,\beta)}[f]$ of an entire function f are defined as:

$$\begin{aligned} \varrho_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ and} \\ \lambda_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)}, \text{ where } \alpha \in L_1. \end{aligned}$$

If f is a meromorphic function, then

$$\begin{aligned} \varrho_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)} \text{ and} \\ \lambda_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}, \text{ where } \alpha \in L_2. \end{aligned}$$

Using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ $\{cf.$ [3]$\}$, for an entire function f, one may easily verify that

$$\begin{aligned} \varrho_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} = \limsup_{r\to+\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)} \text{ and} \\ \lambda_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} = \liminf_{r\to+\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}, \text{ when } \alpha \in L_2. \end{aligned}$$

Definition 1.1 is a special case of Definition 1.2 for $\alpha(r) = \log^{[p]} r$ and $\beta(r) = \log^{[q]} r$.

Now in order to refine the growth scale namely the generalized order (α, β), we introduce the definitions of another growth indicators, called generalized type (α, β) and generalized lower type (α, β) respectively of *a meromorphic* function which are as follows:

Definition 1.3. The generalized type (α, β) denoted by $\sigma_{(\alpha,\beta)}[f]$ and generalized lower type (α, β) denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]$ of a meromorphic function f having finite positive generalized order (α, β) $(0 < \varrho_{(\alpha,\beta)}[f] < \infty)$ are defined as :

$$\sigma_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\varrho_{(\alpha,\beta)}[f]}} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\varrho_{(\alpha,\beta)}[f]}}$$

It is obvious that $0 \leq \overline{\sigma}_{(\alpha,\beta)}[f] \leq \sigma_{(\alpha,\beta)}[f] \leq \infty$.

Analogously, to determine the relative growth of two meromorphic functions having same non zero finite generalized lower order (α, β), one can introduced the definition of generalized weak type (α, β) and generalized upper weak type (α, β) of a meromorphic function f of finite positive generalized lower order (α, β), $\lambda_{(\alpha,\beta)}[f]$ in the following way:

Definition 1.4. The generalized upper weak type (α, β) denoted by $\overline{\tau}_{(\alpha,\beta)}[f]$ and generalized weak type (α, β) denoted by $\tau_{(\alpha,\beta)}[f]$ of a meromorphic function f having finite positive generalized lower order (α, β) $(0 < \lambda_{(\alpha,\beta)}[f] < \infty)$ are defined as :

$$\overline{\tau}_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}} \text{ and } \tau_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}}$$

It is obvious that $0 \leq \tau_{(\alpha,\beta)}[f] \leq \overline{\tau}_{(\alpha,\beta)}[f] \leq \infty$.

Now one may give the definitions of generalized hyper order (α,β) and generalized hyper lower order (α,β) of entire and meromorphic functions in the following way:

Definition 1.5. The generalized hyper order (α,β) denoted by $\overline{\varrho}_{(\alpha,\beta)}[f]$ and generalized hyper lower order (α,β) denoted by $\overline{\lambda}_{(\alpha,\beta)}[f]$ of an entire function f are defined as:

$$\overline{\varrho}_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty}\frac{\alpha(\log(M_f(r)))}{\beta(r)} \text{ and}$$
$$\overline{\lambda}_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty}\frac{\alpha(\log(M_f(r)))}{\beta(r)} \text{ where } \alpha\in L_1.$$

If f is a meromorphic function, then

$$\overline{\varrho}_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty}\frac{\alpha(T_f(r))}{\beta(r)} \text{ and}$$
$$\overline{\lambda}_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty}\frac{\alpha(T_f(r))}{\beta(r)}, \text{ where } \alpha\in L_2.$$

Using the inequality $T_f(r)\le\log M_f(r)\le 3T_f(2r)$ $\{cf.$ [3]$\}$, for an entire function f, one may easily verify that

$$\overline{\varrho}_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty}\frac{\alpha(\log(M_f(r)))}{\beta(r)} = \limsup_{r\to+\infty}\frac{\alpha(T_f(r))}{\beta(r)} \text{ and}$$
$$\overline{\lambda}_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty}\frac{\alpha(\log(M_f(r)))}{\beta(r)} = \liminf_{r\to+\infty}\frac{\alpha(T_f(r))}{\beta(r)} \text{ when } \alpha\in L_2.$$

In this paper we intend to establish some results relating to the growth properties of composite entire and meromorphic functions on the basis of generalized order (α,β) and generalized type (α,β).

2 Main Results

In this section we present the main results of the paper.

Theorem 2.1. *Let f be a meromorphic function and g be an entire function such $0<\lambda_{(\alpha_1,\beta_1)}[f\circ g]\le\varrho_{(\alpha_1,\beta_1)}[f\circ g]<\infty$ and $0<\lambda_{(\alpha_2,\beta_2)}[f]\le\varrho_{(\alpha_2,\beta_2)}[f]<\infty$. Then*

$$\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]}{\varrho_{(\alpha_2,\beta_2)}[f]} \le \liminf_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))}$$
$$\le \min\left\{\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[f]},\frac{\varrho_{(\alpha_1,\beta_1)}[f\circ g]}{\varrho_{(\alpha_2,\beta_2)}[f]}\right\} \le \max\left\{\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[f]},\frac{\varrho_{(\alpha_1,\beta_1)}[f\circ g]}{\varrho_{(\alpha_2,\beta_2)}[f]}\right\}$$
$$\le \limsup_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))} \le \frac{\varrho_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[f]}.$$

Proof. From the definition of $\varrho_{(\alpha_2,\beta_2)}[f]$ and $\lambda_{(\alpha_1,\beta_1)}[f\circ g]$, we have for arbitrary positive ε and for all sufficiently large positive numbers of r that

$$\alpha_1(\exp(T_{f\circ g}(r))) \geqslant (\lambda_{(\alpha_1,\beta_1)}[f\circ g]-\varepsilon)\beta_1(r) \tag{2.1}$$

and

$$\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r))))) \le (\varrho_{(\alpha_2,\beta_2)}[f]+\varepsilon)\beta_1(r)\ . \tag{2.2}$$

Now from (2.1) and (2.2), it follows for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))} \geqslant \frac{(\lambda_{(\alpha_1,\beta_1)}[f\circ g]-\varepsilon)\beta_1(r)}{(\varrho_{(\alpha_2,\beta_2)}[f]+\varepsilon)\beta_1(r)}\ .$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))}\geqslant\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]}{\varrho_{(\alpha_2,\beta_2)}[f]}\,. \tag{2.3}$$

Again we get for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(\exp(T_{f\circ g}(r)))\leq(\lambda_{(\alpha_1,\beta_1)}[f\circ g]+\varepsilon)\beta_1(r) \tag{2.4}$$

and for all sufficiently large positive numbers of r that

$$\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))\geqslant(\lambda_{(\alpha_2,\beta_2)}[f]-\varepsilon)\beta_1(r)\,. \tag{2.5}$$

Combining (2.4) and (2.5), we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{(\lambda_{(\alpha_1,\beta_1)}[f\circ g]+\varepsilon)\beta_1(r)}{(\lambda_{(\alpha_2,\beta_2)}[f]-\varepsilon)\beta_1(r)}\,.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[f]}\,. \tag{2.6}$$

Also for a sequence of positive numbers of r tending to infinity that

$$\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))\leq(\lambda_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(r)\,. \tag{2.7}$$

Now from (2.1) and (2.7), we obtain for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))}\geq\frac{(\lambda_{(\alpha_1,\beta_1)}[f\circ g]-\varepsilon)\beta_1(r)}{(\lambda_{(\alpha_2,\beta_2)}[f]+\varepsilon)\beta_1(r)}\,.$$

As $\varepsilon(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))}\geq\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[f]}\,. \tag{2.8}$$

Also we obtain for all sufficiently large positive numbers of r that

$$\alpha_1(\exp(T_{f\circ g}(r)))\leq(\varrho_{(\alpha_1,\beta_1)}[f\circ g]+\varepsilon)\beta_1(r)\,. \tag{2.9}$$

Now it follows from (2.5) and (2.9) for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{(\varrho_{(\alpha_1,\beta_1)}[f\circ g]+\varepsilon)\beta_1(r)}{(\lambda_{(\alpha_2,\beta_2)}[f]-\varepsilon)\beta_1(r)}\,.$$

Since $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{\varrho_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[f]}\,. \tag{2.10}$$

Further from the definition of $\varrho_{(\alpha_2,\beta_2)}[f]$, we get for a sequence of positive numbers of r tending to infinity that

$$\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))\geqslant(\varrho_{(\alpha_1,\beta_1)}[f]-\varepsilon)\beta_1(r)\,. \tag{2.11}$$

Now from (2.9) and (2.11), it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{(\varrho_{(\alpha_1,\beta_1)}[f\circ g]+\varepsilon)\beta_1(r)}{(\varrho_{(\alpha_2,\beta_2)}[f]-\varepsilon)\beta_1(r)}\,.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{\varrho_{(\alpha_1,\beta_1)}[f\circ g]}{\varrho_{(\alpha_2,\beta_2)}[f]}\ . \tag{2.12}$$

Again we obtain for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(\exp(T_{f\circ g}(r)))\geqslant(\varrho_{(\alpha_1,\beta_1)}[f\circ g]-\varepsilon)\beta_1(r)\ . \tag{2.13}$$

So combining (2.2) and (2.13), we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))}\geqslant\frac{(\varrho_{(\alpha_1,\beta_1)}[f\circ g]-\varepsilon)\beta_1(r)}{(\varrho_{(\alpha_2,\beta_2)}[f]+\varepsilon)\beta_1(r)}\ .$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r)))))}\geqslant\frac{\varrho_{(\alpha_1,\beta_1)}[f\circ g]}{\varrho_{(\alpha_2,\beta_2)}[f]}\ . \tag{2.14}$$

Thus the theorem follows from (2.3), (2.6), (2.8), (2.10), (2.12) and (2.14). □

The following theorem can be proved in the line of Theorem 2.1 and so its proof is omitted.

Theorem 2.2. *Let f be a meromorphic function and g be an entire function such $0<\lambda_{(\alpha_1,\beta_1)}[f\circ g]\leq\varrho_{(\alpha_1,\beta_1)}[f\circ g]<\infty$ and $0<\lambda_{(\alpha_2,\beta_2)}[g]\leq\varrho_{(\alpha_2,\beta_2)}[g]<\infty$. Then*

$$\begin{aligned}\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]}{\varrho_{(\alpha_2,\beta_2)}[g]}&\leq\liminf_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_g(\beta_2^{-1}(\beta_1(r)))))}\\&\leq\min\left\{\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]},\frac{\varrho_{(\alpha_1,\beta_1)}[f\circ g]}{\varrho_{(\alpha_2,\beta_2)}[g]}\right\}\leq\max\left\{\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]},\frac{\varrho_{(\alpha_1,\beta_1)}[f\circ g]}{\varrho_{(\alpha_2,\beta_2)}[g]}\right\}\\&\leq\limsup_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_g(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{\varrho_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]}.\end{aligned}$$

We may now state the following two theorems without proof based on Definition 1.5.

Theorem 2.3. *Let f be a meromorphic function and g be an entire function such $0<\overline{\lambda}_{(\alpha_1,\beta_1)}[f\circ g]\leq\overline{\varrho}_{(\alpha_1,\beta_1)}[f\circ g]<\infty$ and $0<\overline{\lambda}_{(\alpha_2,\beta_2)}[f]\leq\overline{\varrho}_{(\alpha_2,\beta_2)}[f]<\infty$. Then*

$$\begin{aligned}\frac{\overline{\lambda}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\varrho}_{(\alpha_2,\beta_2)}[f]}&\leq\liminf_{r\to+\infty}\frac{\alpha_1(T_{f\circ g}(r))}{\alpha_2(T_f(\beta_2^{-1}(\beta_1(r))))}\\&\leq\min\left\{\frac{\overline{\lambda}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\lambda}_{(\alpha_2,\beta_2)}[f]},\frac{\overline{\varrho}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\varrho}_{(\alpha_2,\beta_2)}[f]}\right\}\leq\max\left\{\frac{\overline{\lambda}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\lambda}_{(\alpha_2,\beta_2)}[f]},\frac{\overline{\varrho}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\varrho}_{(\alpha_2,\beta_2)}[f]}\right\}\\&\leq\limsup_{r\to+\infty}\frac{\alpha_1(T_{f\circ g}(r))}{\alpha_2(T_f(\beta_2^{-1}(\beta_1(r))))}\leq\frac{\overline{\varrho}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\lambda}_{(\alpha_2,\beta_2)}[f]}.\end{aligned}$$

Theorem 2.4. *Let f be a meromorphic function and g be an entire function such $0<\overline{\lambda}_{(\alpha_1,\beta_1)}[f\circ g]\leq\overline{\varrho}_{(\alpha_1,\beta_1)}[f\circ g]<\infty$ and $0<\overline{\lambda}_{(\alpha_2,\beta_2)}[g]\leq\overline{\varrho}_{(\alpha_2,\beta_2)}[g]<\infty$. Then*

$$\begin{aligned}\frac{\overline{\lambda}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\varrho}_{(\alpha_2,\beta_2)}[g]}&\leq\liminf_{r\to+\infty}\frac{\alpha_1(T_{f\circ g}(r))}{\alpha_2(T_g(\beta_2^{-1}(\beta_1(r))))}\\&\leq\min\left\{\frac{\overline{\lambda}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\lambda}_{(\alpha_2,\beta_2)}[g]},\frac{\overline{\varrho}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\varrho}_{(\alpha_2,\beta_2)}[g]}\right\}\leq\max\left\{\frac{\overline{\lambda}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\lambda}_{(\alpha_2,\beta_2)}[g]},\frac{\overline{\varrho}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\varrho}_{(\alpha_2,\beta_2)}[g]}\right\}\\&\leq\limsup_{r\to+\infty}\frac{\alpha_1(T_{f\circ g}(r))}{\alpha_2(T_g(\beta_2^{-1}(\beta_1(r))))}\leq\frac{\overline{\varrho}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\lambda}_{(\alpha_2,\beta_2)}[g]}.\end{aligned}$$

The proofs of the following four theorems can be carried out as of the Theorem 2.1, therefore we omit the details.

Theorem 2.5. *Let f be a meromorphic function and g be an entire function such $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g] \le \sigma_{(\alpha_1,\beta_1)}[f\circ g] < \infty$, $0 < \overline{\sigma}_{(\alpha_2,\beta_2)}[f] \le \sigma_{(\alpha_2,\beta_2)}[f] < \infty$ and $\varrho_{(\alpha_1,\beta_1)}[f\circ g] = \varrho_{(\alpha_2,\beta_2)}[f]$. Then*

$$\begin{aligned}\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g]}{\sigma_{(\alpha_2,\beta_2)}[f]} &\le \liminf_{r\to+\infty}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r))))))}\\ &\le \min\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta_2)}[f]},\frac{\sigma_{(\alpha_1,\beta_1)}[f\circ g]}{\sigma_{(\alpha_2,\beta_2)}[f]}\right\} \le \max\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta_2)}[f]},\frac{\sigma_{(\alpha_1,\beta_1)}[f\circ g]}{\sigma_{(\alpha_2,\beta_2)}[f]}\right\}\\ &\le \limsup_{r\to+\infty}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r))))))} \le \frac{\sigma_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta_2)}[f]}.\end{aligned}$$

Theorem 2.6. *Let f be a meromorphic function and g be an entire function such $0 < \tau_{(\alpha_1,\beta_1)}[f\circ g] \le \overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g] < \infty$, $0 < \tau_{(\alpha_2,\beta_2)}[f] \le \overline{\tau}_{(\alpha_2,\beta_2)}[f] < \infty$ and $\lambda_{(\alpha_1,\beta_1)}[f\circ g] = \lambda_{(\alpha_2,\beta_2)}[f]$. Then*

$$\begin{aligned}\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[f]} &\le \liminf_{r\to+\infty}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r))))))}\\ &\le \min\left\{\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[f]},\frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[f]}\right\} \le \max\left\{\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[f]},\frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[f]}\right\}\\ &\le \limsup_{r\to+\infty}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r))))))} \le \frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[f]}.\end{aligned}$$

Theorem 2.7. *Let f be a meromorphic function and g be an entire function such $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g] \le \sigma_{(\alpha_1,\beta_1)}[f\circ g] < \infty$, $0 < \tau_{(\alpha_2,\beta_2)}[f] \le \overline{\tau}_{(\alpha_2,\beta_2)}[f] < \infty$ and $\varrho_{(\alpha_1,\beta_1)}[f\circ g] = \lambda_{(\alpha_2,\beta_2)}[f]$. Then*

$$\begin{aligned}\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[f]} &\le \liminf_{r\to+\infty}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r))))))}\\ &\le \min\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[f]},\frac{\sigma_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[f]}\right\} \le \max\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[f]},\frac{\sigma_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[f]}\right\}\\ &\le \limsup_{r\to+\infty}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r))))))} \le \frac{\sigma_{(\alpha_1,\beta_1)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[f]}.\end{aligned}$$

Theorem 2.8. *Let f be a meromorphic function and g be an entire function such $0 < \tau_{(\alpha_1,\beta_1)}[f\circ g] \le \overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g] < \infty$, $0 < \overline{\sigma}_{(\alpha_2,\beta_2)}[f] \le \sigma_{(\alpha_2,\beta_2)}[f] < \infty$ and $\lambda_{(\alpha_1,\beta_1)}[f\circ g] = \varrho_{(\alpha_2,\beta_2)}[f]$. Then*

$$\begin{aligned}\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]}{\sigma_{(\alpha_2,\beta_2)}[f]} &\le \liminf_{r\to+\infty}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r))))))}\\ &\le \min\left\{\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta_2)}[f]},\frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]}{\sigma_{(\alpha_2,\beta_2)}[f]}\right\} \le \max\left\{\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta_2)}[f]},\frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]}{\sigma_{(\alpha_2,\beta_2)}[f]}\right\}\\ &\le \limsup_{r\to+\infty}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(\beta_2^{-1}(\beta_1(r))))))} \le \frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta_2)}[f]}.\end{aligned}$$

Analogously one may formulate the following four theorems without their proofs.

Theorem 2.9. *Let f be a meromorphic function and g be an entire function such $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g]$*

$\leq \sigma_{(\alpha_1,\beta_1)}[f\circ g] < \infty$, $0 < \overline{\sigma}_{(\alpha_2,\beta_2)}[g] \leq \sigma_{(\alpha_2,\beta_2)}[g] < \infty$ *and* $\varrho_{(\alpha_1,\beta_1)}[f\circ g] = \varrho_{(\alpha_2,\beta_2)}[g]$. *Then*

$$\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g]}{\sigma_{(\alpha_2,\beta_2)}[g]} \leq \liminf_{r\to+\infty} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_g(\beta_2^{-1}(\beta_1(r))))))}$$
$$\leq \min\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta_2)}[g]}, \frac{\sigma_{(\alpha_1,\beta_1)}[f\circ g]}{\sigma_{(\alpha_2,\beta_2)}[g]}\right\} \leq \max\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta_2)}[g]}, \frac{\sigma_{(\alpha_1,\beta_1)}[f\circ g]}{\sigma_{(\alpha_2,\beta_2)}[g]}\right\}$$
$$\leq \limsup_{r\to+\infty} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_g(\beta_2^{-1}(\beta_1(r))))))} \leq \frac{\sigma_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta_2)}[g]}.$$

Theorem 2.10. *Let f be a meromorphic function and g be an entire function such* $0 < \tau_{(\alpha_1,\beta_1)}[f\circ g] \leq \overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g] < \infty$, $0 < \tau_{(\alpha_2,\beta_2)}[g] \leq \overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ *and* $\lambda_{(\alpha_1,\beta_1)}[f\circ g] = \lambda_{(\alpha_2,\beta_2)}[g]$. *Then*

$$\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]} \leq \liminf_{r\to+\infty} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_g(\beta_2^{-1}(\beta_1(r))))))}$$
$$\leq \min\left\{\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}, \frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]}\right\} \leq \max\left\{\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}, \frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]}\right\}$$
$$\leq \limsup_{r\to+\infty} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_g(\beta_2^{-1}(\beta_1(r))))))} \leq \frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}.$$

Theorem 2.11. *Let f be a meromorphic function and g be an entire function such* $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g] \leq \sigma_{(\alpha_1,\beta_1)}[f\circ g] < \infty$, $0 < \tau_{(\alpha_2,\beta_2)}[g] \leq \overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ *and* $\varrho_{(\alpha_1,\beta_1)}[f\circ g] = \lambda_{(\alpha_2,\beta_2)}[g]$. *Then*

$$\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]} \leq \liminf_{r\to+\infty} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_g(\beta_2^{-1}(\beta_1(r))))))}$$
$$\leq \min\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}, \frac{\sigma_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]}\right\} \leq \max\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_1)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}, \frac{\sigma_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]}\right\}$$
$$\leq \limsup_{r\to+\infty} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_g(\beta_2^{-1}(\beta_1(r))))))} \leq \frac{\sigma_{(\alpha_1,\beta_1)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}.$$

Theorem 2.12. *Let f be a meromorphic function and g be an entire function such* $0 < \tau_{(\alpha_1,\beta_1)}[f\circ g] \leq \overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g] < \infty$, $0 < \overline{\sigma}_{(\alpha_2,\beta_2)}[g] \leq \sigma_{(\alpha_2,\beta_2)}[g] < \infty$ *and* $\lambda_{(\alpha_1,\beta_1)}[f\circ g] = \varrho_{(\alpha_2,\beta_2)}[g]$. *Then*

$$\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]}{\sigma_{(\alpha_2,\beta_2)}[g]} \leq \liminf_{r\to+\infty} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_g(\beta_2^{-1}(\beta_1(r))))))}$$
$$\leq \min\left\{\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta_2)}[g]}, \frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]}{\sigma_{(\alpha_2,\beta_2)}[g]}\right\} \leq \max\left\{\frac{\tau_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta_2)}[g]}, \frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]}{\sigma_{(\alpha_2,\beta_2)}[g]}\right\}$$
$$\leq \limsup_{r\to+\infty} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_g(\beta_2^{-1}(\beta_1(r))))))} \leq \frac{\overline{\tau}_{(\alpha_1,\beta_1)}[f\circ g]}{\overline{\sigma}_{(\alpha_2,\beta_2)}[g]}.$$

3 Conclusion

The main aim of this paper is to develop some results relating to the growth rates of composite entire and meromorphic functions with their corresponding left and right factors on the basis of their generalized order (α,β) and generalized type (α,β), which have not studied previously and the results obtained has a great significient in the filed of growth analysis of entire and meromorphic functions. But still there remains some problems to be investigated for future researchersin this field.

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