Palestine Journal of Mathematics

Vol. 11(1)(2022) , 363–369

THE SINGULAR KIRCHHOFF EQUATION UNDER SMALL TENSION

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Communicated by Suheil Khoury

MSC 2010 Classifications: Primary 35J25; Secondary 35J15, 37L65, 35J60, 65N30.

I dedicate this paper to the memory of Prof. Fernando Torres and acknowledge the support of CNPq and FAPESP.

Keywords and phrases: Existence of solution, Kirchhoff equation, tension, singular, sublinear, superlinear.

Abstract We show existence of a positive solution for the Kirchhoff equation with small tension forces and general nonlinearities, including singular terms. We use an approximation scheme of Galerkin.

1 Introduction

The generalized wave equation

$$u_{tt} - (a + b\int_\Omega |\nabla u|^2)\Delta u = f(x,u), \tag{1.1}$$

describes changes in length u when a string with fixed extremes subject to transversal vibrations due to a force f. The term $(a + b\int_\Omega |\nabla u|^2)$ as well as the constants $a > 0$ and $b \geq 0$ stand for horizontal tensions. Equation (1.1) was introduced a long time ago in the paper [13], and it can be viewed as an extension of D'Alembert's wave equation for free vibration strings, where $a = 1$ is a normalization constant and $b = 0$. Equation (1.1) also serves as a prototypal to study some parabolic equations $u_t - \phi(\int_\Omega |\nabla u|^2)\Delta u = f(x,u)$ and stationary nonlocal equations $-\phi(\int_\Omega |\nabla u|^2)\Delta u = f(x,u)$, according to [3, 4, 5, 6, 7], where, say, the tension function ϕ has suitable assumptions.

Notice that if $u \in H_0^1(\Omega)$, then $(a + b\int_\Omega |u|^2)^\gamma \leq (a + b\lambda_1^{-1}\int_\Omega |\nabla u|^2)^\gamma$ for $\gamma \geq 0$. This is due to the Poincaré inequality that reads as $\int_\Omega |u|^2 \leq \lambda_1^{-1}\int_\Omega |\nabla u|^2$ for every $u \in H_0^1(\Omega)$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ with corresponding eigenfunction $\varphi_1 > 0$. And for this reason the tension term $(a+b\int_\Omega |u|^2)^\gamma$ can be smaller when compared to other situations as well as with (1.1).

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain with smooth boundary $\partial\Omega$. Our aim is to solve the problem.

$$\begin{cases} -(a + b\int_\Omega |u|^2)^\gamma \Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The classical Kirchhoff equation $-(a + b\int_\Omega |\nabla u|^2)^\gamma \Delta u = f(u)$ with zero boundary condition has been treated in many papers, we quote [1, 8, 10] for variational techniques, and also [16, 17] with dimensional restrictions were explored. A fixed point approach was developed in [11, 12], see also [2]. The global solvability was addressed in [9].

Define

$$f(t) = \alpha\frac{1}{t^\theta} + \lambda t^q + \mu t + g(t) \quad \text{for } t \geq 0 \tag{1.3}$$

where

$$\alpha > 0,\, \lambda > 0,\, \mu \geq 0,\, 0 < \theta < 1,\, 0 < q < 1. \tag{1.4}$$

The constants in the equation (1.2) satisfy

$$\gamma \geq 0,\, a > 0,\, b \geq 0\,. \tag{1.5}$$

The continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfies

$$|g(t)| \leq k_0|t|^p \text{ for } t \in \mathbb{R} \text{ and } 1 \leq p < 2N/(N-2), \text{ where } k_0 \text{ is a constant.} \tag{1.6}$$

By a solution of (1.2) we mean a function $u \in H_0^1(\Omega)$ such that

$$(a + b\int_\Omega |u|^2)^\gamma \nabla u \nabla \phi - f(u)\phi = 0, \quad \forall \phi \in H_0^1(\Omega). \tag{1.7}$$

We prescribe weaker assumptions on f compared to the quoted papers, since we do not need the so-called Ambrosetti–Rabinowitz condition that guarantees convergence of specific sequences when applying variational methods, see also [14]. Moreover we do not need to assume f close to zero near the origin, in fact our f is singular at zero. Moreover, assume $g \equiv 0$ for simplicity in the expression of f at (1.3), then the functional $I : H_0^1(\Omega) \to \mathbb{R}$ corresponding to (1.2) is given by

$$I(u) = (a + b\int_\Omega |u|^2)^\gamma \int_\Omega |\nabla u|^2 - \left(\frac{\alpha}{1-\theta}|u|^{1-\theta} + \frac{\lambda}{q+1}|u|^{q+1} + \frac{\mu}{2}|u|^2\right).$$

Since $0 < \theta < 1$, then I is not C^1, thus the classical variational theory cannot be employed in order to find critical points, and ensure the validity of the weak solution relation (1.7). The approach we perform in the present paper does not permit us to find multiple solutions, as the variational theory does. A possible way to extend the method of this paper is to combine it with the ideas of [15], where parabolic and wave equations are studied by means of the Galerkin scheme with f depending only on x.

We deal with (1.2) by solving a sequence of problems in finite dimension by means of a fixed point theorem, and finally we appeal to a convergence argument that leads us to a solution of (1.2).

Firstly we solve the Kirchhoff model problem (1.2) with f as in (1.3) with $g \equiv 0$, this case allow the parameters to vary in large range. Secondly, we solve (1.2) with a more general f with $g \not\equiv 0$, in this situation some parameters cannot be large. We also analyze what happens with the solution when $\lambda \to 0$ and $\lambda \to \infty$. We state the main results.

Theorem 1.1. *Assume (1.3)–(1.5) and $g \equiv 0$. There is $\mu^* > 0$ such that for $0 \leq \mu < \mu^*$ and for every $\alpha, \lambda > 0$, equation (1.2) has a positive solution.*

Theorem 1.2. *Assume (1.3)–(1.6). Then there exist $\alpha^*, \lambda^*, \mu^* > 0$ such that for every $0 < \alpha < \alpha^*$, $0 < \lambda < \lambda^*$ and $0 \leq \mu < \mu^*$ equation (1.2) has a positive solution.*

Theorem 1.3. *Let f be such that $\alpha = \mu = 0$ and $g(t) = t^p$ for $t \geq 0$ with $1 < p < 2N/(N-2)$. And let $u_\lambda > 0$ be the solution obtained in Theorem 1.2. Then $\|u_\lambda\|_{H_0^1} \to 0$ as $\lambda \to 0$.*

Theorem 1.4. *Let $f(t) = \lambda\left(\frac{1}{t^\theta} + t^q + t\right) + t^p$ for $t \geq 0$ with $1 < p < 2N/(N-2)$. And let $u_\lambda > 0$ be the solution obtained in Theorem 1.2. If u_λ exists for every large $\lambda > 0$, then $\|u_\lambda\|_{H_0^1} \to \infty$ as $\lambda \to \infty$.*

2 Preliminaries

The space $H_0^1(\Omega)$ is Hilbert with inner product $(u,v) = \int_\Omega \nabla u \nabla v$ and norm $\|u\|_{H_0^1} = \left(\int_\Omega |\nabla u|^2\right)^{1/2}$. The spectrum of $-\Delta$ in $H_0^1(\Omega)$ is given by the numbers λ_i, $i \in \mathbb{N}$, where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \ldots$. The corresponding eigenfunctions are $\varphi_i \in H_0^1(\Omega)$, $i \in \mathbb{N}$. The first eigenfunction corresponding to λ_1 is $\varphi_1 > 0$. For every $i \in \mathbb{N}$ one has

$$\begin{cases} -\Delta \varphi_i = \lambda_i \varphi_i & \text{in } \Omega \\ \quad \varphi_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

By elliptic regularity $\varphi_i \in C^\infty(\overline{\Omega})$, $i \in \mathbb{N}$. The following orthogonality relations take place

$$\int_\Omega \nabla\varphi_i \nabla\varphi_j = \lambda_j \int_\Omega \varphi_i\varphi_j = 0 \quad \text{if} \;\; i \neq j. \tag{2.2}$$

The set of eigenfunctions can be normalized as $\|\varphi_i\|_{H_0^1} = 1$, $i \in \mathbb{N}$. Hence $\mathcal{B} = \{\varphi_1, \varphi_2, \ldots, \varphi_m, \ldots\}$ is an orthonormal basis of $H_0^1(\Omega)$.

A result that will be useful is Brouwer's Theorem [15]. The statement is: Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous function such that $(F(\xi), \xi) \geq 0$ for every $\xi \in \mathbb{R}^m$ with $|\xi| = r$ for some $r > 0$. Then, there exists $y_0 \in \mathbb{R}^m$ with $|y_0| \leq r$ such that $F(y_0) = 0$.

3 Existence of solution

We now prove Theorem 1.1.

Proof. Let $f_\varepsilon(t) = \alpha\frac{1}{(t+\varepsilon)^\theta} + \lambda t^q + \mu t$ with $0 < \varepsilon < 1$ and let $\mathcal{B} = \{\varphi_1, \varphi_2, \ldots, \varphi_m, \ldots\}$ be an orthonormal basis of $H_0^1(\Omega)$, for instance see (2.2). Define

$$\mathcal{W}_m = [\varphi_1, \varphi_2, \ldots, \varphi_m],$$

to be the space spanned by $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$. Define the function $F : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$F(\xi) = (F_1(\xi), F_2(\xi), \ldots, F_m(\xi))$$

where $\xi = (\xi_1, \xi_2, ..., \xi_m) \in \mathbb{R}^m$,

$$F_j(\xi) = (a + b\int_\Omega |u|^2)^\gamma \int_\Omega \nabla u \nabla\varphi_j - \int_\Omega f_\varepsilon(|u|)\varphi_j, \quad j = 1, 2, \ldots, m$$

and

$$u = \sum_{i=1}^m \xi_i\varphi_i \quad \in \mathcal{W}_m.$$

Therefore

$$(F(\xi), \xi) = (a + b\int_\Omega |u|^2)^\gamma \int_\Omega |\nabla u|^2 - \int_\Omega f_\varepsilon(|u|)u \geq \tag{3.1}$$

$$\geq a^\gamma \|u\|_{H_0^1}^2 - \alpha|\Omega|^\theta C_1^{1-\theta}\|u\|_{H_0^1}^{1-\theta} - \lambda C_{q+1}^{q+1}\|u\|_{H_0^1}^{q+1} - \mu C_2^2\|u\|_{H_0^1}^2. \tag{3.2}$$

The function F is continuous because each F_j is continuous by Sobolev embedding and dominated convergence theorem. Here C_1, C_2 and C_{q+1} are Sobolev constants related to $\|u\|_{H_0^1} \leq C_\sigma\|u\|_{L^\sigma}$ with $1 \leq \sigma < 2N/(N-2)$, which are independent on m and ε. Hence, there is $R_0 > 0$ such that

$$(F(\xi), \xi) > 0 \quad \text{for} \;\; \|u\|_{H_0^1} = |\xi| = R_0. \tag{3.3}$$

Brouwer's Theorem asserts that there exists $u_{m,\varepsilon} \in H_0^1$ with $\|u_{m,\varepsilon}\|_{H_0^1} \leq R_0$ satisfying

$$(a + b\int_\Omega |u_{m,\varepsilon}|^2)^\gamma \int_\Omega \nabla u_{m,\varepsilon}\nabla\varphi_j - \int_\Omega f_\varepsilon(|u_{m,\varepsilon}|)\varphi_j = 0, \quad j = 1, 2, \ldots, m. \tag{3.4}$$

Hence

$$(a + b\int_\Omega |u_{m,\varepsilon}|^2)^\gamma \int_\Omega \nabla u_{m,\varepsilon}\nabla\zeta_m - \int_\Omega f_\varepsilon(|u_{m,\varepsilon}|)\zeta_m = 0, \quad \forall\zeta_m \in \mathcal{W}_m.$$

Let $k \in \mathbb{N}$, then for every $m \geq k$ we obtain

$$(a + b\int_\Omega |u_{m,\varepsilon}|^2)^\gamma \int_\Omega \nabla u_{m,\varepsilon}\nabla\zeta_k - \int_\Omega f_\varepsilon(|u_{m,\varepsilon}|)\zeta_k = 0, \quad \forall\zeta_k \in \mathcal{W}_k. \tag{3.5}$$

Since $\|u_{m,\varepsilon}\|_{H_0^1} \leq R_0$ and H_0^1 is reflexive, there exists $u_\varepsilon \in H_0^1$ such that

$$(a_1) \quad u_{m,\varepsilon} \rightharpoonup u_\varepsilon \quad \textit{weakly in } H_0^1 \text{ as } m \to \infty$$
$$(a_2) \quad u_{m,\varepsilon} \to u_\varepsilon \quad \textit{in } L^\sigma \text{ for } 1 \le \sigma < 2N/(N-2) \text{ as } m \to \infty$$

Letting $m \to \infty$ in the expression (3.5), with the aid of (a_1)-(a_2), we get

$$(a + b\int_\Omega |u_\varepsilon|^2)^\gamma \int_\Omega \nabla u_\varepsilon \nabla \zeta_k - \int_\Omega f_\varepsilon(|u_\varepsilon|)\zeta_k = 0, \quad \forall \zeta_k \in \mathcal{W}_k.$$

Since the space of all subspace $[\mathcal{W}_m]_{k\in\mathbb{N}}$ is dense in H_0^1, then

$$(a + b\int_\Omega |u_\varepsilon|^2)^\gamma \int_\Omega \nabla u_\varepsilon \nabla \zeta - \int_\Omega f_\varepsilon(|u_\varepsilon|)\zeta = 0, \quad \forall \zeta \in H_0^1. \tag{3.6}$$

Hence u_ε is a weak solution of

$$\begin{cases} -(a + b\int_\Omega |u_\varepsilon|^2)^\gamma \Delta u_\varepsilon = f_\varepsilon(|u_\varepsilon|) & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

Observe that $f_\varepsilon(u_\varepsilon) > 0$, hence we are in position to use the maximum principle. Consequently, $u_\varepsilon > 0$ in Ω. Hence u_ε satisfies

$$\begin{cases} -(a + b\int_\Omega |u_\varepsilon|^2)^\gamma \Delta u_\varepsilon = f_\varepsilon(u_\varepsilon) & \text{in } \Omega \\ u_\varepsilon > 0 & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

We claim that $u_\varepsilon \ge \delta_0\varphi_1$ in Ω for some $\delta_0 > 0$. Indeed, notice that there is a constant $\varrho > 0$, which does not depend on ε, such that

$$f_\varepsilon(t) = \alpha\frac{1}{(t+\varepsilon)^\theta} + \lambda t^q + \mu t \ge \alpha\frac{1}{(t+1)^\theta} + \lambda t^q \ge \varrho \quad \text{for } t \ge 0.$$

Let $\psi = \delta\varphi_1$ with $\delta > 0$ and notice that $\|u_\varepsilon\|_{H_0^1} \le \liminf_{m\to\infty} \|u_{m,\varepsilon}\|_{H_0^1} \le R_0$, then

$$-(a + b\int_\Omega |u_\varepsilon|^2)^\gamma \Delta\psi = \delta\varphi_1\lambda_1(a + b\int_\Omega |u_\varepsilon|^2)^\gamma \le \delta\varphi_1\lambda_1(a + b\frac{R_0^2}{\lambda_1})^\gamma \le \varrho \tag{3.8}$$

where the last inequality (3.8) is valid by taking $\delta > 0$ small enough, and it is independent on ε. Comparing with (3.7) and using the boundary condition $v = \psi = 0$ on $\partial\Omega$, we obtain by the maximum principle that there is $\delta_0 > 0$ such that $u_\varepsilon \ge \delta_0\varphi_1$ in Ω.

Since $\|u_\varepsilon\|_{H_0^1} \le R_0$. By Sobolev embedding and still denoting a subsequence $\varepsilon = \varepsilon_n \to 0$, then

$$(b_1) \quad u_\varepsilon \rightharpoonup u_0 \quad \textit{weakly in } H_0^1 \textit{ as } \varepsilon \to 0$$
$$(b_2) \quad u_\varepsilon \to u_0 \quad \textit{in } L^\sigma \text{ for } 1 \le \sigma < 2N/(N-2) \textit{ as } \varepsilon \to 0$$
$$(b_3) \quad u_\varepsilon \to u_0 \quad \textit{a.e. in } \Omega \textit{ as } \varepsilon \to 0$$
$$(b_4) \quad |u_\varepsilon| \le h(x) \quad \textit{a.e. in } \Omega, \text{ for } \textit{some } h \in L^\sigma,\ 1 \le \sigma < 2N/(N-2)$$

We conclude that $u_0 \ge \delta_0\varphi_1$ in Ω. We rewite (3.6) next

$$(a + b\int_\Omega |u_\varepsilon|^2)^\gamma \int_\Omega \nabla u_\varepsilon \nabla\zeta - \int_\Omega (\alpha\frac{1}{(u_\varepsilon+\varepsilon)^\theta} + \lambda u_\varepsilon^q + \mu u_\varepsilon)\zeta = 0, \quad \forall\zeta \in H_0^1. \tag{3.9}$$

Using (b_1)-(b_4) and letting $\varepsilon \to 0$ in (3.9) we obtain

$$(a + b\int_\Omega |u_0|^2)^\gamma \int_\Omega \nabla u_0 \nabla\zeta - \int_\Omega (\alpha\frac{1}{u_0^\theta}\zeta + \lambda u_0^q + \mu u_0)\zeta = 0, \quad \forall\zeta \in H_0^1. \tag{3.10}$$

The first integral of (3.10) is a consequence of (b_1)-(b_2). The integral of u_0^q follows from (b_3)-(b_4) and dominated convergence theorem. The integral involving μ follows by (b_2). Notice that $\int_\Omega u_0^{-\theta} \le \delta_0^{-\theta} \int_\Omega \varphi_1^{-\theta} < \infty$. Then,

$$\int_\Omega \frac{1}{(u_\varepsilon+\varepsilon)^\theta}\zeta \to \int_\Omega \frac{1}{u_0^\theta}\zeta, \quad \forall \zeta \in H_0^1, \tag{3.11}$$

since by dominated convergence theorem we can write (3.11) with $\zeta \in C_0^\infty(\Omega)$, and since H_0^1 is the completion of $C_0^\infty(\Omega)$, we can take $\zeta \in H_0^1$, hence (3.11) holds for every $\zeta \in H_0^1$.

□

We proceed to prove of Theorem 1.2.

Proof. Let $\mathcal{B}$, $\mathcal{W}_m$ and F as defined in the proof of Theorem 1.1. Define the approximation $f_\varepsilon(t) = \alpha\frac{1}{(t+\varepsilon)^\theta} + \lambda t^q + \mu t + g(t)$ with $0<\varepsilon<1$. Estimate (3.1) now transforms into

$$(F(\xi),\xi) = (a + b\int_\Omega |u|^2)^\gamma \int_\Omega |\nabla u|^2 - \int_\Omega f_\varepsilon(|u|)u \ge \tag{3.12}$$

$$\ge a^\gamma \|u\|^2_{H_0^1} - \alpha|\Omega|^\theta C_1^{1-\theta}\|u\|^{1-\theta}_{H_0^1} - \lambda C_{q+1}^{q+1}\|u\|^{q+1}_{H_0^1} - k_0 C_{p+1}^{p+1}\|u\|^{p+1}_{H_0^1} - \mu C_2^2\|u\|^2_{H_0^1}.$$

Hence, there is a constant $K_0>0$ such that

$$(F(\xi),\xi) \ge a^\gamma\|u\|^2_{H_0^1} - K_0\left(\alpha\|u\|^{1-\theta}_{H_0^1} + \lambda\|u\|^{q+1}_{H_0^1} + \|u\|^{p+1}_{H_0^1} + \mu\|u\|^2_{H_0^1}\right). \tag{3.13}$$

We will choose R_0, α^*, μ^* and λ^* according to our needs. Let $\|u\|_{H_0^1} = R_0$. Thus we require

$$R_0 = \min\{1, \frac{1}{2}(\frac{2}{3K_0})^{1/(p-1)}\}.$$

We want

$$\alpha < (\frac{1}{2})^{1+\theta}(\frac{2}{3K_0})^{1+\theta/(p-1)}\frac{2}{3K_0}.$$

Hence $\alpha<\alpha^*$ if we take

$$\alpha^* = \frac{1}{2}(\frac{1}{2})^{1+\theta}(\frac{2}{3K_0})^{1+\theta/(p-1)}\frac{2}{3K_0}.$$

We need

$$\mu < 2/3K_0,$$

thus $\mu<\mu^*$ is satisfied if one has

$$\mu^* = \frac{1}{3K_0}.$$

Once R_0 has been chosen, we seek λ^* such that $R_0^2 - K_0\lambda R_0^{q+1} > 0$, i.e., $\lambda < R_0^{1-q}/K_0$ for $\lambda<\lambda^*$. Hence we take

$$\lambda^* = \frac{1}{K_0}(\frac{1}{2})^{2-q}(\frac{2}{3K_0})^{(1-q)/(p-1)}.$$

Thus, let

$$\Pi = R_0^2 - K_0\lambda^* R_0^{q+1} > 0.$$

Therefore,

$$(F(\xi),\xi) > \Pi \quad \text{for } \|u\|_{H_0^1} = |\xi| = R_0. \tag{3.14}$$

Brouwer's Theorem states that there exists $u_{m,\varepsilon} \in H_0^1$ with $\|u_{m,\varepsilon}\|_{H_0^1} \le R_0$ satisfying (3.4). Notice that there is a constant $\varrho>0$, which does not depend on ε such that

$$f_\varepsilon(t) = \alpha\frac{1}{(t+\varepsilon)^\theta} + \lambda t^q + \mu t + g(t) \ge \alpha\frac{1}{(t+1)^\theta} + \lambda t^q \ge \varrho \quad \text{for } t\ge 0.$$

The accomplishment of the proof follows almost verbatim the steps (3.4) to (3.11).

□

4 Behavior of the solution

We prove Theorem 1.3 next.

Proof. Here $f(t) = \lambda t^q + t^p$. The solution u_λ satisfies

$$a^\gamma \|u_\lambda\|^2_{H_0^1} \le (a + b\int_\Omega |u_\lambda|^2)^\gamma \int_\Omega |\nabla u_\lambda|^2 = \int_\Omega f(u_\lambda)u_\lambda =$$

$$\int_\Omega \lambda u_\lambda^{q+1} + u_\lambda^{p+1} \le \lambda C_{q+1}^{q+1}\|u_\lambda\|^{q+1}_{H_0^1} + C_{p+1}^{p+1}\|u_\lambda\|^{p+1}_{H_0^1}.$$

Then

$$\|u_\lambda\|^{1-q}_{H_0^1} \le \frac{\lambda C_{q+1}^{q+1}}{1 - C_{p+1}^{p+1}\|u_\lambda\|^{p-1}_{H_0^1}}.$$

By the choice of $R_0 \le 1$ we get

$$1 - C_{p+1}^{p+1}\|u_\lambda\|^{p-1}_{H_0^1} \ge 1/2.$$

Hence $\|u_\lambda\|_{H_0^1} \le \big(2\lambda C_{q+1}^{q+1}\big)^{1/(1-q)} \to 0$ as $\lambda \to 0$. □

The proof of Theorem 1.4 is as follows.

Proof. We denote the solution by u_λ. Since $\|u_\lambda\|_{H_0^1} \le R_0$, the term $(a+b\int_\Omega |u_\lambda|^2)^\gamma$ is bounded. Suppose on the contrary that for every $\lambda > 0$, by Theorem 1.1, there is a solution u_λ. Multiply the equation (1.2) by φ_1, integrate and use (2.1), hence

$$\int_\Omega f(u_\lambda)\varphi_1 = \lambda_1 \int_\Omega u_\lambda\varphi_1(a + b\int_\Omega |u_\lambda|^2)^\gamma \le \lambda_1 M_0 \int_\Omega u_\lambda\varphi_1, \tag{4.1}$$

for a constant $M_0 > 0$ independent on λ. Since $f(t) = \lambda\big(\frac{1}{t^\theta} + t^q + t\big) + t^p \ge \lambda t^q + t^p$ for $t \ge 0$, then $f(t) \ge \lambda^{(p-1)(p-q)}C_{p,q}t$ for $t \ge 0$, where $C_{p,q} > 0$ is a constant depending only on p and q. Hence by (4.1) one obtains

$$\lambda^{(p-1)(p-q)}C_{p,q}\int_\Omega u_\lambda\varphi_1 \le \lambda_1 M_0 \int_\Omega u_\lambda\varphi_1.$$

Thus λ is bounded, which a contradiction.

□

Concluding remarks. We solved equation (1.2) when $0 \le \mu < \mu^*$, it would be interesting to know if a solution exists whether $\mu \ge \mu^*$, as well as the behavior of the H_0^1-norm of a such solution with respect to the parameters $\alpha > 0$ and $\lambda > 0$. Another remaining question is related to the applicability of the methods of the present paper to find a solution when $p = 2N/(N-2)$, that is, when f has critical growth. The problem in dimension 2 is also challenging.

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Received: October 22, 2020.
Accepted: January 7, 2021.