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# Numerical solution of a seventh order boundary value problem by splitting coupled finite difference method

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**Abstract** In the present article we are concerned with the numerical solution of a seventh order boundary value problem. We have proposed a novel finite difference method, derived from the splitting coupled equations approach. Under appropriate conditions, we have established the convergence of the proposed method. We also acquired the numerical value of the derivative of the solution to the problem, which is practically useful for some modeling problems. Numerical results are in good agreement with the theoretical findings.

## 1 Introduction

In the present article we consider seventh order boundary value problem of the following form:

$$u^{(7)}(x) = f(x, u), \quad a < x < b, \tag{1.1}$$

subject to the boundary conditions

$$\begin{gathered} u(a) = \alpha_1, u'(a) = \alpha_2, u''(a) = \alpha_3, u^{(3)}(a) = \alpha_4, \\ u(b) = \beta_1, u'(b) = \beta_2 \quad \text{and} \quad u''(b) = \beta_3, \end{gathered}$$

where $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$, $\beta_1$, $\beta_2$ and $\beta_3$ are real constant.
The problems in physical sciences can be modeled mathematically and formulated by differential equations. The problems in engineering sciences deal with the formulation and solution of higher order differential equation. The higher order differential equation and boundary value problem studied and discussed in [1]. In particular seventh order boundary value problems arise in mathematical modeling of induction motors with two rotor circuits [2]. To ensure the existence and uniqueness of the solution of the problem (1.1), we presume the smoothness of the forcing function $f(x, u)$. However for the detail discussion on the existence and uniqueness of the solution of higher order differential equations and corresponding BVPs, we can refer [3]. In the present article, we concerned with numerical solution of reference problem instead of the analytical solution.

In the literature on the numerical solutions of BVPs, several numerical methods have been reported for seventh order boundary value problems. We can list some of them for instance Variational Iteration Method [4], Variation of Parameters Method [5], Differential Transformation Method [6], Reproducing Kernel Space [7], Collocation Method using Sextic B- Splines [8], Homotopy Analysis Method [9], Optimal Homotopy Asymptotic Method [10] and references there in.

Some advance numerical techniques for numerical solution of boundary value problems have been reported in the literature. These techniques are very satisfactory and yield a highly accurate numerical solution. Hence, the purpose of this article is to incorporate these advancements in developing numerical technique for numerical solution of seventh order boundary value problems (1.1). So we incorporated the those ideas in developing an accurate and convergent finite difference method for numerical solution of seventh order boundary value problem by splitting method, a system of boundary value problems. We hope that others may find the proposed method as an improvement in numerical technique to those existing techniques for the seventh order boundary value problems in the literature.

We shall present our work in this article as follows: In Section 2 the finite difference method, in Section 3 the derivation of the proposed finite difference method. In Section 4, the convergence analysis of the proposed method under appropriate condition. The numerical experiment on model problems and short discussion on numerical results are presented in Section 5. A summary on the overall development and performance of the proposed method are presented in Section 6.

## 2 The Difference Method

Let us assume problem (1.1) posses solution and it will be $u(x)$ such that

$$u^{(4)}(x) = v(x), \quad a < x < b \tag{2.1}$$

and the boundary conditions are

$$u(a) = \alpha_1, \quad u'(a) = \alpha_2, \quad u(b) = \beta_1 \quad \text{and} \quad u'(b) = \beta_2$$

where augment fuction $v(x)$ is regular and differentiable in $[a, b]$. Further we have following third order boundary value problem,

$$v^{(3)}(x) = f(x, u), \quad a < x < b \tag{2.2}$$

and the boundary conditions are

$$u''(a) = \alpha_3, \quad u^{(3)}(a) = \alpha_3 \quad \text{and} \quad u''(b) = \beta_3$$

To incorporate these boundary conditions, let us define

$$v(x) = u^{(4)}(x) - \lambda u''(x) \tag{2.3}$$

where $\lambda$ is coupling constant and $\lambda \in (0, 1)$. So we get problems (2.1)-(2.3), a system of boundary value problems by splitting method from problem (1.1). Thus the seventh order boundary value problem (1.1) has been transformed into a system of boundary value problems (2.1)-(2.3). Solving numerically problem (1.1) is equivalent to solve numerically system of problems (2.1)-(2.3).

We partition the interval [a,b] in which the solution of problem (1.1) is desired to introduce finite number of mesh points. In these subintervals mesh points $a \leq x_0 < x_1 < x_2 < ...... < x_{N+1} \leq b$ are generated by using uniform step length $h$ such that $x_i = a + ih, \quad i = 0, 1, 2, ....., N + 1$. We wish to determine the numerical solution $u(x)$ of the problem (1.1) at these mesh points $x_i$. We denote the numerical approximation of $u(x)$ and $f(x, u(x))$ respectively by $u_i$ and $f_i$ at these mesh point $x = x_i, \quad i = 1, 2, ....., N$. Also the boundary value problem (1.1) replaced by the system of boundary value problems (2.1)-(2.3) may be written as under

$$\begin{aligned} u_i^{(4)} &= v_i, \\ v_i^{(3)} &= f_i \end{aligned} \tag{2.4}$$

at these node $x = x_i, i = 0, .., N + 1$. Following the ideas in [11, 12], we propose our finite difference method for a numerical solution of problem (2.4),

$$-2(u_{i-1} - 2u_i - u_{i+1}) + h(u'_{i+1} - u'_{i-1}) = \frac{h^4}{90}(v_{i+1} + 13v_i + v_{i-1}), \tag{2.5}$$

$$-3(u_{i+1} - u_{i-1}) + h(u'_{i+1} + 4u'_i + u'_{i-1}) = \frac{h^4}{60}(v_{i+1} - v_{i-1}), \tag{2.6}$$

$$-3v_{i-1} + 4v_i - v_{i+1} = 2hv'_{i-1} + \frac{h^3}{6}(3f_i + f_{i+1}), \quad i = 1 \tag{2.7}$$

$$v_{i-2} - 3v_{i-1} + 3v_i - v_{i+1} = \frac{h^3}{2}(-3f_i + f_{i+1}), \quad 2 \le i \le N$$

If the source function $f(x,u)$ in problem (1.1) is linear then the system of equations (2.5)-(2.7) will be linear otherwise we will obtain nonlinear system of equations.

## 3 Derivation of the Difference Method

In this section we out line the derivation of the proposed method, we have followed the same approach as given in [11, 12, 13]. Let us write a linear combination of solution $u(x)$ , $u'(x)$ and $v(x)$ at nodes $x_{x\pm1}$,and $x_i$,

$$a_2u_{i+1} + a_1u_{i-1} + a_0u_i + h(b_2u'_{i+1} + b_1u'_{i-1}) + h^4(c_2v_{i+1} + c_0v_i + c_1v_{i-1}) = 0 \tag{3.1}$$

where $a_0 - c_0$ are constants to be determined. to determine these constants, we expanding each term on the left hand side of (3.1) in Taylor series about the point $x_i$. Using method of undetermine coefficients, compare the coefficients of $h^p, p = 0, 1, .., 7$ on both side we get a system of equations. Solving this system of equations, we get

$$(a_2, a_1, a_0, b_2, b_1, c_2, c_0, c_1) = (-2, -2, 4, 1, -1, -\frac{1}{90}, -\frac{13}{90}, -\frac{1}{90}) \tag{3.2}$$

On substitution of these constants $a_0 - c_0$ from (3.1)into (3.2) and simplify, we have

$$-(v_{i-1} + 2v_i - v_{i+1}) + h(u'_{i+1} - u'_{i-1}) - \frac{h^4}{90}(v_{i+1} + 13v_i + v_{i-1}) + tu_i = 0 \tag{3.3}$$

where $tu_i, i = 1, .., N$ is local truncation error and equal to $-\frac{19h^8}{30240}u_i^{(8)}$. Similarly we can derive the following equations

$$-3(u_{i+1} - u_{i-1}) + h(u'_{i+1} + 4u'_i + u'_{i-1}) - \frac{h^4}{60}(v_{i+1} - v_{i-1}) + tu'_i, \tag{3.4}$$

where local local error $tu'_i$ is equal to $-\frac{h^5}{504}u_i^{(7)}, i = 1, .., N$ and

$$-3v_{i-1} + 4v_i - v_{i+1} - 2hv'_{i-1} - \frac{h^3}{6}(3f_i + f_{i+1}) + tv_i, \quad i = 1 \tag{3.5}$$

$$v_{i-2} - 3v_{i-1} + 3v_i - v_{i+1} - \frac{h^3}{2}(-3f_i + f_{i+1}) + tv_i, \quad 2 \le i \le N$$

where local truncation error $tv_i$ are respectively equal to $-\frac{3h^5}{20}v_i^{(5)}, i = 1$ and $-\frac{h^5}{2}v_i^{(5)}, 2 \le i \le N$.

Thus by neglecting the local error terms in (3.3)-(3.5), we will get our proposed difference method for the numerical solution of the problem (1.1). Moreover we are getting the numerical value of the derivative of the solution of the problem (1.1) as a by product of the method. Some times we need it which otherwise get approximated.

## 4 Convergence Analysis

In this section we will discuss the convergence of the method proposed in section 1. Thus for the discussion of convergence let us consider following test equation.

$$u^{(7)}(x) = f(x,u), \qquad a < x < b. \tag{4.1}$$

$$u(a) = \alpha_1,\ u'(a) = \alpha_2,\ u''(a) = \alpha_3,\ u^{(3)}(a) = \alpha_4,$$
$$u(b) = \beta_1,\ u'(b) = \beta_2 \text{ and } u''(b) = \beta_3$$

Let **s** be the approximate solution of difference method (2.4-2.5) for numerical solution of the problem (4.1), we can write this in the matrix form

$$\mathbf{Js} = \mathbf{Rh} \tag{4.2}$$

where **J** is coefficient matrix, $\mathbf{s} = [\mathbf{u}, \mathbf{u}', \mathbf{v}]^T$ and $\mathbf{Rh} = [\mathbf{rh}_1, \mathbf{rh}_2, \mathbf{rh}_3]^T$. These matrix are

$$\mathbf{rh}_3 = \begin{pmatrix} 3v_0 + 2hv_0' + \frac{h^3}{6}(3f_1 + f_2) \\ -v_0 + \frac{h^3}{2}(-3f_2 + f_3) \\ \frac{h^3}{2}(-3f_3 + f_4) \\ \vdots \\ v_{N+1} + \frac{h^3}{2}(-3f_N + f_{N+1} + \lambda\beta_3) \end{pmatrix}_{N\times 1}, \mathbf{rh}_2 = \begin{pmatrix} -3\alpha_1 - h\alpha_2 - \frac{h^4}{60}v_0 \\ 0 \\ \vdots \\ 3\beta_1 - h\beta_2 + \frac{h^4}{60}v_{N+1} \end{pmatrix}_{N\times 1},$$

$$\mathbf{rh}_1 = \begin{pmatrix} 2\alpha_1 + h\alpha_2 + \frac{h^4}{90}v_0 \\ 0 \\ \vdots \\ 2\beta_1 - h\beta_2 + \frac{h^4}{90}v_{N+1} \end{pmatrix}_{N\times 1}, \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}_{N\times 1}, \mathbf{u}' = \begin{pmatrix} u_1' \\ \vdots \\ u_N' \end{pmatrix}_{N\times 1}, \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}_{N\times 1},$$

and let us define the coefficients matrix **J** in terms of block matrix,

$$\mathbf{J} = \begin{pmatrix} \mathbf{C}_{1,1} & \vdots & \mathbf{C}_{1,2} & \vdots & \mathbf{C}_{1,3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{C}_{2,1} & \vdots & \mathbf{C}_{2,2} & \vdots & \mathbf{C}_{2,3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{C}_{3,1} & \vdots & \mathbf{C}_{3,2} & \vdots & \mathbf{C}_{3,3} \end{pmatrix}_{3N\times 3N}$$

where

$$\mathbf{C}_{1,1} = 2\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}_{N\times N}, \mathbf{C}_{1,2} = h\begin{pmatrix} 0 & 1 & & & 0 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 0 & & & -1 & 0 \end{pmatrix}_{N\times N},$$

$$\mathbf{C}_{1,3} = -\frac{h^4}{90}\begin{pmatrix} 13 & 1 & & & 0 \\ 1 & 13 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 13 & 1 \\ 0 & & & 1 & 13 \end{pmatrix}_{N\times N}, \mathbf{C}_{2,1} = \frac{-3}{h}\mathbf{C}_{1,2}$$

$$\mathbf{C}_{2,2} = h\begin{pmatrix} 4 & 1 & & & 0 \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ 0 & & & 1 & 4 \end{pmatrix}_{N\times N}, \mathbf{C}_{2,3} = \frac{-h^3}{60}\mathbf{C}_{1,2},$$

$$\mathbf{C}_{3,3} = \begin{pmatrix} 4 & -1 & & & & 0 \\ -3 & 3 & -1 & & & \\ 1 & -3 & 3 & -1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -3 & 3 & -1 \\ & & & -1 & 3 & 3 \end{pmatrix}_{N\times N},$$

and matrices $(\mathbf{C}_{3,1})_{N\times N}$ and $(\mathbf{C}_{3,2})_{N\times N}$ depend on forcing function $f(x,u)$. These matrices are well defined. The exact solution $\mathbf{S} = [\mathbf{U},\mathbf{U}',\mathbf{V}]^T$ of the difference method (2.4-2.5) will satisfy the following equation

$$\mathbf{JS} = \mathbf{Rh} + \mathbf{T} \tag{4.3}$$

where $\mathbf{T} = [\mathbf{tu},\mathbf{tu}',\mathbf{tv}]^T$ is truncation error and will be defined as,

$$\mathbf{tu} = \begin{pmatrix} \frac{19h^8}{30240}u_1^{(8)} \\ \vdots \\ \frac{19h^8}{30240}u_N^{(8)} \end{pmatrix}_{N\times 1}, \mathbf{tu}' = \begin{pmatrix} \frac{h^7}{504}u_1^{(7)} \\ \vdots \\ \frac{19h^8}{30240}u_N^{(8)} \end{pmatrix}_{N\times 1}, \mathbf{tv} = \begin{pmatrix} \frac{3h^5}{20}v_1^{(5)} \\ \frac{h^5}{2}v_2^{(5)} \\ \vdots \\ \frac{h^5}{2}v_N^{(5)} \end{pmatrix}_{N\times 1}$$

Let us define an error function the difference between approximate and exact solution of the difference method (2.4-2.5) i.e. $\mathbf{E} = \mathbf{s} - \mathbf{S}$. To introduce and calculate so defined error function let subtract (4.3) from (4.2), we will obtain following error equation

$$\mathbf{JE} = -\mathbf{T} \tag{4.4}$$

Thus from (4.4), we observe that the convergence of the proposed method depends on the properties of coefficients matrix $\mathbf{J}$. We will prove under appropriate assumptions that the coefficient matrix $\mathbf{J}$ is invertible. Let us test the invertibility of coefficient matrix $\mathbf{J}$. The diagonal matrices $\mathbf{C}_{1,1}$, $\mathbf{C}_{2,2}$ and $\mathbf{C}_{3,3}$ of matrix $\mathbf{J}$ have different structure. The matrix $\mathbf{C}_{1,1}$ is invertible [14]. Matrix $\mathbf{C}_{2,2}$ is strictly diagonally dominant so it will invertible. For matrix $\mathbf{C}_{3,3}$, we have to rely on computation of explicit inverse. Let explicit inverses of $\mathbf{C}_{3,3}$ be $\mathbf{C}_{3,3}^{-1} = (k_{i,j})_{N\times N}$, where

$$k_{i,j} = \begin{cases} \frac{i^2(N-j+1)(N-j+2)}{2(N+1)^2}, & i \le j \le N \\ \frac{(N-i)(N-i+1)}{2}k_{N-1,j} - ((N-i)^2-1)k_{N,j}, & j < i \end{cases} \tag{4.5}$$

$$k_{N,j} = \begin{cases} \frac{(4N(N+2)(2N-1)-(N-2j)((N-2)^2(N-2j+2)+8N)}{32(N+1)^2}, & j \le \frac{N}{2} \\ \frac{N(2N^2+3N+2)+(N-2j+2)((2N+1)(2j-N)-2N)}{8(N+1)^2}, & \frac{N}{2} < j \end{cases}$$

$$k_{N-1,j} = \begin{cases} \frac{N^3-2N+2-(N-2j)(N(N-2j+2)-2)}{2(N+1)^2}, & j \le \frac{N}{2} \\ \frac{N^3-2N-2+(N-2j+2)(N(2j-N)+2)}{2(N+1)^2}, & \frac{N}{2} < j. \end{cases}$$

Thus from (4.5) we can verify that matrix $\mathbf{C}_{3,3}$ is invertible. Let us define following terms [15],

$$v_k^{up} = \max_{j=1,2..,k-1} ||A_{jk}A_{kk}^{-1}||, \text{k=2,3} \quad , \quad v_k^{low} = \max_{j=k+1,3} ||A_{jk}A_{kk}^{-1}||, \text{k=1,2} \quad ,$$

$$M^* = \prod_{2\le k\le 3}(1+v_k^{up}) \quad \text{and} \quad M_* = \prod_{1\le k\le 2}(1+v_k^{low}).$$

Let us assume

$$M_*M^* < M_* + M^* \quad \text{and} \quad M = \max_{p=1,2,3} ||\mathbf{C}_{p,p}^{-1}||$$

then matrix $\mathbf{J}$ is invertible [15] and moreover

$$||\mathbf{J}^{-1}|| \le \frac{MM_*M^*}{M_* + M^* - M_*M^*}. \tag{4.6}$$

Thus from (4.4) and (4.6), we have

$$||\mathbf{E}|| = ||\mathbf{J}^{-1}\mathbf{T}|| \le \frac{MM_*M^*}{M_* + M^* - M_*M^*}||T|| \tag{4.7}$$

It is easy to prove that $\frac{MM_*M^*}{M_*+M^*-M_*M^*}$ is finite. Thus $\|\mathbf{E}\|$ is bounded. Also it is easy to prove $||\mathbf{E}||$ tends to zero as $h \to 0$. So we can conclude that finite difference method (2.5-2.7) converge. The order of the convergence of the difference method (2.5-2.7) is at least $O(h^2)$.

## 5 Numerical Results

We have considered four model problems to establish the computational efficiency of proposed method (2.5-2.7). We have considered uniform step size *h* in computation of numerical solution of each model problem. In Table 1 and Table 2, we have shown $MAEU$, the maximum absolute error in the solution $u(x)$ and $MAEV$, the derivatives of solution $v(x)$ of the problems (1.1) for different values of *N*. We have used the following formulas in computation of $MAEU$ and $MAEV$:

$$MAEU = \max_{1 \le i \le N} |u(x_i) - u_i|$$

$$MAEV = \max_{1 \le i \le N} |u'(x_i) - v_i|$$

We have used Gauss Seidel iterative method to solve linear system of equations (2.5-2.7). All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 GHz PC. The solutions are computed on *N* nodes and iteration is continued until either the maximum difference between two successive iterates is less than $10^{-6}$ or the number of iteration reached $10^3$.
**Problem 1.** The model linear problem given by

$$u^{(7)}(x) = -u(x) - (35 + 12x + 2x^2)\exp(x), \quad 0 < x < 1$$

subject to boundary conditions

$$u(0) = 1, \quad u'(0) = 0, \quad u''(0) = -1, \quad u'''(0) = -3 \quad ,$$
$$u(1) = 0, \quad u'(1) = -\exp(1) \quad \text{and} \quad u''(1) = -4\exp(1).$$

The analytical solution of the problem is $u(x) = x(1 - x)\exp(x)$. The $MAEU$ and $MAEV$ computed by method (2.5-2.7) for coupling constant $C = .40199$ and different values of *N* are presented in Table 1.

**Problem 2.** The model linear problem given by

$$u^{(7)}(x) = u(x)u'(x) + (2 - 3x + x^2 + (x - 8)\exp(x))\exp(-2x), \quad 0 < x < 1$$

subject to boundary conditions

$$u(0) = 1, \quad u'(0) = 0, \quad u''(0) = -1, \quad u'''(0) = 2 \quad ,$$
$$u(1) = 2\exp(-1), \quad u'(1) = -\exp(-1) \quad \text{and} \quad u''(1) = 0.$$

The analytical solution of the problem is $u(x) = (1 + x)\exp(-x)$. The $MAEU$ and $MAEV$ computed by method (2.4-2.5) for coupling constant $C = .4099$ and different values of *N* are presented in Table 2.

**Table 1.** Maximum absolute error (Problem 1).

| | N | | |
|---|---|---|---|
| | 32 | 64 | 128 |
| MAEU | .28539286(-2) | .14184146(-5) | .55249515(-7) |
| MAEV | .10026446(-1) | .32737342(-4) | .99397312(-5) |

**Table 2.** Maximum absolute error (Problem 2).

| | N | | |
|---|---|---|---|
| | 16 | 32 | 64 |
| MAEU | .10952180(-3) | .11572996(-6) | .97807437(-7) |
| MAEV | .44546052(-3) | .30212423(-5) | .56484023(-5) |

The numerical results obtained in numerical experiment on considered model problems are satisfactory. The error in numerical result decreases as step size *h* decreases. We observed in numerical experiment that the convergence of the proposed method depends on the consideration of the coupling constant. In our numerical experiments, we have estimated the value of the coupling constant by guess and simulation. However, the accurate value of the coupling constant may possibly increase the accuracy of the results produced by the proposed method. We have obtained numerical approximation of the derivative of solution of problem as a byproduct the proposed method (2.5-2.7).

## 6 Conclusion

In the present article, we have developed and discussed the numerical technique using finite differences and splitting method for the numerical solution of seventh order differential equations and corresponding boundary value problem. We transformed the problem into system of problems by introducing a smooth augment function. The continuous system of problems at nodal points $x = x_i$, $i = 1, 2.., N$ reduced to a discrete system of algebraic equations (2.5-2.7). If source function $f(x, u)$ is linear then we obtained linear discrete system of algebraic equations. The proposed method in numerical experiments has shown its efficiency, also we got a numerical approximation of the derivative of the solution as an intermediate result. In future work, we shall work with an improvement in present idea. Work in this direction is in progress.

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