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Some new fixed point results in b–metric–like spaces

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Abstract In this paper, we prove some fixed point and common fixed theorems in b–metric–like spaces. Some of our results generalize, extends and improves some well known fixed point results in existing literature. Illustrative example is also provided.

1 Introduction and preliminaries

There exists many generalizations of the concept of metric spaces in the literature. The idea of partial metric space was presented by Matthews [16], [17], as a part of the study of denotational semantics of dataflow networks. A lot of fixed point results were studied in partial metric spaces by many other researchers see [1], [2], [3], [4], [12], [15], [18], [19]. The idea of b–metric spaces [8] and metric like spaces [6] were introduced in the literature, which comes from the generalizations of metric spaces. The generalizations of metric like spaces, b–metric spaces and partial metric spaces which leads to the concept of b–metric–like spaces is lately introduced in literature by [5]. The topological structure of b–metric–like spaces is presently investigated by Hussain et al. [13] as well as they proved some fixed point theorems in b–metric–like spaces. For detailed study, we recommend some other papers such as [20], [21], [22], [23], [10], [11]. In this paper, we established some fixed point and common fixed point theorems in b–metric–like spaces which generalize, extends and improves some fixed point results in existing literature.

Definition 1.1. [5] Let d be a distance function on a nonempty set $\mathcal{X}$ i.e. $d : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ we give the following conditions:
(c_1) $d(\xi, \eta) = 0$ then $\xi = \eta$;
(c_2) $d(\xi, \eta) = d(\eta, \xi)$;
(c_3) $d(\xi, \zeta) \leq b\,[d(\xi, \eta) + d(\eta, \zeta)]$ for all $\xi, \eta, \zeta \in \mathcal{X}$ and $b \geq 1$ be a constant.

If d satisfies $c_1 - c_3$, then d is called b-metric-like on $\mathcal{X}$. The pair $(\mathcal{X}, d)$ is then called b–metric–like space.
Alghamdi et al. [5] introduced some concepts in b–metric–like spaces as follows.
Each b–metric–like d on $\mathcal{X}$ generalizes a topology τ_d on $\mathcal{X}$ whose base is the family of open d–balls $B_d(\xi, \epsilon) = \{y \in \mathcal{X} : |d(\xi, y) - d(\xi, \xi)| < \epsilon\}$, for all $\xi \in \mathcal{X}$ and $\epsilon > 0$.

Definition 1.2. [5] A sequence $\{\xi_n\}$ in the b-metric-like space $(\mathcal{X}, d)$ converges to a point $\xi \in \mathcal{X}$ if and only if $d(\xi, \xi) = \lim_{n\to+\infty} d(\xi, \xi_n)$.

Definition 1.3. [5] A sequence $\{\xi_n\}$ in the b-metric-like space $(\mathcal{X}, d)$ is called a Cauchy sequence if there exists $\lim_{m,n\to+\infty} d(\xi_m, \xi_n)$(and it is finite).

Definition 1.4. [5] A b-metric-like space $(\mathcal{X}, d)$ is called complete if every Cauchy sequence $\{\xi_n\}$ in $\mathcal{X}$ converges with respect to τ_d to a point $\xi \in \mathcal{X}$ such that $\lim_{n\to+\infty} d(\xi, \xi_n) = d(\xi, \xi) = \lim_{m,n\to+\infty} d(\xi_m, \xi_n)$.

Definition 1.5. [5] Suppose that $(\mathcal{X}, d)$ is a b-metric-like space. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is said to be continuous at $\xi \in \mathcal{X}$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that $T(B_d(\xi, \delta)) \subseteq B_d(\mathcal{T}\xi, \epsilon)$. We say that $\mathcal{T}$ is continuous on $\mathcal{X}$ if $\mathcal{T}$ is continuous at all $\xi \in \mathcal{X}$.

Definition 1.6. [5] Let $(\mathcal{X}, d)$ be a b–metric–like space, and let ξ_n be a sequence of points of $\mathcal{X}$. A point $\xi \in \mathcal{X}$ is said to be the limit of the sequence ξ_n if $\lim_{n\to+\infty} d(\xi_n, \xi) = d(\xi, \xi)$, and we say that the sequence ξ_n is convergent to ξ and denote it by $\xi_n \to \xi$ as $n \to \infty$.

Definition 1.7. [14] Let f and g be two self-mappings on a set $\mathcal{X}$. If $\omega = f\xi = g\xi$ for some ξ in $\mathcal{X}$, then ξ is called a coincidence point of f and g, where ω is called a point of coincidence of f and g.

Definition 1.8. [14] Let f and g be two self-mappings defined on a set $\mathcal{X}$. Then f and g are said to be weakly compatible if they commute at every coincidence point, i.e., if $f\xi = g\xi$ for some $\xi \in \mathcal{X}$, then $fg\xi = gf\xi$.

Lemma 1.9. [5] *Let $(\mathcal{X}, d)$ be a b-metric-like space with the constant $b \geq 1$. Let $\{\eta_n\}$ be a sequence in $(\mathcal{X}, d)$ such that*

$$d(\eta_n, \eta_{n+1}) \leq \alpha d(\eta_{n-1}, \eta_n),$$

for some α, $0 < \alpha < \frac{1}{b}$, and each $n = 1, 2, \ldots$ Then $\lim_{m,n\to+\infty} d(\eta_m, \eta_n) = 0$.

Corollary 1.10. [7] *Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a surjective mapping and $(\mathcal{X}, d)$ be a complete b-metric-like space with $b \geq 1$ be a constant such that*

$$d(\mathcal{T}\xi, \mathcal{T}\eta) \geq \alpha d(\xi, \eta),$$

for all $\xi, \eta \in \mathcal{X}$ and $\alpha > b$, then $\mathcal{T}$ has a unique fixed point.

2 Main Results

In this section, we prove some fixed point and common fixed point theorems in b–metric–like spaces. Our first main result is stated as:

Theorem 2.1. *Let $(\mathcal{X}, d)$ be a complete b-metric-like space with $b \geq 1$ be a constant. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a surjection such that,*

$$\begin{aligned} d(\mathcal{T}\xi, \mathcal{T}\eta) \geq\ & a_1 d(\xi, \eta) + a_2 d(\xi, \mathcal{T}\xi) + a_3 d(\eta, \mathcal{T}\eta) + a_4 d(\xi, \mathcal{T}\eta) \\ & + a_5 d(\eta, \mathcal{T}\xi) + a_6 d(\xi, \xi) + a_7 d(\eta, \eta), \end{aligned} \tag{2.1}$$

for all $\xi, \eta \in \mathcal{X}$, $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \geq 0$, satisfy $b(a_1 + a_2 + a_5 + 2a_6 + a_7) + a_4 + b^2(a_3 - a_4 + a_5 + a_7) > b^2$ and $1 - a_3 + a_4 - a_5 - a_7 > 0$, then $\mathcal{T}$ has a fixed point.

Proof. Let $\xi_0 \in \mathcal{X}$. Since the mapping $\mathcal{T}$ is surjective, choose $\xi_1 \in \mathcal{X}$ such that $\mathcal{T}\xi_1 = \xi_0$. Continuing this process, we can define a sequence $\{\xi_n\}$ such that $\xi_{n-1} = \mathcal{T}\xi_n$, $n \geq 1$, $n \in N$. Without loss of generality, we suppose that $\xi_{n-1} \neq \xi_n$ for all $n \geq 1$, $n \in N$.
By using condition (2.1), we have

$$\begin{aligned} d(\xi_n, \xi_{n-1}) &= d(\mathcal{T}\xi_{n+1}, \mathcal{T}\xi_n) \\ &\geq a_1 d(\xi_{n+1}, \xi_n) + a_2 d(\xi_{n+1}, \mathcal{T}\xi_{n+1}) + a_3 d(\xi_n, \mathcal{T}\xi_n) \\ &\quad + a_4 d(\xi_{n+1}, \mathcal{T}\xi_n) + a_5 d(\xi_n, \mathcal{T}\xi_{n+1}) + a_6 d(\xi_{n+1}, \xi_{n+1}) \\ &\quad + a_7 d(\xi_n, \xi_n). \\ &= a_1 d(\xi_{n+1}, \xi_n) + a_2 d(\xi_{n+1}, \xi_n) + a_3 d(\xi_n, \xi_{n-1}) \\ &\quad + a_4 d(\xi_{n+1}, \xi_{n-1}) + a_5 d(\xi_n, \xi_n) + a_6 d(\xi_{n+1}, \xi_{n+1}) \\ &\quad + a_7 d(\xi_n, \xi_n). \\ &= a_1 d(\xi_{n+1}, \xi_n) + a_2 d(\xi_{n+1}, \xi_n) + a_3 d(\xi_n, \xi_{n-1}) \\ &\quad + a_4 d(\xi_{n+1}, \xi_{n-1}) + a_5 d(\xi_{n-1}, \xi_n) \\ &\quad + a_5 d(\xi_n, \xi_{n+1}) + a_6 d(\xi_{n+1}, \xi_n) + a_6 d(\xi_n, \xi_{n+1}) \\ &\quad + a_7 d(\xi_{n-1}, \xi_n) + a_7 d(\xi_n, \xi_{n+1}). \end{aligned} \tag{2.2}$$

By $d(\xi_{n+1},\xi_{n-1}) \geq \frac{d(\xi_n,\xi_{n+1})-b\,d(\xi_n,\xi_{n-1})}{b}$, (2.2) implies that

$$d(\xi_{n+1},\xi_n) \leq \frac{b-ba_3+ba_4-ba_5-ba_7}{ba_1+ba_2+a_4+ba_5+2ba_6+ba_7}\, d(\xi_n,\xi_{n-1}). \tag{2.3}$$

Letting $\alpha = \frac{b-ba_3+ba_4-ba_5-ba_7}{ba_1+ba_2+a_4+ba_5+2ba_6+ba_7}$, by $b(a_1+a_2+a_5+2a_6+a_7)+a_4+b^2(a_3-a_4+a_5+a_7) > b^2$, we have $0 < \alpha < \frac{1}{b}$. Applying Lemma 1.9, we see that $\lim_{m,n\to+\infty} d(\xi_m,\xi_n) = 0$ and $\{\xi_n\}$ is a Cauchy sequence. Since $(\mathcal{X},d)$ is complete there exists $p \in \mathcal{X}$ such that $\lim_{n\to+\infty} d(\xi_n,p) = d(p,p) = \lim_{m,n\to+\infty} d(\xi_m,\xi_n) = 0$. Consequently, we can find some $v \in \mathcal{X}$ such that $p = \mathcal{T}v$. Next, we show that $p = v$. From condition (2.1), we get

$$\begin{aligned} d(\xi_n,p) &= d(\mathcal{T}\xi_{n+1},\mathcal{T}v)\\ &\geq a_1 d(\xi_{n+1},v) + a_2 d(\xi_{n+1},\mathcal{T}\xi_{n+1}) + a_3 d(v,\mathcal{T}v)\\ &\quad + a_4 d(\xi_{n+1},\mathcal{T}v) + a_5 d(v,\mathcal{T}\xi_{n+1}) + a_6 d(\xi_{n+1},\xi_{n+1})\\ &\quad + a_7 d(v,v).\\ &\geq a_1 d(\xi_{n+1},v) + a_2 d(\xi_{n+1},\xi_n) + a_3 d(v,p)\\ &\quad + a_4 d(\xi_{n+1},p) + a_5 d(v,\xi_n) + 2a_6 d(\xi_{n+1},\xi_n)\\ &\quad + 2a_7 d(v,p). \end{aligned} \tag{2.4}$$

Also

$$\begin{aligned} d(p,\xi_n) &= d(\mathcal{T}v,\mathcal{T}\xi_{n+1})\\ &\geq a_1 d(v,\xi_{n+1}) + a_2 d(v,\mathcal{T}v) + a_3 d(\xi_{n+1},\mathcal{T}\xi_{n+1})\\ &\quad + a_4 d(v,\mathcal{T}\xi_{n+1}) + a_5 d(\xi_{n+1},\mathcal{T}v) + a_6 d(v,v)\\ &\quad + a_7 d(\xi_{n+1},\xi_{n+1}).\\ &\geq a_1 d(v,\xi_{n+1}) + a_2 d(v,p) + a_3 d(\xi_{n+1},\xi_n)\\ &\quad + a_4 d(v,\xi_n) + a_5 d(\xi_{n+1},p) + 2a_6 d(v,p)\\ &\quad + 2a_7 d(\xi_{n+1},\xi_n). \end{aligned} \tag{2.5}$$

Adding (2.4) and (2.5), we get

$$\begin{aligned} 2d(p,\xi_n) &\geq 2a_1 d(v,\xi_{n+1}) + (a_2+a_3) d(\xi_{n+1},\xi_n) + (a_2+a_3) d(v,p)\\ &\quad + a_4 d(v,\xi_n) + a_4 d(\xi_{n+1},p) + a_5 d(\xi_{n+1},p) + a_5 d(v,\xi_n)\\ &\quad + (2a_6+2a_7) d(v,p) + (2a_6+2a_7) d(\xi_{n+1},\xi_n). \end{aligned} \tag{2.6}$$

Since

$$d(v,\xi_{n+1}) \geq \frac{d(v,p) - b\,d(\xi_{n+1},p)}{b}$$

and

$$d(v,\xi_n) \geq \frac{d(v,p) - b\,d(\xi_n,p)}{b}.$$

Inequality (2.6) implies,

$$\begin{aligned} 2d(p,\xi_n) &\geq 2a_1 \frac{d(v,p) - b\,d(\xi_{n+1},p)}{b} + (a_2+a_3) d(\xi_{n+1},\xi_n)\\ &\quad + (a_2+a_3) d(v,p) + a_4 \frac{d(v,p) - b\,d(\xi_n,p)}{b}\\ &\quad + a_4 d(\xi_{n+1},p) + a_5 d(\xi_{n+1},p) + a_5 \frac{d(v,p) - b\,d(\xi_n,p)}{b}\\ &\quad + (2a_6+2a_7) d(v,p) + (2a_6+2a_7) d(\xi_{n+1},\xi_n). \end{aligned} \tag{2.7}$$

Letting $n \to +\infty$ on inequality (2.7), we get

$$0 \geq \Big(\frac{2a_1}{b} + a_2 + a_3 + \frac{a_4}{b} + \frac{a_5}{b} + 2a_6 + 2a_7\Big) d(v,p),$$

implies that $d(v,p) = 0$, hence $v = p$, that is, $v = p = \mathcal{T}v$. This shows that v is a fixed point of $\mathcal{T}$. □

Remark 2.2. If we put $a_5 = a_6 = a_7 = 0$ in our main Theorem 2.1, we get the Theorem 2.2 of Chen et al. [7].

Corollary 2.3. *Let $(\mathcal{X}, d)$ be a complete b–metric–like space with $b \geq 1$ be a constant. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a surjection satisfying the following condition,*

$$d(\mathcal{T}\xi, \mathcal{T}\eta) \quad \geq \quad a_1 d(\xi,\eta) + a_2 d(\xi, \mathcal{T}\xi) + a_3 d(\eta, \mathcal{T}\eta) + a_4 d(\xi, \mathcal{T}\eta),$$

for all $\xi, \eta \in \mathcal{X}$, $a_1, a_2, a_3, a_4 \geq 0$, satisfy $b(a_1 + a_2) + a_4 + b^2(a_3 - a_4) > b^2$ and $1 - a_3 + a_4 > 0$, then $\mathcal{T}$ has a fixed point.

Lemma 2.4. [9] *Let $\mathcal{X}$ be a nonempty set and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a function. Then there exists a subset $E \subseteq \mathcal{X}$ such that $\mathcal{T}(E) = \mathcal{T}(\mathcal{X})$ and $\mathcal{T} : E \to \mathcal{X}$ is one–to–one.*

Corollary 2.5. *Let $(\mathcal{X}, d)$ be a complete b–metric–like space with a constant $b \geq 1$ and the two self mappings $\mathcal{F}$ and $\mathcal{T}$ satisfy the following condition,*

$$d(\mathcal{F}\xi, \mathcal{F}\eta) \geq \alpha\, d(\mathcal{T}\xi, \mathcal{T}\eta)$$

for all $\xi, \eta \in \mathcal{X}$, where $\alpha > b$ be a constant. If $\mathcal{F}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$ is complete subset of $\mathcal{X}$, then the self maps $\mathcal{F}$ and $\mathcal{T}$ have a unique point of coincidence in $\mathcal{X}$. Moreover, if the two maps $\mathcal{F}$ and $\mathcal{T}$ are weakly compatible, then $\mathcal{F}$ and $\mathcal{T}$ have a unique common fixed point.

Proof. By the above Lemma 2.4, there exists $E \subseteq \mathcal{X}$ such that $\mathcal{T}(E) = \mathcal{T}(\mathcal{X})$ and $\mathcal{T} : E \to \mathcal{X}$ is one–to–one. Define a mapping $g : \mathcal{T}(E) \to \mathcal{T}(E)$ by $g(\mathcal{T}\xi) = \mathcal{F}\xi$. As we know that $\mathcal{T}$ is one–to–one on E and g is well defined. Note that $d(g(\mathcal{T}\xi), g(\mathcal{T}\eta)) \geq \alpha\, d(\mathcal{T}\xi, \mathcal{T}\eta)$ for all $\mathcal{T}\xi, \mathcal{T}\eta \in \mathcal{T}(E)$. Since $\mathcal{T}(E) = \mathcal{T}(\mathcal{X})$is complete, by using Corollary 1.10, there exists a unique point ξ_0 in $\mathcal{X}$ in a way that $g(\mathcal{T}\xi_0) = \mathcal{T}\xi_0$, hence $\mathcal{F}\xi_0 = \mathcal{T}\xi_0$, which clearly shows that the two mappings $\mathcal{F}$ and $\mathcal{T}$ have a unique point of coincidence in $\mathcal{X}$. Suppose that $\mathcal{F}\xi_0 = \mathcal{T}\xi_0 = p$, since the two maps are weakly compatible, $\mathcal{F}p = \mathcal{T}p$, which implies that $\mathcal{F}p = \mathcal{T}p = p$, hence p is the unique common fixed point of maps $\mathcal{F}$ and $\mathcal{T}$. □

Theorem 2.6. *Let $(\mathcal{X}, d)$ be a complete b-metric-like space with $b \geq 1$ be a constant. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a surjection such that,*

$$\begin{aligned} d(\mathcal{T}\xi, \mathcal{T}\eta) \quad \geq \quad & a_1 d(\xi,\eta) + a_2[d(\xi, \mathcal{T}\xi) + d(\eta, \mathcal{T}\eta)] \\ & + a_3[d(\xi, \mathcal{T}\eta) + d(\eta, \mathcal{T}\xi)] + a_4[d(\xi, \mathcal{T}\xi) + d(\xi,\eta)] \\ & + a_5[d(\xi, \mathcal{T}\eta) + d(\xi,\eta)], \end{aligned} \tag{2.8}$$

for all $\xi, \eta \in \mathcal{X}$, $a_1, a_2, a_3, a_4, a_5 \geq 0$, satisfy $b(a_1 + a_2 + a_3 + 2a_4 + a_5) + a_3 + b^2(a_2 + a_5) > b^2$ and $1 - (a_2 + a_5) > 0$, then $\mathcal{T}$ has a fixed point.

Proof. Let $\xi_0 \in \mathcal{X}$. Since the mapping $\mathcal{T}$ is surjective, choose $\xi_1 \in \mathcal{X}$ such that $\mathcal{T}\xi_1 = \xi_0$. Continuing this process, we can define a sequence $\{\xi_n\}$ such that $\xi_{n-1} = \mathcal{T}\xi_n$, $n \geq 1$, $n \in N$. Without loss of generality, we suppose that $\xi_{n-1} \neq \xi_n$ for all $n \geq 1$, $n \in N$.

By using condition (2.8), we have

$$\begin{aligned}
d(\xi_n,\xi_{n-1}) &= d(\mathcal{T}\xi_{n+1},\mathcal{T}\xi_n)\\
&\geq a_1 d(\xi_{n+1},\xi_n)+a_2[d(\xi_{n+1},\mathcal{T}\xi_{n+1})+d(\xi_n,\mathcal{T}\xi_n)]\\
&\quad +a_3[d(\xi_{n+1},\mathcal{T}\xi_n)+d(\xi_n,\mathcal{T}\xi_{n+1})]\\
&\quad +a_4[d(\xi_{n+1},\mathcal{T}\xi_{n+1})+d(\xi_{n+1},\xi_n)]\\
&\quad +a_5[d(\xi_n,\mathcal{T}\xi_n)+d(\xi_{n+1},\xi_n)]\\
&= a_1 d(\xi_{n+1},\xi_n)+a_2[d(\xi_{n+1},\xi_n)+d(\xi_n,\xi_{n-1})]\\
&\quad +a_3[d(\xi_{n+1},\xi_{n-1})+d(\xi_n,\xi_n)]\\
&\quad +a_4[d(\xi_{n+1},\xi_n)+d(\xi_{n+1},\xi_n)]\\
&\quad +a_5[d(\xi_n,\xi_{n-1})+d(\xi_{n+1},\xi_n)]. \qquad (2.9)
\end{aligned}$$

By $d(\xi_{n+1},\xi_{n-1})\geq \frac{d(\xi_n,\xi_{n+1})-b\,d(\xi_n,\xi_{n-1})}{b}$, (2.9) implies that

$$d(\xi_{n+1},\xi_n)\leq \frac{b-ba_2-ba_5}{ba_1+ba_2+a_3+ba_3+2ba_4+ba_5}\,d(\xi_n,\xi_{n-1}). \qquad (2.10)$$

Letting $\alpha=\frac{b-ba_2-ba_5}{ba_1+ba_2+a_3+ba_3+2ba_4+ba_5}$, by $b(a_1+a_2+a_3+2a_4+a_5)+a_3+b^2(a_2+a_5)>b^2$, we have $0<\alpha<\frac{1}{b}$. Applying Lemma 1.9, we see that $\lim_{m,n\to+\infty} d(\xi_m,\xi_n)=0$ and $\{\xi_n\}$ is a Cauchy sequence. Since $(\mathcal{X},d)$ is complete there exists $p\in\mathcal{X}$ such that $\lim_{n\to+\infty} d(\xi_n,p)=d(p,p)=\lim_{m,n\to+\infty} d(\xi_m,\xi_n)=0$. Consequently, we can find some $v\in\mathcal{X}$ such that $p=\mathcal{T}v$. Next, we show that $p=v$. From condition (2.8), we get

$$\begin{aligned}
d(\xi_n,p) &= d(\mathcal{T}\xi_{n+1},\mathcal{T}v)\\
&\geq a_1 d(\xi_{n+1},v)+a_2[d(\xi_{n+1},\mathcal{T}\xi_{n+1})+d(v,\mathcal{T}v)]\\
&\quad +a_3[d(\xi_{n+1},\mathcal{T}v)+d(v,\mathcal{T}\xi_{n+1})]\\
&\quad +a_4[d(\xi_{n+1},\mathcal{T}\xi_{n+1})+d(\xi_{n+1},v)]\\
&\quad +a_5[d(v,\mathcal{T}v)+d(\xi_{n+1},v)]\\
&= a_1 d(\xi_{n+1},v)+a_2[d(\xi_{n+1},\xi_n)+d(v,p)]\\
&\quad +a_3[d(\xi_{n+1},p)+d(v,\xi_n)]\\
&\quad +a_4[d(\xi_{n+1},\xi_n)+d(\xi_{n+1},v)]\\
&\quad +a_5[d(v,p)+d(\xi_{n+1},v)]\\
&= a_1 d(\xi_{n+1},v)+a_2 d(\xi_{n+1},\xi_n)+a_2 d(v,p)\\
&\quad +a_3 d(\xi_{n+1},p)+a_3 d(v,\xi_n)\\
&\quad +a_4 d(\xi_{n+1},\xi_n)+a_4 d(\xi_{n+1},v)\\
&\quad +a_5 d(v,p)+a_5 d(\xi_{n+1},v). \qquad (2.11)
\end{aligned}$$

Also

$$\begin{aligned}
d(p,\xi_n) &= d(\mathcal{T}v,\mathcal{T}\xi_{n+1})\\
&\geq a_1 d(v,\xi_{n+1})+a_2[d(v,\mathcal{T}v)+d(\xi_{n+1},\mathcal{T}\xi_{n+1})]\\
&\quad +a_3[d(v,\mathcal{T}\xi_{n+1})+d(\xi_{n+1},\mathcal{T}v)]\\
&\quad +a_4[d(v,\mathcal{T}v)+d(v,\xi_{n+1}]\\
&\quad +a_5[d(\xi_{n+1},\mathcal{T}\xi_{n+1})+d(v,\xi_{n+1})]\\
&= a_1 d(v,\xi_{n+1})+a_2 d(v,p)+a_2 d(\xi_{n+1},\xi_n)\\
&\quad +a_3 d(v,\xi_n)+a_3 d(\xi_{n+1},p)\\
&\quad +a_4 d(v,p)+a_4 d(v,\xi_{n+1}]\\
&\quad +a_5 d(\xi_{n+1},\xi_n)+a_5 d(v,\xi_{n+1})]. \qquad (2.12)
\end{aligned}$$

Adding (2.11) and (2.12), we get

$$\begin{aligned} 2d(p,\xi_n) \geq\ & 2a_1 d(v,\xi_{n+1}) + 2a_2 d(\xi_{n+1},\xi_n) + 2a_2 d(v,p) \\ & + 2a_3 d(\xi_{n+1},p) + 2a_3 d(v,\xi_n) + 2a_4 d(\xi_{n+1},v) \\ & + a_4 d(v,p) + a_4 d(\xi_{n+1},\xi_n) + 2a_5 d(v,\xi_{n+1}) \\ & + 2a_5 d(\xi_{n+1},\xi_n) + a_5 d(v,p). \end{aligned} \tag{2.13}$$

Since

$$d(v,\xi_{n+1}) \geq \frac{d(v,p) - b\, d(\xi_{n+1},p)}{b},$$

and

$$d(v,\xi_n) \geq \frac{d(v,p) - b\, d(\xi_n,p)}{b}.$$

Inequality (2.13) implies,

$$\begin{aligned} 2d(p,\xi_n) \geq\ & 2a_1 \frac{d(v,p) - b\, d(\xi_{n+1},p)}{b} + 2a_2 d(\xi_{n+1},\xi_n) + 2a_2 d(v,p) \\ & + 2a_3 d(\xi_{n+1},p) + 2a_3 \frac{d(v,p) - b\, d(\xi_n,p)}{b} \\ & + 2a_4 \frac{d(v,p) - b\, d(\xi_{n+1},p)}{b} + a_4 d(v,p) + a_4 d(\xi_{n+1},\xi_n) \\ & + 2a_5 \frac{d(v,p) - b\, d(\xi_{n+1},p)}{b} + 2a_5 d(\xi_{n+1},\xi_n) + a_5 d(v,p). \end{aligned} \tag{2.14}$$

Letting $n \to +\infty$ on inequality (2.14), we get

$$0 \geq \Big(\frac{2a_1}{b} + 2a_2 + \frac{2a_3}{b} + \frac{2a_4}{b} + a_4 + \frac{2a_5}{b} + a_5\Big) d(v,p),$$

implies that $d(v,p) = 0$, hence $v = p$, that is, $v = p = \mathcal{T}v$. This shows that v is a fixed point of $\mathcal{T}$. □

Now, we introduce an example to illustrate the validity of our main result.

Example 2.7. Let $\mathcal{X} = [0,+\infty)$ and a b-metric-like $d : \mathcal{X} \times \mathcal{X} \to [0,+\infty)$ defined by

$$d(\xi,\eta) = (\xi + \eta)^2,$$

then $(\mathcal{X}, d)$ is a complete b-metric-like space with $b = 2$ a constant. Define $\mathcal{F}\xi = \frac{\xi}{4}$ and $\mathcal{T}\xi = ln(1 + \frac{\xi}{8})$ are self mappings $\mathcal{F}$ and $\mathcal{T}$ on $\mathcal{X}$. Since $k \geq ln(1+k)$ for each $k \in [0,+\infty)$, for all $\xi, \eta \in \mathcal{X}$, we have

$$\begin{aligned} d(\mathcal{F}\xi,\mathcal{F}\eta) &= \Big(\frac{\xi}{4} + \frac{\eta}{4}\Big)^2 = \Big(2\frac{\xi}{8} + 2\frac{\eta}{8}\Big)^2 = 4\Big(\frac{\xi}{8} + \frac{\eta}{8}\Big)^2 \\ &\geq 4\Big(ln\Big(1 + \frac{\xi}{8}\Big) + ln\Big(1 + \frac{\eta}{8}\Big)\Big)^2 = 4d(\mathcal{T}\xi,\mathcal{T}\eta), \end{aligned}$$

which means that $d(\mathcal{F}\xi,\mathcal{F}\eta) \geq \alpha\, d(\mathcal{T}\xi,\mathcal{T}\eta)$, where $\alpha = 4$. Hence all the conditions of Corollary 2.5 are satisfied, hence the mappings $\mathcal{F}$ and $\mathcal{T}$ have unique point of coincidence, actually, 0 is the unique point of coincidence. Further, by $\mathcal{F}\mathcal{T}0 = \mathcal{T}\mathcal{F}0$, we observe that 0 is the unique common fixed point of $\mathcal{F}$ and $\mathcal{T}$.

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