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FIXED POINT THEOREMS FOR SELF AND NON-SELF CONTRACTIONS IN MODULAR METRIC SPACES ENDOWED WITH A GRAPH

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Abstract The main results obtained in this paper are fixed point theorems for self and non-self G_w-contractions on modular metric spaces endowed with a graph. Our new results are extensions of recently fixed point theorems for self-mappings on metric spaces and also fixed point theorems for non-self mappings in Banach spaces or convex metric spaces.

1 Background

The theory of metric modular w was introduced by the Chistyakov [7, 8, 9]. It is well known that metric d attributes a non-negative finite distance between two points in metric spaces. Informally speaking, a metric modular on a non-empty set assigns a non-negative (maybe, infinite valued) "field of (generalized) velocities": to each time $\alpha > 0$ (the absolute value of) an average velocity $w_\alpha(r, s)$ is associated in such a way that in order to cover the "distance" between points $r, s \in S$ it takes time α to move from r to s with velocity $w_\alpha(r, s)$.

The metric fixed point theory has been widely investigated since 1922 when Banach in [1] has proved the contraction theorem for complete metric spaces. Many generalizations of the Banach contraction theorem have been introduced by various authors, see, for example, [3, 16], and Jachymski [11] generalized the Banach contraction theorem for self mappings defined on complete metric spaces endowed with the graph. Berinde in [2] has extended the concept of Banach contraction theorem for non-self mappings in complete metric spaces endowed with the graph by using the inwardness condition defined in [4].

Chistyakov in [6] has laid down the foundation of fixed point theory for modular metric spaces by introducing the Lipschitz as well as contraction condition for modular metric spaces. The author stated that for a (convex) modular on S, a mapping $f : S_w^* \to S_w^*$ is modular contractive if there exists a real number $k \in (0, 1)$ and $\alpha_0(k) > 0$ provided that

$$w_{k\alpha}(fr, fs) \leq w_\alpha(r, s) \text{ for all } 0 < \alpha \leq \alpha_0$$

and $r, s \in S_w^*$. In [6], the author introduced the fixed point theorem for contractive mappings in modular metric spaces for convex modular.

After this initiation of contraction condition for modular metric spaces, many authors have expanded the metric fixed point theory to modular metric spaces. Moreno et al. in [13] have generalized the Banach contraction principle to modular metric spaces which are defined as follows: Let (S_w, w) be a complete modular metric space, then every contraction mapping $f : S_w \to S_w$ has a unique fixed point.

This paper has been organized in the following manner: In Section 2, we will give the brief introduction of modular metrics, modular set S_w and metric d_w induced from modular metric along with modular sequences, its limit, convergence, and completeness of modular metric spaces (S_w, w). In the last section, our main aim is to study the fixed point theorems for self mappings as well as non-self mappings using Moreno et al. contraction principle for modular metric spaces. These theorems are the generalization of fixed point theorems discussed by Berinde [2] on Banach spaces endowed with a graph.

2 Metric Modular

Let S be a nonempty set (having at least two elements) and $\alpha \in (0, \infty)$. Throughout this paper, for the simplicity of arguments, we will denote the function $w : (0, \infty) \times S \times S \to [0, \infty]$ with $w_\alpha (r, s) := w (\alpha, r, s)$ for all $\alpha > 0$ and $r, s \in S$. The set $w = \{w_\alpha\}$ for $\alpha > 0$ is one-parameter family of functions $w_\alpha : S \times S \to [0, \infty]$. For fixed given $r, s \in S$, we may define $w^{r,s} (\alpha) = w (\alpha, r, s)$ for all $\alpha > 0$, so that $w^{r,s} : (0, \infty) \to [0, \infty]$, is another family of functions.

Definition 2.1. ([7])A function $w : (0, \infty) \times S \times S \to [0, \infty]$ satisfying the following three conditions:

(F1) For $r, s \in S$, $w_\alpha (r, s) = 0$ for all $\alpha > 0$ if and only if $r = s$;

(F2) $w_\alpha (r, s) = w_\alpha (s, r)$ for all $\alpha > 0$ and $r, s \in S$;

(F3) $w_{\alpha+\beta} (r, s) \leq w_\alpha (r, t) + w_\beta (t, s)$ for all $\alpha, \beta > 0$ and $r, s, t \in S$.

is called metric modular on S.

In place of (F1), a weaker condition is defined as:

(F1') $w_\alpha (s, s) = 0$ for all $\alpha > 0$ and $s \in S$, then w is called metric pseudomodular for S.

A (pseudo) modular w is called convex if it satisfies following inequality,

$$w_{\alpha+\beta} (r, s) \leq \frac{\alpha}{\alpha + \beta} w_\alpha (r, t) + \frac{\beta}{\alpha + \beta} w_\beta (t, s) \text{ for all } \alpha, \beta > 0 \text{ and } r, s, t \in S.$$

An additional property satisfied by convex pseudo modular w is defined as :

$$w_\alpha (r, s) \leq \frac{\beta}{\alpha} w_\beta (r, s) \leq w_\beta (r, s) \text{ if } \beta \leq \alpha.$$

Some examples are given below to have a better insight of metric modular w.

Example 2.2. The examples of (pseudo) modulars on a set S are denoted by following indexed objects. Let $\alpha > 0$ and $r, s \in S$. We have:

(i) $w^1_\alpha (r, s) = \infty$ if $r \neq s$, and $w^1_\alpha (r, s) = 0$ if $r = s$. Furthermore, if (S, d) is a (pseudo) metric space with (pseudo) metric d, then we also have:

(ii) $w^2_\alpha (r, s) = \frac{d(r,s)}{\Phi(\alpha)}$, where $\Phi : (0, \infty) \to (0, \infty)$ is a non-decreasing function;

(iii) $w^3_\alpha (r, s) = \infty$ if $\alpha < d (r, s)$ and $w^3_\alpha (r, s) = 0$ if $\alpha \geq d (r, s)$;

(iv) $w^4_\alpha (r, s) = \infty$ if $\alpha \leq d (r, s)$ and $w^4_\alpha (r, s) = 0$ if $\alpha > d (r, s)$.

Lemma 2.3. *([5])For given $r, s \in S$, the function $w^{r,s} : (0, \infty) \to [0, \infty]$ is non-increasing on the interval $(0, \infty)$, that is, if $0 < \beta < \alpha$ then the condition (F3) (with $t = r$) and (F1') implies that*

$$w_\alpha (r, s) = w_{(\alpha-\beta)+\beta} (r, s) \leq w_{\alpha-\beta} (r, r) + w_\beta (r, s) = w_\beta (r, s). \quad (2.1)$$

Consequently, for the given $r, s \in S$, at each point $\alpha > 0$, the left limit is defined as

$$\begin{aligned} &(w_{-0})_\alpha (r, s) \equiv w_{\alpha-0} (r, s) \\ &= \lim_{\beta\to\alpha-0} w_\beta (r, s) = \inf \{w_\beta (r, s) : 0 < \beta < \alpha\} \end{aligned} \quad (2.2)$$

and the right limit is defined as

$$\begin{aligned} &(w_{+0})_\alpha (r, s) \equiv w_{\alpha+0} (r, s) \\ &= \lim_{\beta\to\alpha+0} w_\beta (r, s) = \sup \{w_\beta (r, s) : \beta > \alpha\}, \end{aligned} \quad (2.3)$$

exist in the closed interval $[0, \infty]$.

By using (2.2) and (2.3), we can obtain following inequalities, for all $\alpha > \beta > 0$

$$w_{\alpha+0}\left(r,s\right) \leq w_{\alpha}\left(r,s\right) \leq w_{\alpha-0}\left(r,s\right) \leq w_{\beta+0}\left(r,s\right) \leq w_{\beta}\left(r,s\right) \leq w_{\beta-0}\left(r,s\right) \tag{2.4}$$

Chistyakov has introduced the concept of metrizability for modular spaces.

Let us fix an element $r_0 \in S$. The set defined as

$$S_w \equiv S_w\left(r_0\right) = \left\{r \in S : w_\alpha\left(r, r_0\right) \to 0 \text{ as } \alpha \to \infty\right\},$$

is said to be modular space (around r_0), and r_0 is said to be the center of S_w.

There are two more modular spaces defined on S:

$$S_w^* \equiv S_w^*\left(r_0\right) = \left\{r \in S : \exists\, \alpha = \alpha\left(r\right) \text{ such that } w_\alpha\left(r, r_0\right) < \infty\right\}$$

and

$$S_w^{fin} \equiv S_w^{fin}\left(r_0\right) = \left\{r \in S : w_\alpha\left(r, r_0\right) < \infty \text{ for all } \alpha > 0\right\}.$$

Clearly, $S_w \subset S_w^*$ and it is shown in [7] that this inclusion is proper in general.

Chistyakov in [7] has introduced the essential metrics on modular sets induced from metric modular. In this sequel, we define modular spaces, metric on modular spaces, sequences, along with their convergence and completeness of modular metric spaces.

Lemma 2.4. *If the function w be a metric (pseudo) modular on S, then the (pseudo) metric space defined on modular set S_w is given by*

$$d_w\left(r,s\right) = \inf\left\{\alpha > 0 : w_\alpha\left(r,s\right) \leq \alpha\right\},$$

for $r, s \in S_w$. On the other hand, if w is a convex (pseudo) modular defined on S, then the convex (pseudo) metric space defined on modular set S_w^* is given by

$$d_w^*\left(r,s\right) = \inf\left\{\alpha > 0 : w_\alpha\left(r,s\right) \leq 1\right\},$$

for $r, s \in S_w^*$.

According to [7], in convex modular w, two modular sets are equal, that is, $S_w = S_w^*$, and this standard set is endowed with metric d_w^*. Moreover, metric space d_w can be defined on the largest set S_w^*.

The next lemma shows the relationship of convergence between the metric d_w and modular w.

Lemma 2.5. *Let w be a modular on S, $\{r_n\} \subset S_w$ and $r \in S_w$, we have*

$$\lim_{n\to\infty} d_w\left(r_n, r\right) = 0 \text{ if and only if } \lim_{n\to\infty} w_\alpha\left(r_n, r\right) = 0 \text{ for all } \alpha > 0.$$

A similar conclusion holds for Cauchy sequences concerning d_w.

The above lemma also holds for d_w^* on the modular set S_w^* for convex modular w.

Since the classical theory of metric spaces cannot be applied to modular metric theory in a straight forward way. Therefore, the concept of modular convergence, limit, closedness and completeness has been reintroduced in [6] and [13].

Definition 2.6. Let w be a modular on S. A sequence $\{r_n\}$ from S_w (or S_w^*) is modular convergent to an element $r \in S_w$ (or S_w^*) if $\lim_{n\to\infty} w_\alpha\left(r_n, r\right) = 0$ for all $\alpha > 0$. A sequence $\{r_n\} \subset S_w$ (or S_w^*) is modular Cauchy, If $w_\alpha\left(r_n, r_m\right) \to 0$ as $n, m \to \infty$ and $\alpha > 0$, that is, for $\varepsilon > 0$ there exist $N_0\left(\varepsilon\right) \in \mathbb{N}$ provided that

$$\text{for all } n, m \geq N_0\left(\varepsilon\right) : w_\alpha\left(r_n, r_m\right) < \varepsilon.$$

If a sequence $\{r_n\}$ from S_w (or S_w^*) is modular convergent to an element $r \in S$ then $r \in S_w$ (or $r \in S_w^*$, respectively) and the modular space S_w (or S_w^*) is said to be closed for modular convergence. If every Cauchy sequence is convergent in modular metric space (S_w^*, w) (or (S_w, w)) then it is called complete.

The convergence of sequence $\{r_n\}$ can be weakened if we assume the condition in the above definition to hold only for some $\alpha > 0$ (instead of all $\alpha > 0$), see, for example, [6].

In metric spaces theory, it is well-known that every convergent sequence is Cauchy. Similarly, a modular convergent sequence is modular Cauchy (see [6, p.13]).

Chistyakov has introduced modular entourages that play a significant role in determining the interior and closure of a subset A of S_w^* (or S_w) in metric modular.

Definition 2.7. ([5]) Given $\alpha, \mu > 0$ and $r \in S_w^*$, the modular entourage about r relative to α and μ is the set defined as:

$$B_{\alpha,\mu}(r) = \{s \in S_w^* : w_\alpha(r, s) < \mu\}.$$

The interior and closure of a set $A \subset S_w^*$ are defined as:

$$A^\circ = \{x \in A : B_{\alpha,\alpha}(r) \subset A \text{ for some } \alpha > 0\}$$

and

$$\bar{A} = \{x \in S_w^* : A \cap B_{\alpha,\alpha}(r) \neq \varnothing \text{ for all } \alpha > 0\},$$

respectively. The boundary of A is denoted by ∂A and defined as: $\partial A = \bar{A}/A^\circ$.

Chistyakov in [5, 7] has given the concept that classical modular ρ could be induced from metric modular w defined on real linear space and vice versa. The modular set S_ρ induced from modular function ρ is a subspace of linear space. The criterion for classical modular induced from metric modular guarantees that the sets S_ρ and S_w are equal whenever the center r_0 of S_w is zero. Here, we give a short overview of classical modular ρ.

Orlicz in [15] has defined modular as follows: A modular on a real linear space S is a function $\rho : S \to [0, \infty]$ satisfying the following conditions:

(B1) $\rho(0) = 0$; (B2) If $r \in S$ and $\rho(\lambda r) = 0$ for all $\lambda > 0$, then $r = 0$; (B3) $\rho(-r) = \rho(r)$ for all $r \in S$, (B4) $\rho(\lambda r + \mu s) \leq \rho(r) + \rho(s)$ for all $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$ and $r, s \in S$.

In [14], a modular ρ on S is convex, if it satisfies the following inequality: $\rho(\lambda r + \mu s) \leq \lambda\rho(r) + \mu\rho(s)$.

If ρ is modular on S, then the set described as

$$S_\rho = \left\{r \in S : \lim_{\lambda \to +0} \rho(\lambda r) = 0\right\},$$

is called a modular space. The set S_ρ is a subspace of S, and it can be equipped with F-norm for the rule,

$$|r|_\rho = \inf\left\{\lambda > 0 : \rho\left(\frac{r}{\lambda}\right) \leq \lambda\right\},\ r \in S_\rho.$$

Moreover, if the modular is convex, then the modular space S_ρ coincides with

$$S_\rho^* = \{r \in S : \exists\, \lambda = \lambda(r) > 0 \text{ such that } \rho(\lambda r) < \infty\}$$

and

$$\|r\|_\rho = \inf\left\{\lambda > 0 : \rho\left(\frac{r}{\lambda}\right),\ r \in S_\rho = S_\rho^*\right\},$$

becomes convex F-norm.

The next proposition shows the coherence between metric modular w and classical modular ρ on real linear spaces.

Proposition 2.8. *([5, 7]) Suppose that S is a real linear space. For a given functional $\rho : S \to [0, \infty]$, set*

$$w_\alpha(r, s) := \rho\left(\frac{r - s}{\alpha}\right),\ \alpha > 0,\ r, s \in S. \tag{2.5}$$

Then: ρ is a (convex) modular on S if and only if w is (convex) metric modular on S, respectively.

On the other hand, if the metric modular w satisfies the following two axioms:

(a) $w_\alpha(\beta r, 0) = w_{\frac{\alpha}{\beta}}(r, 0)$ for all $\alpha, \beta > 0$ and $r \in S$.

(b) $w_\alpha(r+z,s+z) = w_\alpha(r,s)$ *for* $\alpha > 0$ *and* $r,s,z \in S$.

For the given $r \in S$, *we set* $w_1(r,0) = \rho(r)$. *Then* w *is (convex) metric modular iff* ρ *is a classic (convex) modular on* S, *respectively. Moreover, the equality (2.5) is also valid, the set* $S_\rho = S_w(0)$ *is subspace of* S *and the functional* $|r|_\rho = d_w(r,0)$, $r \in S_\rho$, *is an* F*-norm on* S_ρ. *If the function* w *is convex, then* $S_w^*(0) \equiv S_\rho^* = S_\rho$ *is the subspace of* S *and the functional* $\|r\|_\rho = d_w^*(r,0)$, $r \in S_\rho^*$, *is a norm on* S_ρ^*. *The same assertions hold for pseudomodular.*

3 Fixed Point Theorems in Modular Metric Spaces

In this section, we will establish fixed point theorems for self as well as non-self mappings of complete metric modular spaces.

Now, we introduce graph theory terminologies, which will be used in this section. Let S_w be the metric modular set. The diagonal of Cartesian product $S_w \times S_w$ is denoted by $\triangle$. Now, consider a directed graph $G_w = (V(G_w), E(G_w))$ such that the set of its vertices, $V(G_w)$, coincides with S_w and the edge set $E(G_w)$ consists of all loops, that is, $\triangle \subset E(G_w)$. Let G_w has no parallel edges (arcs). These graph theory terminologies and notations are standard and can be found in every graph theory book (see, for example, [10, 12]).

The converse graph of G_w is denoted by G_w^{-1}, that is, the graph obtained by G_w by reversing its edges, defined as

$$E\left(G_w^{-1}\right) = \{(s,r) \in S_w \times S_w : (r,s) \in E(G_w)\}$$

If r, s are vertices in the graph G_w, then a path from r to s of length M is a sequence $\{r_i\}_{i=1}^{M}$ of $M+1$ vertices of G_w such that $r_0 = r$, $r_M = s$ and $(r_{i-1}, r_i) \in E(G_w)$, $i = 1, 2...M$.

A graph G_w is called connected if there exists at least a path between two arbitrary vertices. If $\tilde{G}_w = \left(S_w, E\left(\tilde{G}_w\right)\right)$ is the symmetric graph obtained by placing together the vertices of both G_w and G_w^{-1}, that is,

$$E\left(\tilde{G}_w\right) = E(G_w) \cup E\left(G_w^{-1}\right),$$

then G_w is said to be weakly connected whenever $\tilde{G}_w$ is connected.

If $G_w = (V(G_w), E(G_w))$ is a graph and $V(G_w) \supset H$, then the graph $(H, E(G_w))$ with

$$E(H) = E(G_w) \cap (H \times H),$$

is said to be the subgraph of G_w determined by H, denoted by G_{wH}.

3.1 Self contraction case

A mapping $\Upsilon : S_w \to S_w$ is said to be defined on a metric modular space endowed with a graph G_w if it satisfies:

$$\forall\, r,s \in S_w, \ (r,s) \in E(G_w) \text{ implies } (\Upsilon r, \Upsilon s) \in E(G_w). \tag{3.1}$$

A mapping $\Upsilon : S_w \to S_w$ which is defined on metric modular space endowed with a graph G_w, is said to be a G_w-contraction, if there is a constant $k \in (0,1)$ such that $\forall\, r,s \in S_w$ with $(r,s) \in E(G_w)$, we have

$$w_{k\alpha}(\Upsilon r, \Upsilon s) \le w_\alpha(r,s), \text{ for } \alpha > 0. \tag{3.2}$$

If $\Upsilon r = r$, then the element $r \in S_w$ is said to be the fixed point of mapping Υ.

Theorem 3.1. *Suppose* (S_w, w, G_w) *be a complete modular metric space endowed with a weakly connected and directed graph* G_w *such that the following property (T) holds, that is, for any sequence* $\{r_n\}_{n=1}^\infty \subset S_w$ *with* $r_n \to r$ *as* $n \to \infty$ *and* $(r_n, r_{n+1}) \in E(G_w)$ *for all* $n \in \mathbb{N}$, *there exist a subsequence* $\{r_{s_n}\}_{n=1}^\infty$ *satisfying*

$$(r_{s_n}, r) \in E(G_w), \ \forall\, n \in \mathbb{N}. \tag{3.3}$$

Let $\Upsilon : S_w \to S_w$ be a G_w-contraction. If the set

$$S_{w\Upsilon} = \{r \in S_w : (r, \Upsilon r) \in E(G_w)\}, \tag{3.4}$$

is nonempty, then the mapping Υ has a fixed point in S_w.

Proof. Let $r_0 \in S_{w\Upsilon}$. It follows from (3.4) that $(r_0, \Upsilon r_0) \in E(G_w)$ and by using (3.1) ,we obtain

$$\left(\Upsilon^n r_0, \Upsilon^{n+1} r_0\right) \in E(G_w), \ \forall\, n \in \mathbb{N}. \tag{3.5}$$

Denote $r_n := \Upsilon^n r_0$ for all $n \in \mathbb{N}$. Then by the fact that Υ is a G_w-contraction and in view of (3.1), we get

$$w_\alpha(r_n, r_{n+1}) \le w_{\frac{\alpha}{k}}(r_{n-1}, r_n), \tag{3.6}$$

for all $n \in \mathbb{N}$. By using (3.6) with induction, it implies that

$$w_\alpha(r_n, r_{n+1}) \le w_{\frac{\alpha}{k^n}}(r_0, r_1), \ \forall\, n \in \mathbb{N} \text{ and } \alpha > 0. \tag{3.7}$$

For any positive integer q, by (3.2) and (2.1), we have

$$\begin{aligned} w_\alpha(r_n, r_{n+q}) &\le w_{\frac{\alpha}{q}}(r_n, r_{n+1}) + \cdots + w_{\frac{\alpha}{q}}(r_{n+q-1}, r_{n+q}) \\ &\le w_{\frac{\alpha}{qk^n}}(r_0, r_1) + \cdots + w_{\frac{\alpha}{qk^{n+q-1}}}(r_0, r_1) \\ &\le q\omega_{\frac{\alpha}{qk^n}}(r_0, r_1). \end{aligned} \tag{3.8}$$

By using (3.8) and according to the fact that

$$\lim_{\alpha\to\infty} w_\alpha(r, s) = 0, \text{ for all } r, s \in S_w, \tag{3.9}$$

we have

$$\lim_{n\to\infty} w_\alpha(r_n, r_{n+q}) \le 0 + \cdots + 0 = 0, \tag{3.10}$$

i.e., $\{r_n\}$ is Cauchy, hence convergent in (S_w, w, G_w). The limit of this sequence is denoted as:

$$\lim_{n\to\infty} r_n = r^*. \tag{3.11}$$

By the property (T) of (S_w, w, G_w), there exists a subsequence $\{r_{s_n}\}$ satisfying

$$(r_{s_n}, r^*) \in E(G_w), \ \forall\, n \in \mathbb{N}.$$

Hence, by contraction condition (3.2) and in view of (3.1), we get

$$w_\alpha(\Upsilon r_{s_n}, \Upsilon r^*) \le w_{\frac{\alpha}{k}}(r_{s_n}, r^*). \tag{3.12}$$

Therefore, by property (F3) of Definition 2.1, we have

$$\begin{aligned} w_\alpha(r^*, \Upsilon r^*) &\le w_{\frac{\alpha}{2}}(r^*, r_{s_n+1}) + w_{\frac{\alpha}{2}}(r_{s_n+1}, \Upsilon r^*) \\ &= w_{\frac{\alpha}{2}}(r^*, r_{s_n+1}) + w_{\frac{\alpha}{2}}(\Upsilon r_{s_n}, \Upsilon r^*). \end{aligned} \tag{3.13}$$

By using (3.12) in inequality (3.13) yields

$$w_\alpha(r^*, \Upsilon r^*) \le w_{\frac{\alpha}{2}}(r^*, r_{s_n+1}) + w_{\frac{\alpha}{2k}}(r_{s_n}, r^*), \tag{3.14}$$

for all $n \ge 1$. In equation (3.14), assuming $n \to \infty$ and using (3.11), we have $w_\alpha(r^*, \Upsilon r^*) \le 0$, then by condition (F1) we obtain $r^* = \Upsilon r^*$, which is a fixed point of mapping Υ. □

Remark 3.2. For $\alpha > 0$, consider the modular function,

$$w_\alpha(r, s) = \frac{d(r, s)}{\alpha}$$

for $r, s \in S$, where (S, d) is a metric space. This modular is convex and for any fixed r_0, the modular set $S_w^*(r_0) = S_w(r_0) = S$. Since it is convex modular, therefore, $d_w^*(r, s) = d(r, s)$.

If $\alpha = 1$, then the contraction condition (2.1) used in [2] is equivalent to contraction condition (3.2) used in Theorem 3.1, i.e., if there is a constant $k \in (0,1)$ for a mapping $\Upsilon : S_w \to S_w$ such that $\forall\, r, s \in S_w$ with $(r,s) \in E(G_w)$, we have

$$w_{k\alpha}(\Upsilon r, \Upsilon s) \le w_\alpha(r,s), \text{ for } \alpha > 0,$$

implies

$$d(\Upsilon r, \Upsilon s) \le kd(r,s).$$

Therefore, Theorem 3.1 is the extension of [2, Theorem 2.1].

Example 3.3. Let $S = [1,\infty) \subset \mathbb{R}$, (S,d) be the usual metric and according to Example 2.2 for $\Phi(\alpha) = \alpha$, we have

$$w_\alpha(r,s) = \frac{|r-s|}{\alpha}.$$

This is convex metric modular. In this case, $d_w(r,s) = |r-s|$ as well as modular and metric convergence is equivalent to usual d-convergence in S. The modular set $S_w^* = S_w = S$. Since $[1,\infty)$ is closed subset of $\mathbb{R}$ and $\mathbb{R}$ is complete with respect to usual metric. Therefore (S_w, w) is complete metric modular. Let the mapping $\Upsilon : S_w \to S_w$ be defined as $\Upsilon r = \frac{r}{2} + \frac{1}{r}$ and G_w be the complete graph on the set S_w that is, $E(G_w) = S_w \times S_w$. Since G_w is a complete graph, so it satisfies the property (3.1) for all $r, s \in S_w$. Moreover, G_w is weakly connected. Υ is a G_w-contraction with contraction coefficient $k \ge 0.5$. For any convergent sequence $\{r_n\} \subset S_w$ converging to some point $r \in S_w$ property (T) holds due to complete graph. Thus, by Theorem 3.1 mapping Υ has a fixed point which is $r^* = 1.414$.

3.2 Non-self contraction case

Let the function w satisfying the additional property defined as:

$$w_\alpha(r,s) = w_\alpha(r,u) + w_\alpha(u,s), \tag{3.15}$$

for all $\alpha > 0$ and $r, s, u \in S$, then it is said to be metric modular having convexity property.

Example 3.4. Let (S,d) be convex metric spaces and from Example 2.2, we define $\Phi(\alpha) = \alpha^n$ where $n = 1, 2, 3, \cdots$. The function defined as

$$w_\alpha(r,s) = \frac{d(r,s)}{\alpha^n},$$

for $\alpha > 0$ and $r, s \in S$ is a metric modular having convexity property.

Definition 3.5. Let (S_w, w) be a complete metric modular space and $A \subset S_w$. A mapping $\Upsilon : A \to S_w$ is modular metrically inward if for each $r \in A$ there exist an element $u \in A$ such that

$$w_\alpha(r, \Upsilon r) = w_\alpha(r,u) + w_\alpha(u, \Upsilon r), \tag{3.16}$$

for all $\alpha > 0$, where $u = r$ if and only if $r = \Upsilon r$.

By following the Proposition 2.8, for a (convex) modular on S, the modular set $S_w(0) = S_\rho$ ($S_w^*(0) = S_\rho^*$) is a linear subspace of S. Throughout this section, the function w is metric modular having covexity property and $S_w \equiv S_w(0)$, for fixed $r_0 = 0$.

Let $A \subset S_w$ and $r \in S_w$; for a given sequence $\{r_n\} \subset A$ and $\lim_{n\to\infty} w_\alpha(r_n, r) = 0$, if $r \in A$ then this set A is said to be closed. Let A be a nonempty, closed subset of S_w and $\Upsilon : A \to S_w$ be a non-self mapping. We choose $s \in A$ such that $\Upsilon s \notin A$, then there is an element $z \in \partial A$ such that

$$z = (1-\mu)s + \mu \Upsilon s \text{ where } \mu \in (0,1),$$

which represents the fact that

$$w_\alpha(s, \Upsilon s) = w_\alpha(s,z) + w_\alpha(z, \Upsilon s), \ z \in \partial A \text{ for all } \alpha > 0. \tag{3.17}$$

The inward condition used in the Definition 3.5 is more general since it does not require z in equality (3.16) to belong to ∂A.

A non-self mapping $\Upsilon : A \to S_w$ is said to be defined on the modular metric space (S_w, w) endowed with a graph G_w if it satisfies the property that

$$\begin{array}{l} \text{for all } r, s \in A \ \ (r, s) \in E\left(G_w\right) \\ \text{with } \Upsilon r, \Upsilon s \in A, \text{ implies } \left(\Upsilon r, \Upsilon s\right) \in E\left(G_w\right) \cap \left(A \times A\right), \end{array} \tag{3.18}$$

for the subgraph of G_w induced by A.

Theorem 3.6. *Suppose (S_w, w, G_w) be a complete modular metric space endowed with a weakly connected and directed graph G_w provided that following property (T) holds, that is, for any sequence $\{r_n\} \subset S_w$ along with $r_n \to r$ as $n \to \infty$ and*

$$\left(r_n, r_{n+1}\right) \in E\left(G_w\right), \ \forall\, n \in \mathbb{N},$$

there exist a subsequence $\{r_{s_n}\}$ satisfying

$$\left(r_{s_n}, r\right) \in E\left(G_w\right), \ \forall\, n \in \mathbb{N}. \tag{3.19}$$

Let A be a nonempty, closed subset of S_w and $\Upsilon : A \to S_w$ be a G_{wA}-contraction, that is, there exist a constant $k \in (0, 1)$ such that

$$w_{k\alpha}\left(\Upsilon r, \Upsilon s\right) \leq w_{\alpha}\left(r, s\right) \textit{ for all } (r, s) \in E\left(G_{wA}\right) \textit{ and } \alpha > 0, \tag{3.20}$$

where G_{wA} is the subgraph of G_w determined by A. If the set

$$A_{\Upsilon} := \left\{r \in \partial A : \left(r, \Upsilon r\right) \in E\left(G_w\right)\right\},$$

is nonempty and Υ satisfies Rothe's boundary condition

$$\Upsilon\left(\partial A\right) \subset A. \tag{3.21}$$

Then the mapping Υ has a fixed point.

Proof. If $\Upsilon(A) \subset A$, then Υ is a self-map of the closed set A and the conclusion follows by Theorem 3.1. Now, we consider the case that $\Upsilon(A) \not\subset A$. Let $r_0 \in A_{\Upsilon}$. It follows that $(r_0, \Upsilon r_0) \in E\left(G_w\right)$ and in view of equation (3.1) ,we have

$$\left(\Upsilon^n r_0, \Upsilon^{n+1} r_0\right) \in E\left(G_w\right), \text{ for all } n \in \mathbb{N}. \tag{3.22}$$

Let us denote $r_n := \Upsilon^n r_0$, for all $n \in \mathbb{N}$. By virtue of (3.21) $\Upsilon r_0 \in A$.

Consider $r_1 \equiv s_1 = \Upsilon r_0$. Let $\Upsilon r_1 \in A$, set $r_2 \equiv s_2 = \Upsilon r_1$. If $\Upsilon r_1 \notin A$, then we can select an element $r_2 \in \partial A$ on the segment $[r_1, \Upsilon r_1]$, that is,

$$r_2 = \left(1 - \mu\right) r_1 + \mu \Upsilon r_1, \text{ where } \mu \in \left(0, 1\right).$$

By following the same method we obtain two sequences $\{r_n\}$ and $\{s_n\}$ whose terms satisfy one of the succeeding properties:

(i) $r_n \equiv s_n = \Upsilon r_{n-1}$, if $\Upsilon r_{n-1} \in A$;

(ii) $r_n = \left(1 - \mu\right) r_{n-1} + \mu \Upsilon r_{n-1} \in \partial A, \mu \in \left(0, 1\right), \Upsilon r_{n-1} \notin A$.

For the simplicity of arguments in the proof, let us denote

$$U = \left\{r_a \in \{r_n\} : r_a = s_a = \Upsilon r_{a-1}\right\}$$

and

$$Z = \left\{r_a \in \{r_n\} : r_a \neq \Upsilon r_{a-1}\right\}.$$

Note that $\{r_n\} \subset A$ for all $n \in \mathbb{N}$. Moreover if $r_a \in Z$, then both r_{a-1} and r_{a+1} belong to set U. The sequence $\{r_n\}$ can have consecutive terms r_a and r_{a+1} in set U, but this assertion is not true for the set Z. First of all we have to prove that

$$r_a \neq \Upsilon r_{a-1} \text{ implies } r_{a-1} = \Upsilon r_{a-2}.$$

Suppose contrarily that $r_{a-1} \neq \Upsilon r_{a-2}$ then $r_{a-1} \in \partial A$. Since $\Upsilon(\partial A) \subset A$ then $\Upsilon r_{a-1} \in A$, hence $r_a = \Upsilon r_{a-1}$ which is a contradiction.

Here, we have three different cases to show that $\{r_n\}$ is Cauchy which are following:

Case 1. $r_n, r_{n+1} \in U$.

Since both elements belong to set U, therefore, we have $r_n = s_n = \Upsilon r_{n-1}$ and $r_{n+1} = s_{n+1} = \Upsilon r_n$. Hence,

$$\begin{aligned} w_\alpha(r_{n+1}, r_n) &= w_\alpha(s_{n+1}, s_n) \\ &= w_\alpha(\Upsilon s_n, \Upsilon s_{n-1}), \end{aligned}$$

where $(s_n, s_{n-1}) \in E(G_w)$ by virtue of (3.22), we have the following inequality by using contraction condition (3.20)

$$\begin{aligned} w_\alpha(\Upsilon s_n, \Upsilon s_{n-1}) &= w_\alpha(\Upsilon r_n, \Upsilon r_{n-1}) \\ &\leq w_{\frac{\alpha}{k}}(r_n, r_{n-1}). \end{aligned}$$

Therefore, we have

$$w_\alpha(r_{n+1}, r_n) \leq w_{\frac{\alpha}{k}}(r_n, r_{n-1}) < w_{\frac{\alpha}{2k}}(r_n, r_{n-1}), \tag{3.23}$$

by virtue of inequality (2.1).

Case 2. $r_n \in U, r_{n+1} \in Z$.

In this case, we have $r_n = s_n = \Upsilon r_{n-1}$, but $r_{n+1} \neq s_{n+1} = \Upsilon r_n$, therefore we have

$$w_\alpha(r_n, \Upsilon r_n) = w_\alpha(r_n, r_{n+1}) + w_\alpha(r_{n+1}, \Upsilon r_n) \text{ for all } \alpha > 0.$$

The above equality implies $w_\alpha(r_{n+1}, \Upsilon r_n) \neq 0$. Therefore,

$$\begin{aligned} w_\alpha(r_n, r_{n+1}) &= w_\alpha(r_n, \Upsilon r_n) - w_\alpha(r_{n+1}, \Upsilon r_n) \\ &< w_\alpha(r_n, \Upsilon r_n) \\ &= w_\alpha(\Upsilon r_{n-1}, \Upsilon r_n), \end{aligned} \tag{3.24}$$

since $r_n \in U$. By using (3.24) we obtain

$$\begin{aligned} w_\alpha(r_n, r_{n+1}) &< w_\alpha(\Upsilon r_{n-1}, \Upsilon r_n) \\ &= w_\alpha(\Upsilon s_{n-1}, \Upsilon s_n). \end{aligned}$$

We can obtain again inequality (3.23) by using the similar arguments to that in Case 1.

Case 3. $r_n \in Z, r_{n+1} \in U$.

In this case, we have $r_{n+1} = \Upsilon r_n$, and $r_n \neq s_n = \Upsilon r_{n-1}$. Since $r_n \in Z$, so we have

$$w_{\frac{\alpha}{2}}(r_{n-1}, \Upsilon r_{n-1}) = w_{\frac{\alpha}{2}}(r_{n-1}, r_n) + w_{\frac{\alpha}{2}}(r_n, \Upsilon r_{n-1}) \text{ for all } \alpha > 0. \tag{3.25}$$

Hence, by inequality (F3) from Definition 2.1,

$$\begin{aligned} w_\alpha(r_n, r_{n+1}) &\leq w_{\frac{\alpha}{2}}(r_n, \Upsilon r_{n-1}) + w_{\frac{\alpha}{2}}(\Upsilon r_{n-1}, r_{n+1}) \\ &= w_{\frac{\alpha}{2}}(r_n, \Upsilon r_{n-1}) + w_{\frac{\alpha}{2}}(\Upsilon r_{n-1}, \Upsilon r_n) \\ &= w_{\frac{\alpha}{2}}(r_n, \Upsilon r_{n-1}) + w_{\frac{\alpha}{2}}(\Upsilon s_{n-1}, \Upsilon s_n). \end{aligned} \tag{3.26}$$

By virtue of (3.22) $(s_{n-1}, s_n) \in E(G_w)$, and the following inequality is obtained by the contraction condition (3.20)

$$w_{\frac{\alpha}{2}}(\Upsilon s_{n-1}, \Upsilon s_n) \leq w_{\frac{\alpha}{2k}}(s_{n-1}, s_n) = w_{\frac{\alpha}{2k}}(r_{n-1}, r_n). \tag{3.27}$$

Thus, by using (2.1), (3.25), (3.27) in inequality (3.26) and the fact that $k \in (0,1)$, we have

$$\begin{aligned} w_\alpha(r_n, r_{n+1}) &\le w_{\frac{\alpha}{2}}(r_n, \Upsilon r_{n-1}) + w_{\frac{\alpha}{2k}}(r_{n-1}, r_n) \\ &< w_{\frac{\alpha}{2}}(r_n, \Upsilon r_{n-1}) + w_{\frac{\alpha}{2}}(r_{n-1}, r_n) \\ &= w_{\frac{\alpha}{2}}(r_{n-1}, \Upsilon r_{n-1}). \end{aligned}$$

By using (3.22), $(r_{n-2}, r_{n-1}) = (s_{n-2}, s_{n-1}) \in E(G_w)$ and by virtue of contraction condition (3.20), we get

$$\begin{aligned} w_\alpha(r_n, r_{n+1}) &\le w_{\frac{\alpha}{2}}(r_{n-1}, \Upsilon r_{n-1}) \\ &= w_{\frac{\alpha}{2}}(\Upsilon r_{n-2}, \Upsilon r_{n-1}) \\ &\le w_{\frac{\alpha}{2k}}(r_{n-2}, r_{n-1}). \end{aligned} \tag{3.28}$$

Now, we summarize all above mentioned three cases, by virtue of (3.23) and (3.28), it follows that the sequence $\{w_\alpha(r_n, r_{n+1})\}$ satisfies the inequality

$$w_\alpha(r_n, r_{n+1}) \le \max\left\{w_{\frac{\alpha}{2k}}(r_{n-2}, r_{n-1}), w_{\frac{\alpha}{2k}}(r_{n-1}, r_n)\right\}, \tag{3.29}$$

for all $n \ge 2$. We obtain the following inequality by simple induction for $n \ge 2$, and using (3.29)

$$w_\alpha(r_n, r_{n+1}) \le \max\left\{w_{\frac{\alpha}{2k^{[n/2]}}}(r_0, r_1), w_{\frac{\alpha}{2k^{[n/2]}}}(r_1, r_2)\right\}, \tag{3.30}$$

where $\left[\frac{n}{2}\right]$ denotes the greatest integer not exceeding $\frac{n}{2}$.

Further, by using (3.30) and (2.1), we have

$$\begin{aligned} w_\alpha(r_n, r_{n+q}) &\le w_{\frac{\alpha}{q}}(r_n, r_{n+1}) + \cdots + w_{\frac{\alpha}{q}}(r_{n+q-1}, r_{n+q}) \\ &\le \max\left\{w_{\frac{\alpha}{2qk^{[n/2]}}}(r_0, r_1), w_{\frac{\alpha}{2qk^{[n/2]}}}(r_1, r_2)\right\} + \cdots \\ &+ \max\left\{w_{\frac{\alpha}{2qk^{[n+q-1/2]}}}(r_0, r_1), w_{\frac{\alpha}{2qk^{[n+q-1/2]}}}(r_1, r_2)\right\} \\ &\le q \max\left\{w_{\frac{\alpha}{2qk^{[n/2]}}}(r_0, r_1), w_{\frac{\alpha}{2qk^{[n/2]}}}(r_1, r_2)\right\}, \end{aligned} \tag{3.31}$$

for any positive integer q.

By using (3.31) and (3.9), we have

$$\lim_{n\to\infty} w_\alpha(r_n, r_{n+q}) \le 0 + \cdots + 0 = 0,$$

which shows that $\{r_n\}$ is a Cauchy sequence, hence convergent. Since $\{r_n\} \subset A$ and A is closed, $\{r_n\}$ converges to some point $r' \in A$, i.e., $\lim_{n\to\infty} r_n = r'$.

By property (T), there exist a subsequence $\{r_{s_n}\}$ satisfying

$$(r_{s_n}, r') \in E(G_w), \text{for all } n \in \mathbb{N}.$$

Hence, by the contraction condition (3.20),

$$w_\alpha\left(\Upsilon r_{s_n}, \Upsilon r'\right) \le w_{\frac{\alpha}{k}}\left(r_{s_n}, r'\right). \tag{3.32}$$

Therefore, by (F3) from Definition 2.1, we have

$$\begin{aligned} w_\alpha\left(r', \Upsilon r'\right) &\le w_{\frac{\alpha}{2}}\left(r', r_{s_n+1}\right) + w_{\frac{\alpha}{2}}\left(r_{s_n+1}, \Upsilon r'\right) \\ &= w_{\frac{\alpha}{2}}\left(r', r_{s_n+1}\right) + w_{\frac{\alpha}{2}}\left(\Upsilon r_{s_n}, \Upsilon r'\right). \end{aligned}$$

By using (3.32), the above inequality yields

$$w_\alpha\left(r', \Upsilon r'\right) \le w_{\frac{\alpha}{2}}\left(r', r_{s_n+1}\right) + w_{\frac{\alpha}{2k}}\left(r_{s_n}, r'\right), \tag{3.33}$$

for all $n \ge 1$. Taking limit $n \to \infty$ and using (3.33), we obtain $w_\alpha\left(r', \Upsilon r'\right) \le 0$ and, then by (F1) we get $r' = \Upsilon r'$, which shows that r' is a fixed point of Υ. □

Remark 3.7. Let S be a Banach space, $d(r,s)$ is the metric induced from norm and the modular function is defined as: $w_\alpha(r,s) = \dfrac{d(r,s)}{\alpha}$, for $\alpha > 0$. This modular is convex and satisfies axioms (a) and (b) of Proposition 2.8. Consequently, for center $r_0 = 0$, $S_w^*(0) = S_\rho^*(0)$ and the functional $\|r-s\|_\rho = d_w^*(r,s)$, $r \in S_\rho^*(0)$. The modular set $S_w^*(0) = S_w(0) = S$ and $d_w^*(r,s) = d(r,s)$. If $\alpha = 1$, then the contraction condition (3.4) used in [2] is equivalent to contraction (3.20) condition used in Theorem 3.6, i.e., for a nonempty, closed $A \subset S_w$, if there exist a constant $k \in (0,1)$ for a mapping $\Upsilon : A \to S_w$ such that

$$w_{k\alpha}(\Upsilon r, \Upsilon s) \leq w_\alpha(r,s) \text{ for all } (r,s) \in E(G_{wA}) \text{ and } \alpha > 0,$$

implies

$$d(\Upsilon r, \Upsilon s) \leq kd(r,s).$$

Therefore, Theorem 3.6 is the extension of [2, Theorem 3.1].

Example 3.8. Let $S = \mathbb{R}$, (S,d) be the usual metric. In view of Example 2.2, the modular metric function is defined as

$$w_\alpha(r,s) = \frac{|r-s|}{\alpha}, \text{ for } \alpha > 0.$$

This function satisfies the axioms (a) and (b) of Proposition 2.8 and convexity property (3.15) for all $\alpha > 0$ and $r, s \in S$ (cf. Example 3.4). It is convex metric modular, so we have $S_w^*(0) = S_w(0) = \mathbb{R}$ and $d_w(r,s) = |r-s|$ as well as modular and metric convergence is equivalent to usual d-convergence in S. Since $\mathbb{R}$ is complete, (S_w, w) is a complete metric modular and $A = (-\infty, 0]$ is a closed subset of S_w. Let the mapping $\Upsilon : A \to S_w$ be defined as

$$\Upsilon r = \begin{cases} 0 & \text{if } r \in [-1,0] \\ 0.5 & \text{if } r \in (-\infty,-1). \end{cases}$$

The edge set of graph G_w and the subgraph G_{wA} determined by A is defined as

$$E(G_w) = \{(r,s) \in S_w \times S_w : r \leq s\}$$

and

$$E(G_{wA}) = \{(r,s) \in A \times A : r \leq s\},$$

respectively. It is easy to check that (3.18) holds, that is, for all $r, s \in A$ $(r,s) \in E(G_w)$ with $\Upsilon r, \Upsilon s \in A$, implies $(\Upsilon r, \Upsilon s) \in E(G_w) \cap (A \times A)$. In view of (3.18), for $t, u \in (-\infty, -1)$ and $r, s \in [-1, 0]$, the edges (t,u), (t,r) has to be removed and for the rest of edges we have

$$(\Upsilon r, \Upsilon s) = (0,0) \in E(G_{wA}).$$

Moreover, G_w is a weakly connected and Υ is a non-self G_{wA}-contraction on A with contraction cofficient $k = \frac{1}{4}$, since

$$\frac{|\Upsilon r - \Upsilon s|}{\alpha} = \frac{1}{2\alpha} < \frac{1}{4} \times \frac{|r-s|}{\alpha} \text{ for } r \in (-\infty,-1) \text{ and } s \in [-1,0].$$

(for the rest of edges of $E(G_{wA})$, the contraction condition (3.20) is obvious, since the quantity in its left-hand side is always zero). Property (T) holds with constant sequences $\{r_n = r\}$ satisfying the property $(r_n, r_{n+1}) \in E(G_{wA})$, for all $n \in \mathbb{N}$. Rothe's boundary condition is also satisfied, as $\partial A = \{0\}$ and so $\Upsilon(\partial A) \subset A$. Finally, since we also have $A_\Upsilon = \{0\} \neq \varnothing$, all assumptions in Theorem 3.6 are satisfied, and $r' = 0$ is the fixed point of Υ.

4 Conclusion

In this paper, we have presented the fixed point theorems for self and non-self G_w-contractions on modular metric spaces endowed with a graph. This immediately implies the generalization of recently fixed point theorems for self mappings on metric spaces and also fixed point theorems for non-self mappings in Banach spaces or convex metric spaces.

Concluding this paper, we remark that all statements remain true if we replace the modular metric spaces by metric spaces.

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