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Reliable analysis for the Drinfeld-Sokolov-Wilson equation in mathematical physics

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Abstract The present paper studies the Drinfeld-Sokolov-Wilson (DSW) equation. We perform the $S(\xi)$-expansion method to take some exact solutions and create different solitary wave aspects for each equation. The received perspectives provide the firm mathematical foundation as well as describe the wave generation in soliton physics. As a result, we get some new soliton solutions. Finally, the exact solution and its geometrical properties are constructed, considering Mean curvature and Gaussian curvature as for the DSW equation. The $S(\xi)$-expansion method analyzes the solution follow through instantaneously with this equation.

1 Introduction

Soliton theory has fascinated the observation of experts from all over the world. For constructing soliton solutions in numerous forms, different efficient techniques, such as the Hirota bilinear method [1], Bilinear neural network method [2], $(G'/G, 1/G)$-expansion method [3], Improved (G'/G)-expansion method [4], the modified exp-function method [5], extended Exp-function method [6], transformed rational function method [7], New generalized (G'/G)-expansion method [8, 9] and many more have been developed.

Drinfel'd and Sokolov [10] and Wilson [11] has been proposed the DSW model for dispersive water waves and that play an important role in fluid dynamics [12]. To study of the model is given by

$$\frac{\delta U}{\delta t} + \alpha_1 V \frac{\delta V}{\delta x} = 0, \tag{1.1a}$$

$$\frac{\delta V}{\delta t} + \alpha_2 \frac{\delta^3 V}{\delta x^3} + \alpha_3 U \frac{\delta V}{\delta x} + \alpha_4 V \frac{\delta U}{\delta x} = 0, \tag{1.1b}$$

where α_1, α_2, α_3, and α_4 are nonzero parameters. Recently, several researchers have been showed their interested on this model [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Qin and Yan [13] construct doubly periodic solutions of the coupled DSW equation by using an improve F-expansion method. Ayub et al. [14] have introduced the Exp-function method to look for solitary solutions of the generalized DSW system. Sweet and Gorder [15] applied the method of homotopy analysis to obtain analytical solutions of the generalized DSW system. Jawad [16] has introduced the traveling wave solutions and new solitary wave solutions of the DSW equation, Fornberg-Whitham equation, potential-TSF equation, Jimbo-Miwa equation, Modified Zakharov-Kuznetsov equation, and (2 + 1)-dimensional Konopelchenko-Dubrovsky equation via the tanh and Sech function methods. Cesar [17] obtained exact solutions of this model by applying the improved tanh-coth. Abdelaziz and Ibrahim [18] has been proposed the enhanced of the G'/G-expansion method combined with Liu's theorem to find new exact solutions of the nonlinear (1+1)-dimensional DSW equation. Zhang [19] established variational principles of the DSW equation Via the semi-inverse method and also obtained an exact solitary solution and exact singular periodic wave solution using the variational scheme. Gurefe and Misirli [20] applied the Exp-function method to obtain generalized solitary solutions of the generalized (2+1)dimensional Burgers-type equation and the generalized DSW system.

Niu and Liu [21] re-examined the well known coupled system as the DSW equation. Its proper Darboux transformation is constructed with the help of a Lax operator of fourth order and some solutions are calculated and a nonlinear superposition formula is worked out for the associated Backlund transformation. Arnous et al. [22] employed two integration schemes to draw solitons, singular periodic waves and other types of solutions of the DSW equation. Jin and Lu [23] applied the variational iteration method to solve the classical DSW equation. Hirota et al. [24] present this equation a novel type of solutions called static solitons and this static solutions interact with moving solitons without deformations.

Therefore, Our aim of this article to establish exact solutions and geometrical formation namely, Normal curvature, Gaussian curvatures and Mean curvature of the selected modal as the DSW model through the $S(\xi)$-expansion method.

The synopsis of this paper as seen below In Section 2, we have given the algorithm of the $S(\xi)$-expansion method and obtained new solutions of the DSW equation through the $S(\xi)$-expansion method. In Section 3, we firstly provides some basic definitions of the differential geometry and Minkowski space $\mathbb{R}^3$ and then derived Normal curvature, Mean curvature and Gaussian curvature for the exact solution of DSW equation. In Section 2.2, graphical representations and numerical experiment of the derived solutions are depicted. Finally, the conclusion of our study is given.

2 The $S(\xi)$-expansion method

In this section, we discussed the main features of analytical methods as considering the $S(\xi)$-expansion method [25, 26, 27].

- **Phase 1:** Regarding the general NLEE with space variables $x_1, x_2, x_3, ..., x_n$ and time variable t form

$$\mathrm{P}(V, \frac{\partial V}{\partial t}, \frac{\partial V^2}{\partial t^2}, \frac{\partial V}{\partial x_1}, \frac{\partial V^2}{\partial x_1 t}, \frac{\partial V}{\partial x_2}, \frac{\partial V^2}{\partial x_1 x_2}, \frac{\partial V^2}{\partial x_2 x_1}, ...) = 0, \tag{2.1}$$

where $V(x_1, x_2, x_3, ..., x_n, t)$ is an unknown function and P is a polynomial in

$$V(x_1, x_2, x_3, ..., x_n, t).$$

- **Phase 2:** The traveling wave variable

$$\mathrm{V} = V(x_1, x_2, x_3, ..., x_n, t) = v(\xi), \xi = \sum_{i=1}^{n} k_i x_i + \beta t, \tag{2.2}$$

where k_i for all $i = 1, 2, ..., n$ are constants and β is the speed of the traveling wave. From Eq. 2.2 and Eq. 2.1, we have

$$\mathrm{R}(V, \beta\frac{dV}{d\xi}, k_1\frac{dV}{d\xi}, k_2\frac{dV}{d\xi}, \beta^2\frac{d^2V}{d\xi^2}, k_1\beta\frac{d^2V}{d\xi^2}, ...) = 0. \tag{2.3}$$

- **Phase 3:** Considering the traveling wave solution of Eq. 2.3 can be expressed as the form

$$\mathrm{V}(\xi) = \sum_{i=0}^{N} s_i S(\xi)^i, \tag{2.4}$$

where $S(\xi) = e^{-T(\xi)}$ and $s_i (i = 0, ..., n)$ are constants to be determined, such that $s_N \neq 0$ and $T = T(\xi)$ satisfies the following auxiliary equation:

$$(T(\xi))\prime = e^{-T(\xi)} + \mu e^{T(\xi)} + \lambda, \tag{2.5}$$

where $s_N ...., \beta, \lambda, \mu$ are constants to be determined latter.

Next, we have five solutions of Eq. 2.5.

- when $\mu \neq 0$, $\lambda^2 - 4\mu > 0$,, the solution of Eq. 2.5 is

$$T(\xi) = log(\frac{-\sqrt{\Psi}tanh(\frac{\sqrt{\Psi}}{2})(\xi + E) - \lambda}{2\mu}). \tag{2.6}$$

- when $\mu \neq 0$, $\lambda^2 - 4\mu < 0$, the solution of Eq. 2.5 is

$$T(\xi) = log(\frac{\sqrt{-\Psi}tan(\frac{\sqrt{\Psi}}{2})(\xi + E) - \lambda}{2\mu}). \tag{2.7}$$

- when $\mu = 0$, $\lambda \neq 0$, $\lambda^2 - 4\mu > 0$,, the solution of Eq. 2.5 is

$$T(\xi) = log(\frac{\lambda}{e^{\lambda(\xi+E)} - 1}). \tag{2.8}$$

- when $\mu \neq 0$, $\lambda \neq 0$, $\lambda^2 - 4\mu = 0$,, the solution of Eq. 2.5 is

$$T(\xi) = log(-\frac{2}{\lambda} + \frac{4}{\lambda^2(\xi + E)}). \tag{2.9}$$

- when $\mu = 0$, $\lambda = 0$, $\lambda^2 - 4\mu = 0$,, the solution of Eq. 2.5 is

$$T(\xi) = log(\xi + E). \tag{2.10}$$

- **Phase 4:** Eq. 2.3 can be constructed as analytical methods, i.e.,the $S(\xi)$-expansion method.
- **Phase 5:** The form of V is obvious and making the hight order derivatives and the nonlinear term of Eq. 2.3 to find the coefficients of V.
- **Phase 6:** Finally, the solitary wave solutions of Eq. 2.1 have been obtained by combining consequently given steps.

2.1 The Drinfel'd-Sokolov-Wilson (DSW) equation via the $S(\xi)$-expansion method

Herein, applied traveling wave variable $V(\xi) = V(x,t)$, $\xi = x + \beta t$, Eq. 1.1a and 1.1b converts into a nonlinear ODE

$$\beta\frac{dU}{d\xi} + \alpha_1 V\frac{dV}{d\xi} = 0, \tag{2.11}$$

$$\beta\frac{dV}{d\xi} + \alpha_2\frac{d^3V}{d\xi^3} + \alpha_3 U\frac{dV}{d\xi} + \alpha_4\frac{dU}{d\xi}V = 0. \tag{2.12}$$

Now integrating Eq. 2.11, we have

$$U = -\frac{\alpha_1 V^2}{2\beta}. \tag{2.13}$$

From Eq. 2.13 and Eq. 2.12 , we obtain

$$2\beta\alpha_2\frac{d^3V}{d\xi^3} + 2\beta^2\frac{dV}{d\xi} - \alpha_1(\alpha_3 + 2\alpha_4)V^2\frac{dV}{d\xi} = 0. \tag{2.14}$$

Again integrating Eq. 2.14, we obtain

$$2\beta\alpha_2\frac{d^2V}{d\xi^2} + 2\beta^2 V - \frac{\alpha_1(\alpha_3 + 2\alpha_4)}{3}V^3 = 0. \tag{2.15}$$

Applying the homogeneous balance between V^3 and V'' of Eq. 2.15, we have $N = 1$. substituting the value of N in Eq. (13), we obtain

$$V(\xi) = s_0 + s_1 S(\xi), \tag{2.16}$$

where $S(\xi) = exp(-\phi(\xi))$ and the coefficients s_0 and s_1 are constants to be evaluated.

From Eq. 2.16 and Eq. 2.15 and then equating each coefficients of $T(\xi)$ to zero, we get

$$4\beta\alpha_2 s_1 - (\frac{2}{3}\alpha_1\alpha_4 + \frac{1}{3}\alpha_1\alpha_3)s_1^3 = 0, \tag{2.17}$$

$$-(2\alpha_1\alpha_4 + \alpha_1\alpha_3)s_0 s_1^2 + 6\beta\alpha_2\lambda s_1 = 0, \tag{2.18}$$

$$(2\beta\alpha_2\lambda^2 + 4\beta\alpha_2\mu + 2\beta^2)s_1 - (2\alpha_1\alpha_4 + \alpha_1\alpha_3)s_0^2 s_1 = 0, \tag{2.19}$$

$$2\beta\alpha_2\lambda\mu s_1 + 2\beta^2 s_0 - (\frac{2}{3}\alpha_1\alpha_4 + \frac{1}{3}\alpha_1\alpha_3)s_1^3 = 0. \tag{2.20}$$

Applying Maple, we calculate the Eq. 2.17 to Eq. 2.20

$\beta = \frac{\alpha_2(\lambda^2-4\mu)}{2}$, $s_0 = (\pm\sqrt{\frac{6(\lambda^2-4\mu)}{2\alpha_1\alpha_4+\alpha_1\alpha_3}})\alpha_2\lambda$ and $s_1 = (\pm\sqrt{\frac{6(\lambda^2-4\mu)}{2\alpha_1\alpha_4+\alpha_1\alpha_3}})\alpha_2$, where λ and μ are constants.

Substituting the values of β, s_0, s_1 into Eq. 2.16, we have

$$\mathrm{V}(\xi) = \Phi\lambda + \Phi S(\xi), \tag{2.21}$$

where $\xi = x - \frac{\alpha_2(\lambda^2-4\mu)}{2}t$ and $\Phi = (\pm\sqrt{\frac{6(\lambda^2-4\mu)}{2\alpha_1\alpha_4+\alpha_1\alpha_3}})\alpha_2$.

A substitution of Eq. 2.6 to Eq. 2.10 into Eq. 2.21, leads to the following five traveling wave solutions of the Drinfel'd-Sokolov-Wilson (DSW) equation.

if $\mu \neq 0$ and $\lambda^2 - 4\mu > 0$, then

$$V_1(\xi) = \Phi\lambda - \Phi(\frac{2\mu}{\sqrt{\Psi}tanh(\frac{\sqrt{\Psi}}{2})(\xi+E)+\lambda}). \tag{2.22}$$

if $\mu \neq 0$ and $\lambda^2 - 4\mu < 0$, then

$$V_2(\xi) = \Phi\lambda + \Phi(\frac{2\mu}{\sqrt{-\Psi}tan(\frac{\sqrt{\Psi}}{2})(\xi+E)-\lambda}). \tag{2.23}$$

if $\mu = 0$, $\lambda \neq 0$ and $\lambda^2 - 4\mu > 0$, then

$$V_3(\xi) = \Phi\lambda + \Phi(\frac{\lambda}{e^{\lambda(\xi+E)}-1}). \tag{2.24}$$

if $\mu \neq 0$, $\lambda \neq 0$ and $\lambda^2 - 4\mu = 0$, then

$$V_4(\xi) = \Phi\lambda + \Phi(\frac{2}{\lambda} + \frac{4}{\lambda^2(\xi+E)})^{-1}. \tag{2.25}$$

if $\mu = 0$, $\lambda = 0$ and $\lambda^2 - 4\mu = 0$, then

$$V_5(\xi) = \Phi\lambda + \Phi(\frac{1}{(\xi+E)}. \tag{2.26}$$

2.2 Graphical representations of the obtained solutions

Herein, we have been established five traveling wave solution by using the $S(\xi)$-expansion method. Namely, exponential function, rational function, trigonometric function, and hyperbolic function are obtained from encompassing of explicit solutions. If we provided the particular value of an unknown parameter of the traveling wave solutions, then the solitary wave might be produced. As for, we have illustrated some figure of the solitary wave for considering a particular value of an unknown parameter. In this section, We constructed the graphical representation as regrading the results of DSW equation through the $S(\xi)$-expansion method.

$2D$ **Graph of the exact equation via the $S(\xi)$-expansion method:**

- Referring to Figure 1, $2D$ Graph of the exact solution $V_1(x,t)$ at the particular values of $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 4$, $\lambda = 1$, $E = 1$, $\mu = -\frac{1}{2}$ and $t = 1$.
- Referring to Figure 2, $2D$ Graph of the exact solution $V_2(x,t)$ at the particular values of $\alpha_1 = 1$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = 0.3$, $\alpha_4 = 0.1$, $\lambda = 1$, $E = 1$, $\mu = 1$ and $t = 1$..
- Referring to Figure 3, $2D$ Graph of the exact solution $V_3(x,t)$ at the particular values of $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 4$, $\lambda = 1$, $E = 1$, $\mu = 0$ and $t = 1$.

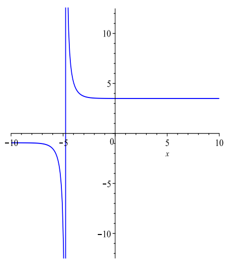


Figure 1. $2D$ graph of the exact solutions for the DSW equation via the $S(\xi)$-expansion method.

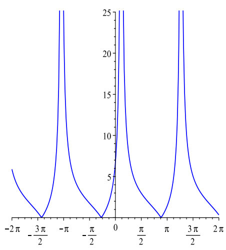


Figure 2. $2D$ graph of the exact solutions for the DSW equation via the $S(\xi)$-expansion method.

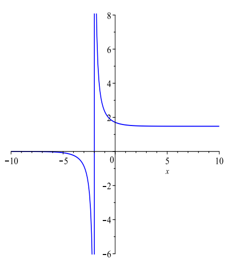
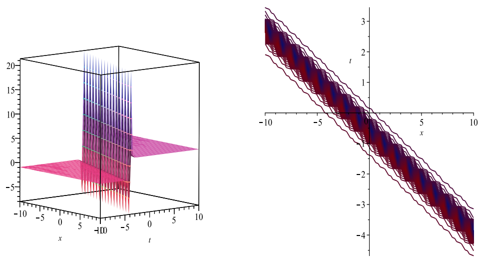


Figure 3. $2D$ graph of the exact solutions for the DSW equation via the $S(\xi)$-expansion method.

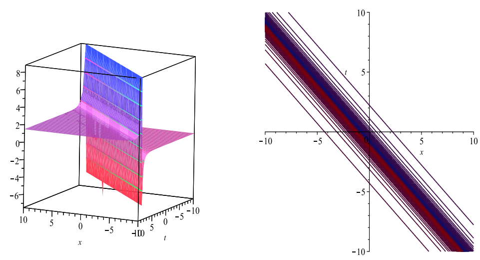
$3D$ and contour plot graph of the exact equation via the $S(\xi)$-expansion method:

- Referring to Figure 4, The shape of the exact solution $V_1(x,t)$ at the particular values of $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 4$, $\lambda = 1$, $E = 1$ and $\mu = -\frac{1}{2}$ within the interval $-10 \leq x, t \leq 10$.
- Referring to Figure 5, Graph of the exact solution $V_3(x,t)$ at the particular values of $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 4$, $\lambda = 1$, $E = 1$ and $\mu = 0$ within the interval $-10 \leq x, t \leq 10$.



a. b.

Figure 4. Graph of the exact solutions for the DSW equation via the $S(\xi)$-expansion method.



a. b.

Figure 5. Graph of the exact solutions for the DSW equation via the $S(\xi)$-expansion method.

3 Geometrical properties through the surfaces of the DSW model

Several researchers have been showed their interested on geometrical properties through the exact solutions in various ways by authors [28, 29, 30]. All the exact solutions of the DSW model try to generalize the surface structures. In this section, we firstly review the basic definitions of the some differential geometry and Minkowski space $\mathbb{R}^3$ and then derived surface normal curvature, mean curvature and gaussian curvature of the surface structures of the DSW model.

3.1 Basic definitions and surface normal of the surface of the DSW model

This section provides some basic definitions of the differential geometry and Minkowski space $\mathbb{R}^3$ [28, 29, 30]

Let $\mathbb{R}^3$ be the vector structure of the real vector space. Let $E(x, y, z) = \{E_1, E_2, E_3\}$ be the canonical basis of $\mathbb{R}^3$ such as $E_1 = (1, 0, 0)$, $E_2 = (0, 1, 0)$ and $E_3 = (0, 0, 1)$ and (x, y, z) the coordinates of a vector with respect to E.

Definition 3.1. let X and T ba any vector fields in the metric space $E_1^3 = (\mathbb{R}^3$.So, the Lorentz-Minkowski space is the metric space $E_1^3 = (\mathbb{R}^3, \langle\rangle)$ if there exist $X = (x_1, x_2, x_3)$ and $T = (t_1, t_2, t_3)$ and the inner product $\langle\rangle$ such that

$$\langle X, T\rangle = x_1t_1 + x_2t_2 - x_3t_3. \tag{3.1}$$

Henceforth X as:

- $\|X\| = \sqrt{\langle X, X\rangle}$ if X is a spacelike vector,
- $\|X\| = -\sqrt{\langle X, X\rangle}$ if X is a timelike vector.

Definition 3.2. A vector $X \in E_1^3$ also called

- spacelike vector if $\langle X, X\rangle > 0$ or $X = 0$,
- timelike vector if $\langle X, X\rangle < 0$ and $X \neq 0$,

- lightlike vector if $\langle X, X\rangle = 0$ and $X \neq 0$.

Definition 3.3. A vector subspace $U \subset \mathbb{R}^3$ is given, so we assume that $x, t \in S$ and the induced metric $\langle , \rangle_S$:

$$\langle x, t\rangle_S = \langle x, t\rangle$$

. The inner product on P classifies of the three types such as

- The metric is positive definite and S is called spacelike.
- The metric has index 1 and S is said timelike.
- The metric is degenerate and U is called lightlike.

Definition 3.4. A smooth surface is a surface whose parametrization consists of regular surface patches.

Definition 3.5. Let S is the smooth function and let the graph of S denoted by :

$$S(x, t) = (x, t, V(x, t)),$$

where $V(x, t)$ is the exact solution of the DSW equation.

Definition 3.6. The first fundamental form of a surface in E^3 is the expression

$$Edx^2 + 2Fdxdt + Gdt^2,$$

such as $E = (V_x, V_x)$, $F = (V_x, V_t)$ and $G = (V_t, V_t)$. The first fundamental form describes the intrinsic geometry of a surface

Definition 3.7. If $S(x, t) = (x, t, V(x, t))$ is a parametrization of a surface in E^3, then the unit vector normal to the surface at any point is given by

$$N = \frac{S_x \wedge S_t}{\|S_x \wedge S_t\|}.$$

where $\wedge$ denotes the wedge product in $\mathbb{R}^3$, $S_x(x, t)$ is the partial derivatives with respect to x and $S_t(x, t)$ is the partial derivatives with respect to t. A surface in E^3 is said to be

- a spacelike surface if N is a timelike,
- a timelike surface if N is a spacelike,
- a lightlike (ot degenerate) surface if N is a lightlike.

Remark 3.8. We note that a point is called regular if $N \neq 0$ and singular if $N = 0$.

Remark 3.9. We note that the regular parameterization that is $\|S_x \wedge S_t\| \neq 0$.

Definition 3.10. The second fundamental form of a surface is the expression:

$$edx^2 + 2fdxdt + gdt^2,$$

such as $e = (V_{xx}, N)$, $f = (V_{xt}, N)$ and $g = (V_{tt}, N)$, where $V_{xx} = \frac{\partial^2 V}{\partial u^2}$, $V_{xt} = \frac{\partial^2 V}{\partial x \partial t}$ and $V_{tt} = \frac{\partial^2 V}{\partial t^2}$.

Definition 3.11. The Gaussian curvature of a surface in E^3 is the function:

$$\kappa(\rho) = \frac{eg - f^2}{EG - F^2}.$$

Definition 3.12. The mean curvature of a surface in E^3 is the function:

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2},$$

where $V(x, t)$ is the exact solutions of the DSW equation, $V_x(x, t)$ is the partial derivative with respect to x and $V_t(x, t)$ is the partial derivative with respect to t.

3.2 Surface normal through the exact solution of the DSW equation

From the definition of Surface normal, we have

$$N = \frac{S_x \times S_t}{\|S_x \times S_t\|},$$

where $S(x,t) = (x, t, V(x,t))$, $S_x(x,t)$ is the partial derivatives with respect to x and $S_t(x,t)$ is the partial derivatives with respect to t. Now from the above equation and Eq. 2.26, we get

$$N = \frac{\frac{\Phi}{(\xi+E)^2}e_1 - \frac{\Phi\beta}{(\xi+E)^2}e_1 + e_3}{\sqrt{(}1+\beta^2)\frac{\Phi}{(\xi+E)^4} + 1}. \tag{3.2}$$

Similarly, we could provide the surface normal of the other exact solution for the DSW equation, which are omitted for convenience.

3.3 Mean and Gaussian curvature through the exact solution of the DSW equation

We compute the curvatures, namely, the Mean curvature and Gaussian curvature $\kappa(\rho)$ of a non-degenerate surface by using a local parametrization. Here we follow the same ideas as in [29]. Consider a local parametrization $V = (x,t)$, where V is spacelike or timelike. Let $S = (V_x, V_t)$ be a local basis of the tangent plane at each point of $V = (x,t)$. With respect to $S = (V_x, V_t)$, let

$$M = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

be the matricial expression of the first fundamental form, where $E = \langle V_x, V_x \rangle$, $F = \langle V_x, V_t \rangle$ and $G = \langle V_t, V_t \rangle$. Denote $U = EG - F^2$. The surface is

- a spacelike surface if $U > 0$
- a timelike surface if $U < 0$

Take the unit normal vector field

$$N = \frac{V_x \wedge V_t}{\|V_x \wedge V_t\|}, \tag{3.3}$$

Again, we use the notation $\langle N, N \rangle = r$. Here $\|V_x \wedge V_t\| = \sqrt{-}r(EG - F^2) = \sqrt{-}rU$. Let

$$\begin{bmatrix} e & f \\ f & g \end{bmatrix}$$

be the matricial expression with respect to $S = (V_x, V_t)$, where $e = \langle V_{xx}, N \rangle$, $f = \langle V_{xt}, N \rangle$ and $g = \langle V_{tt}, N \rangle$.

Therefore the Gauss curvature $\kappa(\rho)$ and the mean curvature H are

$$\kappa(\rho) = \frac{eg - f^2}{EG - F^2}. \tag{3.4}$$

and

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}. \tag{3.5}$$

According to 3.3, we have

$$E = \langle V_x, V_x \rangle;$$

$$F = \langle V_x, V_t \rangle;$$

$$G = \langle V_t, V_t \rangle;$$

$$e = \langle V_{xx}, N \rangle = \frac{det(V_x, V_t, V_{xx})}{\sqrt{-}rU};$$

$$f = \langle V_{xt}, N \rangle = \frac{det(V_x, V_t, V_{xt})}{\sqrt{-r}U};$$

$$g = \langle V_{tt}, N \rangle = \frac{det(V_x, V_t, V_{tt})}{\sqrt{-r}U}.$$

Thus, the Gauss curvature $\kappa(\rho)$ and the mean curvature H can be written as

$$\kappa(\rho) = -\frac{det(V_x, V_t, V_{xx})det(V_x, V_t, V_{tt}) - det(V_x, V_t, V_{xt})^2}{U^2},$$

and

$$H = -\frac{det(V_x, V_t, V_{xx})G - 2det(V_x, V_t, V_{xt})F + det(V_x, V_t, V_{tt})E}{2(-rU)^{\frac{3}{2}}},$$

where $V(x,t)$ is the exact solutions of the DSW equation, $V_x(x,t)$ is the partial derivative with respect to x and $V_t(x,t)$ is the partial derivative with respect to t.

Examples Let V be the exact solutions of the DSW equation and consider the surface $S = V(x,t)$. Let $S(x,t) = (x, t, V(x,t))$. The coefficients of the first fundamental form are $E = 1 - V_x^2$, $F = -V_x V_t$ and $G = 1 - V_t^2$. Thus $EG - F^2 = 1 - V_x^2 - V_t^2 = 1 - |\nabla V|^2$. If the plunge of $S = V(x,t)$ is spacelike (resp. timelike), we have $|\nabla V|^2 < 1$ (resp. > 1). The mean curvature H satisfies

$$(1 - V_t^2)V_{xx} + 2V_x V_t + (1 - V_x^2)V_{xx} = -2H(-r(1 - |\nabla V|^2))^{\frac{3}{2}}.$$

Similarly, the gauss curvature $\kappa(\rho)$ is

$$\kappa(\rho) = -\frac{V_{xx}V_{tt} - V_{xt}^2}{(1 - V_x^2 - V_t^2)^2}.$$

The examples for component $V(x,t)$ are shown in the exact solutions of the DSW equation. Of course, with the help of the results presented in this paper more sophisticated solutions may be calculated.

We note that

- the surface $S(x,t)$ is an elliptic paraboloid near the point ρ, ρ is called an elliptic point if $\kappa(\rho) > 0$.
- the surface $S(x,t)$ is a hyperbolic paraboloid near the point ρ, ρ is called a hyperbolic point if $\kappa(\rho) < 0$,
- the surface $S(x,t)$ is a parabolic cylinder near the point ρ, ρ is called a parabolic point if $\kappa(\rho) < 0$.

4 Conclusion

In this study, some of the new exact solutions are secured from the DSW equation by applying $S(\xi)$-expansion method. Hence, the studied scheme is a useful and outspoken computational scheme that provides outstanding results. Furthermore, the studied process is significantly diminishing the size of the computational work. Finally, we investigated its geometrical characteristics by solving Mean curvature and Gaussian curvature for these surfaces from the DSW equation. As for these methods could be useful for future research.

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