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ON SOME INVARIANT SUBMANIFOLDS OF KENMOTSU MANIFOLDS

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Abstract We show that invariant submanifolds of Kenmotsu manifolds are totally geodesic. When the second fundamental form σ is 2-semiparallel, pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel, 2-Ricci-generalized pseudoparallel and establish their equivalence. Further examples are given.

1 Introduction

In 1972, K. Kenmotsu [5] studied a class of contact Riemannian manifolds called Kenmotsu manifolds, which is not Sasakian. In fact Kenmotsu proved that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kahlerian manifold with a warping function $f(t) = se^t$, where s is a non-zero constant. Hyperbolic space is an example of Kenmotsu manifold.

The study of the geometry of invariant submanifolds of Kenmotsu manifolds is carried out by V.S. Prasad and C.S. Bagewadi [5], Recently A.A. Shaikh, Y. Matsuyama and S.K. Hui [20] studied on invariant submanifolds of $(LCS)_n$-manifolds and S.K. Hui, S. Uddin, A.H. Alkhaldi and P. Mandal [11] have studied on Invariant submanifolds of generalized Sasakian-space-forms. S. Sular and C. Ozgur [21] and M. Kobayashi [12]. The author [12] has shown that the submanifold M of a Kenmotsu manifold $\widetilde{M}$ has parallel second fundamental form if and only if M is totally geodesic. The authors [5] have shown the equivalence of totally geodesicity of M, parallelism and semiparallelism of the second fundamental form σ. Also they have shown that invariant submanifold M of Kenmotsu manifold $\widetilde{M}$ carries Kenmotsu structure and $K \leq \widetilde{K}$, where $K, \widetilde{K}$ are sectional curvature of M and $\widetilde{M}$ respectively and equality holds if M is totally geodesic. Further the authors [21] have shown the equivalence of totally geodesicity of M, recurrency of σ, parallelism of third fundamental form on M and generalized 2-recurrency of σ. In this paper we show that invariant submanifolds of Kenmotsu manifolds are totally geodesic when the second fundamental form σ is 2-semiparallel, pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel, 2-Ricci-generalized pseudoparallel and establish their equivalence.

2 Basic Concepts

The covariant differential of the p^{th} order, $p \geq 1$ of a $(0,k)$-tensor field T, $k \geq 1$ denoted by $\nabla^p T$, defined on a Riemannian manifold (M,g) with the Levi-Civita connection ∇. The tensor T is said to be *recurrent* [22], if the following condition holds on M:

$$(\nabla T)(X_1, ..., X_k; X)T(Y_1, ..., Y_k) = (\nabla T)(Y_1, ..., Y_k; X)T(X_1, ..., X_k) \tag{2.1}$$

respectively.

$$(\nabla^2 T)(X_1, ..., X_k; X, Y)T(Y_1, ..., Y_k) = (\nabla^2 T)(Y_1, ..., Y_k; X, Y)T(X_1, ..., X_k),$$

where $X, Y, X_1, Y_1, ..., X_k, Y_k \in TM$. From (2.1) it follows that at a point $x \in M$, if the tensor T is non-zero, then there exists a unique 1-form ϕ respectively, a $(0,2)$-tensor ψ, defined on a neighborhood U of x such that

$$\nabla T = T \otimes \phi, \quad \phi = d(\log \|T\|) \tag{2.2}$$

respectively.

$$\nabla^2 T = T \otimes \psi, \tag{2.3}$$

holds on U, where $\|T\|$ denotes the norm of T and $\|T\|^2 = g(T,T)$. The tensor T is said to be *generalized* 2*-recurrent* if

$$\begin{aligned} &((\nabla^2 T)(X_1, ..., X_k; X, Y) - (\nabla T \otimes \phi)(X_1, ..., X_k; X, Y))T(Y_1, ..., Y_k) \\ &= ((\nabla^2 T)(Y_1, ..., Y_k; X, Y) - (\nabla T \otimes \phi)(Y_1, ..., Y_k; X, Y))T(X_1, ..., X_k), \end{aligned}$$

holds on M, where ϕ is a 1-form on M. From this it follows that at a point $x \in M$ if the tensor T is non-zero, then there exists a unique $(0,2)$-tensor ψ, defined on a neighborhood U of x, such that

$$\nabla^2 T = \nabla T \otimes \phi + T \otimes \psi, \tag{2.4}$$

holds on U.

Let $f : (M,g) \to (\widetilde{M}, \widetilde{g})$ be an isometric immersion from an n-dimensional Riemannian manifold (M,g) into $(n+d)$-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$, $n \geq 2$, $d \geq 1$. We denote by ∇ and $\widetilde{\nabla}$ as Levi-Civita connection of M^n and $\widetilde{M}^{n+d}$ respectively. Then the formulas of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{2.5}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for any tangent vector fields X, Y and the normal vector field N on M, where σ, A and $\nabla^{\perp}$ are the second fundamental form, the shape operator and the normal connection respectively. If the second fundamental form σ is identically zero then the manifold is said to be *totally geodesic*. The second fundamental form σ and A_N are related by

$$\widetilde{g}(\sigma(X,Y), N) = g(A_N X, Y),$$

for tangent vector fields X, Y. The first and second covariant derivatives of the second fundamental form σ are given by

$$(\widetilde{\nabla}_X \sigma)(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{2.7}$$

$$\begin{aligned} (\widetilde{\nabla}^2 \sigma)(Z,W,X,Y) &= (\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma)(Z,W), \qquad (2.8)\\ &= \nabla^{\perp}_X((\widetilde{\nabla}_Y \sigma)(Z,W)) - (\widetilde{\nabla}_Y \sigma)(\nabla_X Z, W) \\ &\quad -(\widetilde{\nabla}_X \sigma)(Z, \nabla_Y W) - (\widetilde{\nabla}_{\nabla_X Y} \sigma)(Z,W) \end{aligned}$$

respectively, where $\widetilde{\nabla}$ is called the *van der Waerden-Bortolotti connection* of M [7]. If $\widetilde{\nabla}\sigma = 0$, then M is said to have *parallel second fundamental form* [7]. We next define endomorphisms $R(X,Y)$ and $X \wedge_B Y$ of $\chi(M)$ by

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \\ (X \wedge_B Y)Z &= B(Y,Z)X - B(X,Z)Y \qquad (2.9) \end{aligned}$$

respectively. where $X, Y, Z \in \chi(M)$ and B is a symmetric $(0,2)$-tensor.

Now, for a $(0,k)$-tensor field T, $k \geq 1$ and a $(0,2)$-tensor field B on (M,g), we define the tensor $Q(B,T)$ by

$$\begin{aligned} Q(B,T)(X_1, ..., X_k; X, Y) &= -(T(X \wedge_B Y)X_1, ..., X_k) \qquad (2.10)\\ &\quad - \cdots - T(X_1, ..., X_{k-1}(X \wedge_B Y)X_k). \end{aligned}$$

Putting into the above formula $T = \sigma, \widetilde{\nabla}\sigma$ and $B = g$, $B = S$, we obtain the tensors $Q(g,\sigma)$, $Q(S,\sigma)$, $Q(g, \widetilde{\nabla}\sigma)$ and $Q(S, \widetilde{\nabla}\sigma)$.

Definition 2.1. An immersion is said to be semiparallel [8], 2-semiparallel [17], pseudoparallel [3], 2-pseudoparallel [17] and Ricci-generalized pseudoparallel [15] respectively if the following conditions hold for all vector fields X, Y tangent to M

$$\widetilde{R}\cdot\sigma = 0, \quad (2.11)$$
$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = 0, \quad (2.12)$$
$$\widetilde{R}\cdot\sigma = L_1 Q(g,\sigma), \quad (2.13)$$
$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_1 Q(g,\widetilde{\nabla}\sigma) \quad \text{and} \quad (2.14)$$
$$\widetilde{R}\cdot\sigma = L_2 Q(S,\sigma), \quad (2.15)$$

where $\widetilde{R}$ denotes the curvature tensor with respect to connection $\widetilde{\nabla}$. Now we introduce the definition of 2-Ricci-generalized pseudoparallel.

Definition 2.2. An immersion is said to be 2-Ricci-generalized pseudoparallel if

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_2 Q(S,\widetilde{\nabla}\sigma). \quad (2.16)$$

Here L_1 and L_2 are functions depending on σ and $\widetilde{\nabla}\sigma$. From the Gauss and Weingarten formulas, we obtain

$$(\widetilde{R}(X,Y)Z)^T = R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X. \quad (2.17)$$

By (2.11), we have

$$(\widetilde{R}(X,Y)\cdot\sigma)(U,V) = R^{\perp}(X,Y)\sigma(U,V) - \sigma(R(X,Y)U,V) - \sigma(U,R(X,Y)V), \quad (2.18)$$

for all vector fields X, Y, U and V tangent to M, where

$$R^{\perp}(X,Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]}. \quad (2.19)$$

Similarly, we have

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,W) = R^{\perp}(X,Y)(\widetilde{\nabla}\sigma)(U,V,W) - (\widetilde{\nabla}\sigma)(R(X,Y)U,V,W) - (\widetilde{\nabla}\sigma)(U,R(X,Y)V,W) - (\widetilde{\nabla}\sigma)(U,V,R(X,Y)W), \quad (2.20)$$

for all vector fields X, Y, U, V, W tangent to M, where $(\widetilde{\nabla}\sigma)(U,V,W) = (\widetilde{\nabla}_U\sigma)(V,W)$ [2].

3 Preliminaries

Let $\widetilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold with structure (ϕ,ξ,η,g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is the Riemannian metric satisfying

$$\phi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1, \quad \eta\circ\phi = 0, \quad \phi\xi = 0, \quad (3.1)$$
$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \quad (3.2)$$

for all vector fields X, Y on M. If

$$(\nabla_X\phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X, \quad (3.3)$$
$$\nabla_X\xi = X - \eta(X)\xi, \quad (3.4)$$

where ∇ denotes the Riemannian connection of g, then (M,ϕ,ξ,η,g) is called an almost Kenmotsu manifold [4].

Example of Kenmotsu manifold: Consider the 3-dimensional manifold $M = \{(x,y,z)\epsilon R^3; z \neq 0\}$. Let (E_1, E_2, E_3) be linearly independent vectors are given by

$$E_1 = z\frac{\partial}{\partial x}, \quad E_2 = z\frac{\partial}{\partial y}, \quad E_3 = -z\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(E_i, E_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and is given by

$$g = \frac{1}{z^2}(dx^2 + dy^2 + dz^3).$$

(ϕ, ξ, η) is given by $\xi = E_3 = -z\frac{\partial}{\partial z}$, $\eta = -\frac{1}{z}dz$ and $\phi E_1 = -E_2, \phi E_2 = E_1, \phi E_3 = 0$. The above (ϕ, ξ, η, g) satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X.$$

Hence (ϕ, ξ, η, g) is a Kenmotsu structure for $C^* \times R$.

In Kenmotsu manifolds the following relations hold [4]:

$$R(X,Y)Z = \{g(X,Z)Y - g(Y,Z)X\}, \tag{3.5}$$
$$R(X,Y)\xi = \{\eta(X)Y - \eta(Y)X\}, \tag{3.6}$$
$$R(\xi,X)Y = \{\eta(Y)X - g(X,Y)\xi\}, \tag{3.7}$$
$$R(\xi,X)\xi = \{X - \eta(X)\xi\}, \tag{3.8}$$
$$S(X,\xi) = -(n-1)\eta(X), \tag{3.9}$$
$$Q\xi = -(n-1)\xi. \tag{3.10}$$

A submanifold M of a Kenmotsu manifold $\widetilde{M}$ is called an invariant submanifold of $\widetilde{M}$, if for each $x \in M$, $\phi(T_xM) \subset T_xM$. As a consequence, ξ becomes tangent to M. In an invariant submanifold of a Kenmotsu manifold

$$\sigma(X,\xi) = 0, \tag{3.11}$$

for any vector X tangent to M.

4 2-semiparallel, pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel Invariant submanifolds of Kenmotsu manifolds

We consider invariant submanifolds of Kenmotsu manifolds satisfying the conditions $\widetilde{R}\cdot\widetilde{\nabla}\sigma = 0$, $\widetilde{R}\cdot\sigma = L_1 Q(g,\sigma)$, $\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_1 Q(g,\widetilde{\nabla}\sigma)$ $\widetilde{R}\cdot\sigma = L_2 Q(S,\sigma)$ and $\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_2 Q(S,\widetilde{\nabla}\sigma)$.

Theorem 4.1. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$. Then M is 2-semiparallel if and only if it is totally geodesic.*

Proof. Let M be 2-semiparallel $\widetilde{R}\cdot\widetilde{\nabla}\sigma = 0$. Put $X = V = \xi$ in (2.20), we get the relation

$$R^{\perp}(\xi,Y)(\widetilde{\nabla}\sigma)(U,\xi,W) - (\widetilde{\nabla}\sigma)(R(\xi,Y)U,\xi,W) - (\widetilde{\nabla}\sigma)(U,R(\xi,Y)\xi,W) \tag{4.1}$$
$$-(\widetilde{\nabla}\sigma)(U,\xi,R(\xi,Y)W) = 0.$$

In view of (2.7), (3.4), (3.7), (3.8) and (3.11), we have the following equalities:

$$\begin{aligned}&(\widetilde{\nabla}\sigma)(U,\xi,W)=(\widetilde{\nabla}_U\sigma)(\xi,W),\\&=\nabla^{\perp}_U\sigma(\xi,W)-\sigma\left(\nabla_U\xi,W\right)-\sigma\left(\xi,\nabla_U W\right),\\&=-\sigma(U,W),\end{aligned}\tag{4.2}$$

$$\begin{aligned}&(\widetilde{\nabla}\sigma)(R(\xi,Y)U,\xi,W)=(\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,W),\\&=\nabla^{\perp}_{R(\xi,Y)U}\sigma(\xi,W)-\sigma(\nabla_{R(\xi,Y)U}\xi,W)-\sigma(\xi,\nabla_{R(\xi,Y)U}W),\\&=-\eta(U)\sigma(Y,W),\end{aligned}\tag{4.3}$$

$$\begin{aligned}&(\widetilde{\nabla}\sigma)(U,R(\xi,Y)\xi,W)=(\widetilde{\nabla}_U\sigma)(R(\xi,Y)\xi,W),\\&=\nabla^{\perp}_U\sigma(R(\xi,Y)\xi,W)-\sigma(\nabla_U R(\xi,Y)\xi,W)-\sigma(R(\xi,Y)\xi,\nabla_U W),\\&=\nabla^{\perp}_U\sigma\left(\left\{Y-\eta(Y)\xi\right\},W\right)-\sigma\left(\nabla_U\left\{Y-\eta(Y)\xi\right\},W\right)\\&-\sigma(Y,\nabla_U W)\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}&(\widetilde{\nabla}\sigma)(U,\xi,R(\xi,Y)W)=(\widetilde{\nabla}_U\sigma)(\xi,R(\xi,Y)W),\\&=\nabla^{\perp}_U\sigma(\xi,R(\xi,Y)W)-\sigma(\nabla_U\xi,R(\xi,Y)W)-\sigma(\xi,\nabla_U R(\xi,Y)W),\\&=-\eta(W)\sigma(U,Y).\end{aligned}\tag{4.5}$$

Substituting (4.2) – (4.5) into (4.1), we obtain

$$\begin{aligned}&-R^{\perp}(\xi,Y)\sigma(U,W)+\eta(U)\sigma(Y,W)-\nabla^{\perp}_U\sigma\left(\left\{Y-\eta(Y)\xi\right\},W\right)\\&+\sigma\left(\nabla_U\left\{Y-\eta(Y)\xi\right\},W\right)+\sigma(Y,\nabla_U W)+\eta(W)\sigma(U,Y)=0.\end{aligned}\tag{4.6}$$

Replacing W by ξ and using (3.4), (3.11) in (4.6), we get $\sigma(U,Y)=0$. The converse statement is trivial. This proves the theorem. □

Theorem 4.2. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$. Then M is pseudoparallel if and only if it is totally geodesic.*

Proof. Let M be pseudoparallel $\widetilde{R}\cdot\sigma=L_1Q(g,\sigma)$. Setting $X=V=\xi$ in (2.10), (2.18) and adding, it becomes

$$\begin{aligned}&R^{\perp}(\xi,Y)\sigma(U,\xi)-\sigma(R(\xi,Y)U,\xi)-\sigma(U,R(\xi,Y)\xi)=-L_1\left\{g(\xi,\xi)\sigma(U,Y)\right.\\&\left.-\,g(\xi,U)\sigma(\xi,Y)+g(\xi,Y)\sigma(\xi,U)-g(Y,U)\sigma(\xi,\xi)\right\}.\end{aligned}\tag{4.7}$$

With the help of equations (3.1), (3.8) and (3.11) in (4.7), we obtain $\sigma(U,Y)=0$ and if $L_1\neq 1$. The converse statement is trivial and thus we can state the above theorem. □

Theorem 4.3. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$. Then M is 2-pseudoparallel if and only if it is totally geodesic.*

Proof. Let M be 2-pseudoparallel $\widetilde{R}\cdot\widetilde{\nabla}\sigma=L_1Q(g,\widetilde{\nabla}\sigma)$. Putting $X=V=\xi$ in (2.10), (2.20) and adding, by view of (3.1) and (3.11), takes the form

$$\begin{aligned}&R^{\perp}(\xi,Y)(\widetilde{\nabla}\sigma)(U,\xi,W)-(\widetilde{\nabla}\sigma)(R(\xi,Y)U,\xi,W)-(\widetilde{\nabla}\sigma)(U,R(\xi,Y)\xi,W)\\&-(\widetilde{\nabla}\sigma)(U,\xi,R(\xi,Y)W)=-L_1[\eta(W)\left\{\nabla^{\perp}_\xi\sigma(Y,U)-\sigma(\nabla_\xi Y,U)-\sigma(Y,\nabla_\xi U)\right\}\\&-\nabla^{\perp}_W\sigma(Y,U)+\sigma(\nabla_W Y,U)+\sigma(Y,\nabla_W U)-\eta(Y)\left\{\nabla^{\perp}_\xi\sigma(W,U)-\sigma(\nabla_\xi W,U)\right.\\&\left.-\sigma(W,\nabla_\xi U)\right\}-\eta(U)\left\{\nabla^{\perp}_\xi\sigma(Y,W)-\sigma(\nabla_\xi Y,W)-\sigma(Y,\nabla_\xi W)\right\}].\end{aligned}\tag{4.8}$$

Applying (4.2) – (4.5) into (4.8) reduces to

$$\begin{aligned}&-R^{\perp}(\xi,Y)\sigma(U,W)+\eta(U)\sigma(Y,W)-\nabla^{\perp}_{U}\sigma\left(\{Y-\eta(Y)\xi\},W\right)\\&+\sigma\left(\nabla_{U}\{Y-\eta(Y)\xi\},W\right)+\sigma(Y,\nabla_{U}W)+\eta(W)\sigma(U,Y)\\&=-L_{1}[\eta(W)\left\{\nabla^{\perp}_{\xi}\sigma(Y,U)-\sigma(\nabla_{\xi}Y,U)-\sigma(Y,\nabla_{\xi}U)\right\}-\nabla^{\perp}_{W}\sigma(Y,U)\\&+\sigma(\nabla_{W}Y,U)+\sigma(Y,\nabla_{W}U)-\eta(Y)\left\{\nabla^{\perp}_{\xi}\sigma(W,U)-\sigma(\nabla_{\xi}W,U)\right.\\&\left.-\sigma(W,\nabla_{\xi}U)\right\}-\eta(U)\left\{\nabla^{\perp}_{\xi}\sigma(Y,W)-\sigma(\nabla_{\xi}Y,W)-\sigma(Y,\nabla_{\xi}W)\right\}].\end{aligned}\tag{4.9}$$

Which, by $W=\xi$ and using (3.4), (3.11) in (4.9), we procure $\sigma(U,Y)=0$ and the converse statement is trivial. In view of above discussions we can state the above theorem. □

Theorem 4.4. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$. Then M is Ricci-generalized pseudoparallel if and only if it is totally geodesic.*

Proof. Let M be Ricci-generalized pseudoparallel $\widetilde{R}\cdot\sigma=L_2Q(S,\sigma)$. If we choose $X=\xi$ and $V=\xi$ in (2.10), (2.18) and adding, turns to

$$\begin{aligned}&R^{\perp}(\xi,Y)\sigma(U,\xi)-\sigma(R(\xi,Y)U,\xi)-\sigma(U,R(\xi,Y)\xi)=-L_{2}\left\{S(\xi,\xi)\sigma(U,Y)\right.\\&\left.-S(\xi,U)\sigma(\xi,Y)+S(\xi,Y)\sigma(\xi,U)-S(Y,U)\sigma(\xi,\xi)\right\}.\end{aligned}\tag{4.10}$$

Making use of (3.8), (3.9) and (3.11) in (4.10), we get $\sigma(U,Y)=0$ and if $L_2\neq\frac{-1}{(n-1)}$. The converse statement is trivial. □

Theorem 4.5. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$. Then M is 2-Ricci-generalized pseudoparallel if and only if it is totally geodesic.*

Proof. Let M be 2-Ricci-generalized pseudoparallel $\widetilde{R}\cdot\widetilde{\nabla}\sigma=L_2Q(S,\widetilde{\nabla}\sigma)$. Changing X and V with ξ in (2.10), (2.20) and adding, which in view of (3.9) and (3.11), it follows that

$$\begin{aligned}&R^{\perp}(\xi,Y)(\widetilde{\nabla}\sigma)(U,\xi,W)-(\widetilde{\nabla}\sigma)(R(\xi,Y)U,\xi,W)-(\widetilde{\nabla}\sigma)(U,R(\xi,Y)\xi,W)\\&-(\widetilde{\nabla}\sigma)(U,\xi,R(\xi,Y)W)=-L_{2}[-(n-1)\eta(W)\left\{\nabla^{\perp}_{\xi}\sigma(Y,U)-\sigma(\nabla_{\xi}Y,U)\right.\\&\left.-\sigma(Y,\nabla_{\xi}U)\right\}+(n-1)\left\{\nabla^{\perp}_{W}\sigma(Y,U)-\sigma(\nabla_{W}Y,U)-\sigma(Y,\nabla_{W}U)\right\}\\&+(n-1)\eta(Y)\left\{\nabla^{\perp}_{\xi}\sigma(W,U)-\sigma(\nabla_{\xi}W,U)-\sigma(W,\nabla_{\xi}U)\right\}\\&+(n-1)\eta(U)\left\{\nabla^{\perp}_{\xi}\sigma(Y,W)-\sigma(\nabla_{\xi}Y,W)-\sigma(Y,\nabla_{\xi}W)\right\}].\end{aligned}\tag{4.11}$$

Taking (4.2), (4.3), (4.4) and (4.5) into (4.11), we obtain by some calculation

$$\begin{aligned}&-R^{\perp}(\xi,Y)\sigma(U,W)+\eta(U)\sigma(Y,W)-\nabla^{\perp}_{U}\sigma\left(\{Y-\eta(Y)\xi\},W\right)\\&+\sigma\left(\nabla_{U}\{Y-\eta(Y)\xi\},W\right)+\sigma(Y,\nabla_{U}W)+\eta(W)\sigma(U,Y)\\&=-L_{2}[-(n-1)\eta(W)\left\{\nabla^{\perp}_{\xi}\sigma(Y,U)-\sigma(\nabla_{\xi}Y,U)-\sigma(Y,\nabla_{\xi}U)\right\}\\&+(n-1)\left\{\nabla^{\perp}_{W}\sigma(Y,U)-\sigma(\nabla_{W}Y,U)-\sigma(Y,\nabla_{W}U)\right\}\\&+(n-1)\eta(Y)\left\{\nabla^{\perp}_{\xi}\sigma(W,U)-\sigma(\nabla_{\xi}W,U)-\sigma(W,\nabla_{\xi}U)\right\}\\&+(n-1)\eta(U)\left\{\nabla^{\perp}_{\xi}\sigma(Y,W)-\sigma(\nabla_{\xi}Y,W)-\sigma(Y,\nabla_{\xi}W)\right\}].\end{aligned}\tag{4.12}$$

If one substitues $W=\xi$ and fetching equations (3.4), (3.11) in (4.12), we find that $\sigma(U,Y)=0$. The converse statement is trivial. Hence we state the above theorem. □

Combining all the above results and the results of [5, 12, 21], we have the following:

Corollary 4.6. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$. Then the following statements are equivalent:*

(i) σ is parallel;

(ii) σ *is recurrent;*

(iii) M *has parallel third fundamental form;*

(iv) σ *is generalized 2-recurrent;*

(v) M *is semiparallel;*

(vi) M *is 2-semiparallel;*

(vii) M *is pseudoparallel and if* $L_1 \neq 1$*;*

(viii) M *is* 2 *pseudoparallel;*

(ix) M *is Ricci-generalized pseudoparallel and if* $L_2 \neq \frac{-1}{(n-1)}$*;*

(x) M *is* 2*-Ricci-generalized pseudoparallel;*

(xi) $K = \widetilde{K}$*;*

(xii) M *is totally geodesic.*

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