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HIGHER ORDER INVOLUTES OF A CURVE IN QUATERNIONIC SPACES

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1 Introduction

The quaternions were introduced by Irish mathematician Sir William R. Hamilton who discovered that the appropriate generalization in which the real axis is left unchanged whereas the vector(imaginary) axis is supplemented by adding two further vector axis in 1843 [1]. Until the middle of the 20th century, the practical use of quaternions was minimal in comparison with other methods. But, currently this situtation has changed. Recently the theory of quaternion has developed rapidly and many mathematicians focusing on this field by different point of view. One of them is the quaternion valued function of a real variable Serret-Frenet formulae studied by Baharathi and Nagaraj [2]. Also another study on Serret-Frenet formulas of quaternionic curves in Semi-Euclidean space $\mathbb{E}_2^4$ can be found in [14] in detail. Another is Keçilioğlu and İlarslan 's study. They obtained some characterizations for (1,3) type quaternionic Bertrand curves in Euclidean 4-space by means of the curvature functions of the curve[3]. Yılmaz and Külahcı studied on the quaternionic curve in $\mathbb{Q}_4$ and obtained some characterization about rectifying curves [9].

Involutes of a given curve is another attractive research subject among geometers. The idea of a string involute is defined by C. Huygens (1658), who is also known as an optician. He discovered involutes trying to build a more accurate clock [4]. In particularly, the involute-(evolute) of a given curve is a well known concept in the classsical diferential geometry. For a general point of view in [12] Özyılmaz and Yılmaz focused on involute-evolute curve couple in $\mathbb{E}^4$. T. Soyfidan and M. A. Güngör studied a quaternionic curve Euclidean 4-space $\mathbb{E}^4$ and gave the quaternionic involute-evolute curves for quaternionic curves [15]. As and Sarıoğlugil obtained the Bishop curvatures on involute-evolute curve couple in $\mathbb{E}^3$ [5]. In [11] the authors extended involute-evolute concept to the n-dimensional simply isotropic space $\mathbb{I}_n^1$. In [13] Fukunaga and Takahashi studied involutes of fronts in Euclidean plane.

In this paper, Serret-Frenet formulas are re-given for quaternionic curve in $\mathbb{Q}_3$ and $\mathbb{Q}_4$. Firstly, the characterizations for a quaternionic involute curve are given and proved in $\mathbb{Q}_3$. Because of the similarity of Euclidean version, we omit the proofs of the theorems in $\mathbb{Q}_3$. In a similar manner, the characterizations for a quaternionic involute curve are given and proved in $\mathbb{Q}_4$.

2 Preliminaries

In this section we briefly introduce quaternion theory in Euclidean space. Detailed information can be found in [6].

A real quaternion is defined by

$$q = ae_1 + be_2 + ce_3 + de_4$$

(or $q = S_q + V_q$ where the symbols $S_q = d$ and $V_q = ae_1 + be_2 + ce_3$ denote scalar and vectoral part of q) such that

$$\begin{aligned} i)e_i \times e_i &= -e_4, (e_4 = +1, 1 \leq i \leq 3) \\ ii)e_i \times e_j &= e_k = -e_j \times e_i, (1 \leq i, j \leq 3) \end{aligned}$$

where (ijk) is an even permutation of (123) in the Euclidean space $\mathbb{R}^4$. Using these basic products we can now arrange the product of two quaternions to get

$$p \times q = S_p S_q - < V_p, V_q > + S_p V_q + S_q V_p + V_p \wedge V_q, \quad \forall p, q \in \mathbb{Q}$$

where we have used the dot and cross products in Euclidean space $\mathbb{R}^3$ [7]. The conjugate of the quaternion q is denoted by $\widehat{q}$ and defined

$$q = S_q - V_q = de_4 - ae_1 - be_2 - ce_{3.}$$

Therefore, we define the symmetric real-valued, non-degenerate, bilinear form h as follows

$$\begin{aligned} h &: \quad \mathbb{Q} \times \mathbb{Q} \to \mathbb{R} \\ (p, q) &\to \quad h(p, q) = \frac{1}{2}(p \times q + q \times p). \end{aligned}$$

So, it is named the quaternion inner product. The norm of a real quaternion q is

$$\|q\|^2 = h(q, q) = q \times q = a^2 + b^2 + c^2 + d^2.$$

If $\|q\| = 1$, then q is called a unit quaternion. It is known that the groups of unit real quaternions and unitary matrices $SU(2)$ are isomorphic. Therefore, spherical concepts in S^3 such as meridians of longtitude and parallels of latitude are explained with assistance elements of $SU(2)$. Besides, the element of $SU(3)$ can match with each element of S^3 [8].

The sphere $S^3 \subset \mathbb{Q}$ in quaternionic calculus is like the unit circle $S^1 \subset C$ in complex calculus. Indeed, $S^3 = \{q \in \mathbb{Q}, \|q\| = 1\}$ constitutes a group under quaternionic multiplication. q is called spatial quaternion whenever $q + q = 0$ [2]. Furthermore, quaternion product of two spatial is $p \times q = -\langle p, q\rangle + p \wedge q$. q is a temporal quaternion whenever $q - q = 0$. Any q can be written as $q = \frac{1}{2}(q + q) + \frac{1}{2}(q - q)$ [7].

We will deal with involutes of a curve. In what follows the orthogonal trajectories of the first tangents of the curve are called the involutes of x [10]. Let $x = x(s)$ be a regular generic curve in $\mathbb{E}^n$ given with the arclength parameter s (i.e. , $\left\|x'(s) = (1)\right\|$). Then the curves which are orthogonal to the system of k-dimensional osculating hyperplanes of x are called the involutes of order k of the curve x [11]. Simply the curve has 1 st and 2 nd order involutes for $k = 3$. If $k = 4$, then curve has 1 st, 2 nd and 3 rd order involutes. Similar to $\mathbb{E}^3$ and $\mathbb{E}^4$, we obtain involutes of order k of a q curve given in $\mathbb{Q}_3$ and $\mathbb{Q}_4$, respectively.

3 Higher Order Involutes in Quaternionic 3-Space

The 3-dimensional Euclidean space $\mathbb{E}^3$ is identified with the space of spatial quaternions which is denoted by $\mathbb{Q}$.

$$x : I \subset \mathbb{E}^3 \to \mathbb{Q}$$

$$s \to x(s) \to \sum_{i=1}^{3} x_i(s) \quad (1 \leq i \leq 3)$$

be a smooth curve defined over the interval $I = [0, 1]$. Let the parameter s chosen such that the tangent $T = x'(s)$ has unit size. Let $\{V_1(s), V_2(s), V_3(s)\}$ be Serret-Frenet frame of quaternionic curve $x = x(s)$ in $\mathbb{Q}_3$ and $\{K(s), k(s)\}$ be its Frenet curvatures given by

$$\begin{aligned} K(s) &= \frac{\left\| x'(s) \times x''(s) \right\|}{\|x'(s)\|^3} \\ k(s) &= \frac{\langle x'(s) \times x''(s), x'(s) \rangle}{\|x'(s) \times x''(s)\|^2} \end{aligned} \tag{3.1}$$

The following part is anologous to the Euclidean case because of the similarity of Frenet formulas in $\mathbb{E}^3$ and $\mathbb{Q}_3$. Hence, we only give theorems without proofs.

Theorem 3.1. *Let x be a regular unit speed real quaternionic curve in $\mathbb{Q}_3$ and any real quaternionic curve $\overline{x}(s)$ be first order involute of $x(s)$. Frenet curvatures $K_{\overline{x}}$ and $k_{\overline{x}}$ of the 1 st order involute $\overline{x}(s)$ is given as follows:*

$$K_{\overline{x}} = \frac{\sqrt{r^2(s) + k^2(s)}}{|c-s|\, k(s)}, k_{\overline{x}} = \frac{\left(\frac{r(s)}{k(s)}\right)' k(s)}{|c-s|\, r^2(s) + k^2(s)}$$

Theorem 3.2. *Let x be a regular unit speed real quaternionic curve in $\mathbb{Q}_3$ and any real quaternionic curve $\overline{x}(s)$ be the second order involute of $x(s)$. Then Frenet curvatures $K_{\overline{x}}$ and $k_{\overline{x}}$ of $\overline{x}(s)$ is given as follows:*

$$K_{\overline{x}} = \frac{sgn(r(s))}{|\lambda_2(s)|}, k_{\overline{x}} = \frac{k(s)}{\lambda_2(s)\, r(s)}$$

4 Higher Order Involutes in Quaternionic 4-Space

The 4-dimensional Euclidean space $\mathbb{E}^4$ is identified with the space of spatial quaternions which is denoted by $\mathbb{Q}$.

$$x : I \subset \mathbb{E}^4 \to \mathbb{Q}$$

$$s \to x(s) \to \sum_{i=1}^{4} x_i(s) \quad (1 \le i \le 4), \quad e_4 = 1$$

be a smooth curve defined over the interval $I = [0,1]$. Let the parameter s chosen such that the tangent $T = x'(s)$ has the unit size.

Theorem 4.1: Let $\{V_1(s), V_2(s), V_3(s), V_4(s)\}$ be the Serret-Frenet frame of a quaternionic curve $x = x(s)$ in $\mathbb{Q}$. Then the Serret-Frenet equations are

$$V_1'(s) = K(s) V_2(s)$$

$$\begin{aligned} V_2'(s) &= -K(s) V_1(s) + k(s) V_3(s) \\ V_3'(s) &= -k(s) V_2(s) + (r(s) - K(s)) V_4(s) \\ V_4'(s) &= -(r(s) - K(s)) V_3(s) \end{aligned} \tag{4.1}$$

where $K(s) = \left\| T'(s) \right\|$, [2].

The torsion, bitorsion and principal curvature of x is denoted by k, $(r-k)$ and K, respectively. In addition, the Serret-Frenet apparatus of the quaternionic curve x, are given by

$$V_1(s) = \frac{x'(s)}{\|x'(s)\|}$$

$$V_2(s) = \frac{\left\|x'(s)\right\|^2 x''(s) - h\left(x'(s), x''(s)\right)x'(s)}{\left\|\left\|x'(s)\right\|^2 x''(s) - h\left(x'(s), x''(s)\right)x'(s)\right\|}$$

$$V_3(s) = \eta V_4(s) \wedge V_1(s) \wedge V_2(s)$$

$$V_4(s) = \eta \frac{V_1(s) \wedge V_2(s) \wedge x'''(s)}{\left\|V_1(s) \wedge V_2(s) \wedge x'''(s)\right\|}, (\eta = \pm 1) \tag{4.2}$$

$$K(s) = \frac{\left\|\left\|x'(s)\right\|^2 x''(s) - h\left(x'(s), x''(s)\right)x'(s)\right\|}{\left\|x'(s)\right\|^4}$$

$$k(s) = \frac{\left\|V_1(s) \wedge V_2(s) \wedge x'''(s)\right\| \left\|x'(s)\right\|}{\left\|\left\|x'(s)\right\|^2 x''(s) - h\left(x'(s), x''(s)\right)x'(s)\right\|}$$

$$(r - K)(s) = \frac{h\left(x''''(s), V_4(s)\right)}{\left\|V_1(s) \wedge V_2(s) \wedge x'''(s)\right\| \left\|x'(s)\right\|}$$

Definition 4.1: Let $\overline{x}$, $x : I \to \mathbb{Q}_3$ be any regular real quaternionic curve with parameter s^* and s, respectively. In addition, $\{V_{1_{\overline{x}}}(s^*), V_{2_{\overline{x}}}(s^*), V_{3_{\overline{x}}}(s^*), V_{4_{\overline{x}}}(s^*)\}$ and $\{V_{1_x}(s), V_{2_x}(s), V_3(s), V_{4_x}(s)\}$ indicate the Serret-Frenet frame of the $\overline{x}$ and x, respectively. If $h\left(T_{\overline{x}(s^*)}, T_{x(s)}\right) = 0$, then, we call curves $\{\overline{x}, x\}$ as real quaternionic involute-(evolute) curves in $\mathbb{Q}_4$ [1].

Soyfidan and Güngör studied on the properties of a regular unit speed curve in $\mathbb{Q}_4$ and obtain following results for the involute of x (1 st order). We will re-call the theorems without proofs for ensuring the integrity of the higher order involute subject [1].

Theorem 4.2: Let x be a regular unit speed real quaternionic curve in $\mathbb{Q}_4$ and any real quaternionic curve $\overline{x}$ be the involute of x. Then we have $d(x(s), \overline{x}(s^*)) = |c - s|$ where c is real number [1].

Theorem 4.3: Let x be a regular unit speed real quaternionic curve in $\mathbb{Q}_4$ and any real quaternionic curve $\overline{x}$ be the involute of x. The Serret-Frenet apparatus of quaternionic curve $\overline{x}$ can be formed by apparatus of x [1].

In the light of the above concept, we define 2 nd and 3 rd order involute of x in the following.

Theorem 4.4: Let x be a regular unit speed real quaternionic curve in $\mathbb{Q}_4$ and any real quaternionic curve $\overline{x}$ be the 2 nd order involute of x. The Serret-Frenet apparatus of quaternionic curve $\overline{x}$ can be formed by the apparatus of x.

Proof: Let $x = x(s)$ be a unit speed real quaternionic curve. Without loss of generality, suppose that $\overline{x}(s^*)$ is 2 nd order involute of x.

Hence, we obtain

$$\overline{x}(s^*) = x(s) + \lambda_1(s) V_{1_x}(s) + \lambda_2(s) V_{2_x}(s) \tag{4.3}$$

By using and we differentiating (4.3) with respect to s, we obtain

$$\begin{aligned}\frac{d\overline{x}}{ds^*}\frac{ds^*}{ds} &= = \left[1+\lambda_1^{'}(s)-\lambda_2(s)K_x\right]V_{1_x}(s)\\ &\left[\lambda_1(s)K_x+\lambda_2^{'}(s)\right]V_{2_x}(s)\\ &\left[+\lambda_2(s)k\right]V_{3_x}(s)\end{aligned}$$

Moreover, we have

$$\left\langle \overline{x}^{'}(s^*),V_{1_x}(s)\right\rangle = 0,\quad \left\langle \overline{x}^{'}(s^*),V_{2_x}(s)\right\rangle = 0$$

Thus, we get

$$\frac{d\overline{x}}{ds^*}\frac{ds^*}{ds} = \lambda_2(s)kV_{3_x}(s) \tag{4.4}$$

By using equation (4.1) and differentiating (4.3) three time with respect to s, we obtain and

$$\begin{aligned}\frac{d^2\overline{x}}{ds^{*^2}}\frac{ds^{*^2}}{ds} &= -\lambda_2(s)k^2V_{2_x}(s)+\left(\lambda_2^{'}(s)k+\lambda_2(s)k^{'}\right)V_{3_x}(s)\\ &\left(+\lambda_2(s)k(r-K_x)\right)V_{4_x}(s)\end{aligned} \tag{4.5}$$

$$\begin{aligned}\frac{d^3\overline{x}}{ds^{*^3}}\frac{ds^{*^3}}{ds} &= \left[\lambda_2(s)k^2K_x\right]V_{1_x}(s)\\ &\left[-2\lambda_2^{'}(s)k^2-3\lambda_2(s)k^{'}k\right]V_{2_x}(s)\\ &\begin{bmatrix}-\lambda_2(s)k^3+\lambda_2^{''}(s)k\\ +2\lambda_2^{'}(s)k^{'}+\lambda_2(s)k^{''}\\ -\left(\lambda_2(s)k(r-K_x)^2\right)\end{bmatrix}V_{3_x}(s)\\ &\begin{bmatrix}2\lambda_2^{'}(s)k(r-K_x)\\ +2\lambda_2(s)k^{'}(r-K_x)\\ +\lambda_2(s)k(r-K_x)^{'}\end{bmatrix}V_{4_x}(s)\end{aligned} \tag{4.6}$$

$$\begin{aligned}\frac{d^4\overline{x}}{ds^{*^4}}\frac{ds^{*^4}}{ds} &= \begin{bmatrix}3\lambda_2^{'}(s)k^2K_x\\ +5\lambda_2(s)k^{'}kK_x+\lambda_2(s)k^2K_x^{'}\end{bmatrix}V_{1_x}(s)\\ &\begin{bmatrix}\lambda_2(s)k^2K_x^2-3\lambda_2^{''}(s)k^2\\ -9\lambda_2^{'}(s)k^{'}k-4\lambda_2(s)k^{''}k\\ -3\lambda_2(s)\left(k^{'}\right)^2+\lambda_2(s)k^4\\ +\lambda_2(s)k^2(r-K_x)^2\end{bmatrix}V_{2_x}(s)\\ &\begin{bmatrix}-3\lambda_2^{'}(s)k^3-6\lambda_2(s)k^2k^{'}+\lambda_2^{'''}(s)k\\ +3\lambda_2^{''}(s)k^{'}+3\lambda_2^{'}(s)k^{''}+\lambda_2(s)k^{'''}\\ -(r-K_x)^2\left[3\lambda_2^{'}(s)k+3\lambda_2(s)k^{'}\right]\\ -3\lambda_2(s)k(r-K_x)(r-K_x)^{'}\end{bmatrix}V_{3_x}(s)\\ &\begin{bmatrix}-\lambda_2(s)k^3(r-K_x)+3\lambda_2^{''}(s)k(r-K_x)\\ +6\lambda_2^{'}(s)k^{'}(r-K_x)+3\lambda_2(s)k^{''}(r-K_x)\\ +3\lambda_2^{'}(s)k(r-K_x)^{'}+3\lambda_2(s)k^{'}(r-K_x)^{'}\\ +\lambda_2(s)k(r-K_x)^{''}-\lambda_2(s)k(r-K_x)^3\end{bmatrix}V_{4_x}(s)\end{aligned} \tag{4.7}$$

Taking the quaternion norm of the equation (4.4), we have

$$\left\| \frac{d\overline{x}}{ds^*} \frac{ds^*}{ds} \right\|^2 = \lambda_2^2 (s) k^2$$

Thus, we obtain

$$\left\| \frac{d\overline{x}}{ds^*} \frac{ds^*}{ds} \right\| = \lambda_2 (s) k \tag{4.8}$$

and by using the equations (4.2), (4.4) and (4.8), we get

$$V_{1_{\overline{x}}} (s^*) = \frac{\frac{ds^*}{ds} \lambda_2 (s) k V_{3_X} (s)}{\left| \frac{ds^*}{ds} \right| \sqrt{\lambda_2^2 (s) k^2}}$$

Hence, we obtain

$$V_{1_{\overline{x}}} (s^*) = \pm V_{3_x} (s) \tag{4.9}$$

and let us calculate $\pm V_{3_x} (s)$. From the equations (4.4) and (4.5), we get

$$\begin{aligned} h\left(\frac{d\overline{x}}{ds^*}, \frac{d^2\overline{x}}{ds^{*^2}}\right) &= \frac{ds^3}{ds^{*^3}} \left[-\lambda_2^2 (s) k^3\right] \cdot \\ &\quad [\frac{1}{2} \left[V_{3_X} (s) \times V_{2_X} (s) + V_{2_X} (s) \times V_{3_X} (s)\right] \\ &\quad + \left[\lambda_2 (s) \lambda_2^{'} (s) k^2 + \lambda_2^2 (s) k^{'} k\right] \\ &\quad \frac{1}{2} \left[V_{3_X} (s) \times V_{3_X} (s) + V_{3_X} (s) \times V_{3_X} (s)\right] \\ &\quad + \lambda_2^2 (s) k^2 (r - K_x) \\ &\quad \frac{1}{2} \left[V_{3_X} (s) \times V_{4_X} (s) + V_{4_X} (s) \times V_{3_X} (s)\right]] \\ &= \frac{ds^3}{ds^{*^3}} \left[\lambda_2 (s) k \left(\lambda_2^{'} (s) k + \lambda_2 (s) k^{'}\right)\right] \end{aligned}$$

Therefore, by using the equations (4.2), (4.4), (4.5), (4.8) and the last equation, we attain

$$V_{2_{\overline{x}}} (s^*) = \frac{\left(-k V_{2_x} (s) + (r - K_x) V_{4_x} (s)\ \right)}{\sqrt{k^2 + (r - K_x)^2}} \tag{4.10}$$

By using equations (4.6), (4.9) and (4.10), we obtain

$$V_{1_{\overline{x}}} (s^*) \wedge V_{2_{\overline{x}}} (s^*) \wedge \overline{x}^{'''} (s^*) = \frac{k}{\sqrt{k^2 + (r - K_x)^2}} \tag{4.11}$$

$$\begin{bmatrix} \begin{bmatrix} 4\lambda_2^{'} (s) k (r - K_x) \\ +5\lambda_2 (s) k^{'} (r - K_x) \\ +\lambda_2 (s) k (r - K_x)^{'} \end{bmatrix} V_{1_x} (s) \\ \left[-\lambda_2 (s) k K_x (r - K_x)\right] V_{2_x} (s) \\ \left[-\lambda_2 (s) k^2 K_x\right] V_{4_x} (s) \end{bmatrix}$$

Hence, by taking the quaternion norm of equation (4.11), we get

$$\left\| V_{1_{\overline{x}}} (s^*) \wedge V_{2_{\overline{x}}} (s^*) \wedge \overline{x}^{'''} (s^*) \right\| = \frac{k}{\sqrt{k^2 + (r - K_x)^2}} \tag{4.12}$$

$$\begin{gathered} \left(16\lambda_2^{'^2} (s) k^2 (r - K_x)^2 \right. \\ +25\lambda_2^2 (s) k^{'^2} (r - K_x)^2 \\ +\lambda_2^2 (s) k^2 \left((r - K_x)^{'}\right)^2 \\ +\lambda_2^2 (s) k^2 K_x^2 (r - K_x)^2 \\ \left. +\lambda_2^2 (s) k^4 K_x^2\ \right)^{\frac{1}{2}} \end{gathered}$$

Moreover, using the equations (4.2), (4.11) and (4.12), we have

$$V_{4_{\overline{x}}}(s^*) = \eta \frac{\begin{pmatrix} \begin{bmatrix} 4\lambda_2'(s)\,k\,(r-K_x) \\ +5\lambda_2(s)\,k'\,(r-K_x) \\ +\lambda_2(s)\,k\,(r-K_x)' \end{bmatrix} V_{1_x}(s) \\ \left[-\lambda_2(s)\,kK_x\,(r-K_x)\right] V_{2_x}(s) \\ \left[-\lambda_2(s)\,k^2K_x\right] V_{4_x}(s) \end{pmatrix}}{\begin{array}{c} \Big(16\lambda_2^{'2}(s)\,k^2\,(r-K_x)^2 \\ +25\lambda_2^2(s)\,k^{'2}\,(r-K_x)^2+\lambda_2^2(s)\,k^2 \\ \left((r-K_x)'\right)^2+\lambda_2^2(s)\,k^2K_x^2 \\ (r-K_x)^2+\lambda_2^2(s)\,k^4K_x^2\;\Big)^{\frac{1}{2}} \end{array}} \tag{4.13}$$

where $\eta = \pm 1$ providing that $\det\left(V_{1_{\overline{x}}}(s^*), V_{2_{\overline{x}}}(s^*), V_{3_{\overline{x}}}(s^*), V_{4_{\overline{x}}}(s^*)\right) = \pm 1$. Similarly, using the equations (4.2), (4.9), (4.10), (4.13) and essential arrangements, the binormal vector $V_{3_{\overline{x}}}(s^*)$ is obtained as follows that

$$V_{3_{\overline{x}}}(s^*) = \eta \frac{\begin{pmatrix} \left(-\lambda_2(s)\,kK_x\,(r-K_x)^2+\lambda_2(s)\,k^3K_x\right) V_{1_x}(s) \\ \left(-4\lambda_2'(s)\,k\,(r-K_x)^2-5\lambda_2(s)\,k'\,(r-K_x)^2\right) V_{2_x}(s) \end{pmatrix}}{\begin{pmatrix} \Big(16\lambda_2^{'2}(s)\,k^2\,(r-K_x)^2+25\lambda_2^2(s)\,k^{'2} \\ (r-K_x)^2+\lambda_2^2(s)\,k^2K_x^2\left((r-K_x)'\right)^2 \\ +\lambda_2^2(s)\,k^2K_x^2\,(r-K_x)^2+\lambda_2^2(s)\,k^4K_x^2\;\Big) \\ \left(k^2+(r-K_x)^2\right) \end{pmatrix}^{\frac{1}{2}}} \tag{4.14}$$

Conclusion 4.1 Let x be a regular unit speed real quaternionic curve in $\mathbb{Q}_4$ and any real quaternionic curve $\overline{x}$ be the 2 nd order involute of x. The Frenet curvatures of the curve $\overline{x}$ can be given as follows by the virtue of (4.2).

$$K_{\overline{x}}(s^*) = \frac{\sqrt{(r-K_x)^2+k^2}}{\lambda_2(s)\,k} \tag{4.15}$$

$$k^*(s^*) = \frac{\begin{array}{c} \Big(16\lambda_2^{'2}(s)\,k^2\,(r-K_x)^2+25\lambda_2^2(s)\,k^{'2} \\ (r-K_x)^2+\lambda_2^2(s)\,k^2K_x^2\left((r-K_x)'\right)^2 \\ +\lambda_2^2(s)\,k^2K_x^2\,(r-K_x)^2+\lambda_2^2(s)\,k^4K_x^2\;\Big)^{\frac{1}{2}} \end{array}}{\lambda_2^2(s)\,k\,(r-K_x)^2+k^2(s)}$$

$$r^*(s^*) - K_{\overline{x}}(s^*) = \frac{\sqrt{k^2+(r-K_x)^2}}{\begin{array}{c} k^2\lambda_2(s)\Big(16\lambda_2^{'2}(s)\,k^2\,(r-K_x)^2+25\lambda_2^2(s) \\ k^{'2}\,(r-K_x)^2+\lambda_2^2(s)\,k^2\left((r-K_x)'\right)^2 \\ +\lambda_2^2(s)\,k^2K_x^2\,(r-K_x)^2 \\ +\lambda_2^2(s)\,k^4K_x^2\;\Big) \end{array}} \tag{4.16}$$

$$\Bigg(\begin{array}{c} -\lambda_2^2(s)\,k^3K_x^3\,(r-K_x)+3\lambda_2''(s)\,\lambda_2(s)\,k^3K_x \\ (r-K_x) \end{array}$$

$$+9\lambda_2'(s)\,\lambda_2(s)\,k^2k'\,K_x\,(r-K_x)$$

$$+4\lambda_2^2(s)\,k''\,k^2K_x\,(r-K_x)+3\lambda_2^2(s)\,kk^{'2}\,K_x\,(r-K_x)$$

$$-\lambda_2^{2}(s)\,k^{5}K_x(r-K_x)\,\lambda_2^{2}(s)\,k^{3}K_x(r-K_x)^3$$
$$-\lambda_2^{2}(s)\,k^3K_x(r-K_x)-3\lambda_2^{''}(s)\,\lambda_2(s)\,k^3K_x(r-K_x)$$
$$-6\lambda_2^{'}(s)\,\lambda_2(s)\,k^2k^{'}K_x(r-K_x)-3\lambda_2^{2}(s)\,k^2k^{''}K_x(r-K_x)$$
$$-3\lambda_2^{'}(s)\,\lambda_2(s)\,k^3K_x(r-K_x)^{'}-3\lambda_2^2(s)\,k^2k^{'}K_x(r-K_x)^{'}$$
$$-\lambda_2^{2}(s)\,k^3K_x(r-K_x)^{''}+\lambda_2^2(s)\,k^3K_x(r-K_x)^3$$
$$+12\lambda_2^{'2}(s)\,k^3K_x(r-K_x)+35\lambda_2^{'}(s)\,\lambda_2(s)\,k^2k^{'}K_x(r-K_x)$$
$$+3\lambda_2^{'}(s)\,\lambda_2(s)\,k^3K_x(r-K_x)^{'}+5\lambda_2^{2}(s)\,k^2k^{'}K_x$$
$$(r-K_x)'+25\lambda_2^2(s)\,k^{'2}K_x(r-K_x)+4\lambda_2^{'}(s)\,\lambda_2(s)$$
$$k^3K_x^{'}(r-K_x)+5\lambda_2^2(s)\,k^2k^{'}K_x^{'}(r-K_x)$$
$$+\lambda_2(s)\,k^3K_x^{'}(r-K_x)^{'}\ )$$

This completes the proof.

Theorem 4.5: Let x be a regular unit speed real quaternionic curve in $\mathbb{Q}_4$ and any real quaternionic curve $\overline{x}$ be 3 rd order involute of x. The Serret-Frenet apparatus of a quaternionic curve $\overline{x}$ can be formed by apparatus of x.

Proof: Let $x = x(s)$ be a unit speed real quaternionic curve. Without loss of generality, suppose that $\overline{x}(s^*)$ is the 3 rd order involute of x.

Hence, we obtain

$$\overline{x}(s^*) = x(s) + \lambda_1(s)\,V_{1_x}(s) + \lambda_2(s)\,V_{2_x}(s) + \lambda_3(s)\,V_{3_x}(s) \tag{4.17}$$

By using equation (4.1) and we differentiating (4.17) with respect to s, we obtain

$$\begin{aligned}\frac{d\overline{x}}{ds^*}\frac{ds^*}{ds} &= \left[1+\lambda_1^{'}(s)-\lambda_2(s)\,K_x\right]V_{1_x}(s)\\ &\left[\lambda_1(s)\,K_x+\lambda_2^{'}(s)-\lambda_3(s)\,k\right]\\ &\left[\lambda_2(s)\,k+\lambda_3^{'}(s)\right]V_{3_x}(s)+\left[\lambda_3(s)\,(r-K_x)\right]V_{4_x}(s)\end{aligned}$$

Moreover, we have

$$\left\langle \overline{x}^{'}(s^*),V_{1_x}(s)\right\rangle = 0,\quad \left\langle \overline{x}^{'}(s^*),V_{2_x}(s)\right\rangle = 0\quad ,\left\langle \overline{x}^{'}(s^*),V_{3_x}(s)\right\rangle = 0$$

Thus, we get

$$\frac{d\overline{x}}{ds^*}\frac{ds^*}{ds} = \lambda_3(s)\,(r-K_x)\,V_{4_x}(s) \tag{4.18}$$

By using equation (4.1) and we differentiating (4.17) three time with respect to s, we obtain

$$\frac{d^2\overline{x}}{ds^{*2}}\frac{ds^{*2}}{ds} = -\lambda_3(s)\,(r-K_x)^2\,V_{3_x}(s) \tag{4.19}$$
$$\left(\lambda_3^{'}(s)\,(r-K_x)+\lambda_3(s)\,(r-K_x)^{'}\right)V_{4_x}(s)$$

$$\frac{d^3\overline{x}}{ds^{*3}}\frac{ds^{*3}}{ds} = \left[\lambda_3(s)\,(r-K_x)^2\,k\right]V_{2_x}(s) \tag{4.20}$$

$$\left[-2\lambda_3'(s)(r-K_x)^2-3\lambda_3(s)(r-K_x)(r-K_x)'\right]V_{3_x}(s)$$

$$\begin{bmatrix}\lambda_3''(s)(r-K_x)+2\lambda_3'(s)(r-K_x)' \\ +\lambda_3(s)(r-K_x)''-\lambda_3(s)(r-K_x)^3\end{bmatrix}V_{4_x}(s)$$

$$\frac{d^4\overline{x}}{ds^{*4}}\frac{ds^{*4}}{ds}=\left[-\lambda_3(s)K_x(r-K_x)^2k\right]V_{1_x}(s) \tag{4.21}$$

$$\begin{bmatrix}3\lambda_3'(s)k(r-K_x)^2 \\ +5\lambda_3(s)k(r-K_x) \\ +(r-K_x)'+\lambda_3(s)k'(r-K_x)^2\end{bmatrix}V_{2_x}(s)$$

$$\begin{bmatrix}\lambda_3(s)k^2(r-K_x)^2-3\lambda_3''(s)(r-K_x)^2 \\ -9\lambda_3'(s)(r-K_x) \\ +(r-K_x)'-3\lambda_3(s)(r-K_x) \\ -4\lambda_3(s)(r-K_x)(r-K_x)'' \\ +\lambda_3(s)(r-K_x)^4\end{bmatrix}V_{3_x}(s)$$

$$\begin{bmatrix}-3\lambda_3'(s)(r-K_x)^3-3\lambda_3(s)(r-K_x)^2(r-K_x)' \\ +\lambda_3'''(s)(r-K_x)+3\lambda_3''(s)(r-K_x)' \\ +3\lambda_3'(s)(r-K_x)''-3\lambda_3(s)(r-K_x)^2(r-K_x)' \\ +\lambda_3(s)(r-K_x)'''\end{bmatrix}V_{4_x}(s)$$

Taking the quaternion norm of the equation (4.18), we have

$$\left\|\frac{d\overline{x}}{ds^*}\frac{ds^*}{ds}\right\|=|\lambda_3(s)|(r-K_x) \tag{4.22}$$

and by using equations (4.2), (4.18) and (4.22), we get

$$V_{1_{\overline{x}}}(s^*)=\frac{\frac{ds^*}{ds}\lambda_3(s)(r-K_x)}{\left|\frac{ds^*}{ds}\right||\lambda_3(s)|(r-K_x)}V_{4_x}(s)$$

Hence, we obtain

$$V_{1_{\overline{x}}}(s^*)=\pm V_{4_x}(s) \tag{4.23}$$

and let us calculate $\pm V_{4_x}(s)$. From the equations (4.18) and (4.19), we get

$$\begin{aligned} h\left(\frac{d\overline{x}}{ds^*},\frac{d^2\overline{x}}{ds^{*2}}\right) &= \frac{ds^3}{ds^{*3}}\begin{bmatrix}\lambda_3(s)(r-K_x) \\ \left(\lambda_3'(s)(r-K_x)+\lambda_3(s)(r-K_x)'\right) \\ \frac{1}{2}\begin{bmatrix}V_{4_x}(s)\times V_{4_x}(s) \\ +V_{4_x}(s)\times V_{4_x}(s)\end{bmatrix}\end{bmatrix} \\ &= \frac{ds^3}{ds^{*3}}\begin{bmatrix}\lambda_3(s)\lambda_3'(s)(r-K_x)^2 \\ +\lambda_3^2(s)(r-K_x)(r-K_x)'\end{bmatrix}\end{aligned}$$

Therefore, by using the equations (4.2), (4.18), (4.19), (4.22) and the last equation, we attain

$$\begin{aligned} V_{2_{\overline{x}}}(s^*) &= \frac{-\lambda_3^3(s)(r-K_x)^4V_{3_x}(s)}{\lambda_3^3(s)(r-K_x)^4} \\ &= -V_{3_x}(s) \end{aligned} \tag{4.24}$$

By using the equations (4.20), (4.23) and (4.24), we obtain

$$V_{1_{\overline{x}}}(s^*)\wedge V_{2_{\overline{x}}}(s^*)\wedge\overline{x}'''(s^*)=\left[\lambda_3(s)k(r-K_x)^2\right]V_{1_x}(s) \tag{4.25}$$

Hence, by taking the quaternion norm of the equation (4.25), we get

$$\left\| V_{1_{\overline{x}}}(s^*) \wedge V_{2_{\overline{x}}}(s^*) \wedge \overline{x}'''(s^*) \right\| = \lambda_3(s)\, k\,(r - K_x)^2 \tag{4.26}$$

Moreover, using the equations (4.2), (4.25) and (4.26), we have

$$V_{4_{\overline{x}}}(s^*) = \eta V_{1_x}(s)$$

where $\eta = \pm 1$ providing that $\det\left(V_{1_{\overline{x}}}(s^*), V_{2_{\overline{x}}}(s^*), V_{3_{\overline{x}}}(s^*), V_{4_{\overline{x}}}(s^*)\right) = \pm 1$. Similarly, using the equations (4.2), (4.23), (4.24), (4.27) and essential arrangements, the binormal vector $V_{3_{\overline{x}}}(s^*)$ is obtained as follows that

$$V_{3_{\overline{x}}}(s^*) = \eta V_{2_x}(s)$$

Conclusion 4.2 Let x be a regular unit speed real quaternionic curve in $\mathbb{Q}_4$ and any real quaternionic curve $\overline{x}$ be the 3 rd order involute of x. The Frenet curvatures of the curve $\overline{x}$ can be given as follows by the virtue of (4.2).

$$K_{\overline{x}}(s^*) = \frac{1}{\lambda_3(s)} \tag{4.27}$$

$$k^*(s^*) = \frac{k}{\lambda_3(s)(r - K_x)} \tag{4.28}$$

$$r^*(s^*) - K_{\overline{x}}(s^*) = \frac{K_x}{\lambda_3(s)(r - K_x)} \tag{4.29}$$

This completes the proof.

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