Palestine Journal of Mathematics

Vol. 11(1)(2022) , 427–438

r-Almost Newton-Yamabe Solitons on submanifolds of Kenmotsu Space Forms

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Communicated by Zafar Ahsan

MSC 2020 Classifications: Primary 53C44, 53B21, Secondary 53C20, 53C25.

Keywords and phrases: r-Almost Newton Yamabe Solitons, submanifold, Kenmotsu manifold, Locally symmetric spaces, Einstein manifold.

Abstract: In this geometric analysis, we examine the characteristics of Legendrian submanifolds in Kenmotsu space forms conceding the r-almost Newton-Yamabe soliton with a potential function $\psi : M^n \longrightarrow \mathbb{R}$. Also, we discuss the r-almost Newton-Yamabe solitons immersed into a Kenmotsu space form. Furthermore, we establish the conditions for Legendrian submanifolds of Kenmostu space form to be minimal and totally geodesic under Newton transformation . Finally, we exhibit that an 1-almost Newton-Yamabe soliton on Legendrian submanifolds of Kenmotsu space form immersed in a locally symmetric Einstein manifold.

1 Introduction

Geometric flows are most prolific tools to explain the geometric structures in Riemannian geometry. A special class of soultions on which the metric evolves by dilations and diffeomorphisms plays a vital part in the study of singularities of the flows as they apper as possible singularity models. They are often called soliton solutions.

The knowledge of Yamabe flow, which was popularized by Hamilton in his prime research work [13], as a tool for constructing metrics of constant scalar curvature on a Riemannian manifold (M^n, g), $n \geq 3$.The Yamabe flow is an evolution equation for metrics on a Riemannian manifolds as given by

$$\frac{\partial}{\partial t} g(t) = -\rho g, \quad g(0) = g_0, \tag{1.1}$$

where ρ is the scalar curvature corresponding to Riemannian metric g and t is time, which is used to deform a metric by smoothing out its singularities [21].

A Yamabe soliton is a spacial solution of the Yamabe flow that moves by one parameter family of diffeomorphism generated by fixed vector field X on M^n with a constant λ satisfying the following equation

$$\frac{1}{2}\mathcal{L}_X g = (\rho - \lambda)g, \tag{1.2}$$

where $\mathcal{L}_X g$ is the Lie derivative of the metric g. In 2011, Pigola and his coauthors [16] adding the condition for parameter λ in (1.2) to be a real smooth function on M^n. In this more general framework we refer to equation (1.2) as being the fundamental equation of an *almost Yamabe solion*. If $\lambda > 0$, $\lambda < 0$ or $\lambda = 0$, then the (M^n, g) is called *Yamabe shrinker*, *Yamabe expander* or *Yamabe steady soliton* respectively.

In the particular situation when the vector field X is the gradient of a smooth function $\psi : M^n \longrightarrow \mathbb{R}$, the manifold will be called a *gradient almost Yamabe soliton*. The function ψ is called the *potential function* of the gradient almost Yamabe soliton. In this case equation (1.2) becomes

$$Hess\psi = (\rho - \lambda)g, \tag{1.3}$$

where $Hess\psi$ stands for the Hessian of the potential function ψ. The almost gradient Yamabe soliton equation (1.3) links geometric information about the curvature of the manifold through the scalar curvature tensor and the geometry of the level sets of the potential function by means of their second fundamental form. Hence, study almost gradient Yamabe solitons under some curvature conditions is an interesting topic.

On the contrary, Barros and his coauthors [2] discussed the isometric immersions of an almost Ricci soliton (M^n, g, X, λ) in to Riemannian manifold M^{n+p}. In particular , when M^{n+p} has non-positive sectional curvature, they proved that an almost Ricci soliton is a Ricci soliton and the vector field has integrable norm on M^n, then M^n can not be minimal. Moreover, in [24] Wylie proved that if (M^n, g, X, λ) is a shrinking Ricci soliton, with X having bounded norm on M^n, then M^n must be compact. In particular when M^{n+p} is a space form of non-positive sectional curvature, then such immersions can not be minimal. In 2018, Cunah et al, [8] introduced the study of the immersed almost Ricci soliton under *Newton transformation* P_r with second order differential operators L_r and introduced the new notion r-almost Newton-Ricci soliton, for some $0 \leq r \leq n$. If there exist a smooth function $\psi : M^n \longrightarrow \mathbb{R}$ such that the following equation holds

$$Ric + P_r \circ Hess\psi = \lambda g, \tag{1.4}$$

where λ is a smooth function on M^n and $P_r \circ Hess\psi$ stands for the tensor given by

$$P_r \circ Hess\psi(X, Y) = g(P_r \nabla_X \nabla\psi, Y), \tag{1.5}$$

for tangent vector fields $X, Y \in \chi(M)$. In particular, when $r = 0$ we recover the definition of gradient almost Ricci soliton.

On the other side Kenmotsu manifold is an important class of manifolds endowed with geometrical structure Kenmotsu [15] introduced and studied these manifolds. Afterword, many geometers studied the geometry of submanifolds in Kenmostsu manifolds due to its rich geometric importance.

Therefore the present research article inspired by the above literature's, in this frame work we have explore the study of the new notion r-almost Newton-Yamabe soliton on submanifolds of Kenmotsu space form.

2 Preliminaries

A $(2m+1)$-dimensional differentiable manifold M^{2m+1} is called a contact manifold [1] if there exits a globally defined 1-form η such that $\eta \wedge (d\eta)^m \neq 0$. On a contact manifold there exists a unique vector filed ξ satisfying

$$d\eta(\xi, X) = 0, \quad \eta(\xi) = 0 \tag{2.1}$$

for all $X \in T(M^{2m+1})$.

Let M^{2m+1} be a $(2m+1)$-dimensional Riemannian manifold. M is called an almost contact manifold if it is equipped with an almost contact structure (φ, ξ, η), where φ is a $(1,1)$-tensor field, ξ a unit vector field, η a one-form dual to ξ satisfying [1]

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \tag{2.2}$$

$$\varphi(\xi) = 0, \quad \eta(\xi) = 1, \quad g(X, \xi) = \eta(X). \tag{2.3}$$

It is well-known that there exists a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.4}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \tag{2.5}$$

where $X, Y \in \chi(M)$. Moreover, if the almost contact structure (φ, η) is normal, i.e.

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \tag{2.6}$$

$$\nabla_X \xi = X - \eta(X)\xi \tag{2.7}$$

for any vectors X, Y on M, where ∇ denotes the Levi-Civita connection with respect to g, then M is said to be a Kenmostu manifold it is satisfies $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$ on M^{2m+1}, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. For more details and background, (see [1], [25]).

Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a Kenmotsu manifold is a contact manifold with the contact structure (η, g, ξ, φ) and L^n an m-dimensional manifold.

Definition 2.1. An immersion $\varphi : L^m \longrightarrow M^{2m+1}$ is called Legendrian immersion of an m-dimensional compact smooth manifold L into a $(2m+1)$-dimensional contact manifold (M, η) such that [23]

(i) Legendrian $\Longleftrightarrow \varphi * \eta = 0$,

(ii) $dim(L) = m$.

(iii) An legendrain immersion is called L minimal $\Longleftrightarrow div\varphi H = 0$, where H is the mean curvature vector of φ.

We may choose an almost contact metric structure (ξ, g) on M compatible with the contact structure η. A Legendrian deformation of φ is defined as a one-parameter smooth family $\{\varphi_t\}$ of Legendrian immersions $\varphi : L \longrightarrow M$ with $\varphi_0 = \varphi$.

Example 2.2. The odd dimensional Euclidean space admits the standard Kenmotsu structure , we denote by $\mathbb{R}^{2m+1}(3)$.In general, an immersion into $\mathbb{R}^{2n+1}(3)$ which lies in some cylinders and minimal in the cylinder.

3 Legendrian submanifolds of Kenmotsu space form

A plane of T_pM at p is called φ-section if it is spanned by X and φX, where X is orthonormal to ξ. The curvature of φ-section is called φ-sectional curvature.

A $(2m+1)$-Kenmotsu space form is defined as a $(2m+1)$-Kenmotsu manifold with constant φ-sectional curvature c and is denoted by $M^{2n+1}(c)$. As example of Kenmotsu space form, $\mathbb{R}^{2n+1}$ and $\mathbb{S}^{2n+1}$ are equipped with Kenmotsu space form structures(more details in [1] and [25]). The curvature of a Kenmotsu space form [15] $M^{2n+1}(c)$ is given by [25]

$$\bar{R}(X, Y, Z, W) = \frac{c-3}{4}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \tag{3.1}$$

$$+\frac{c+1}{4}[g(X, \varphi W)g(Y, \varphi Z) - g(X, \varphi Z)g(Y, \varphi W)$$

$$-2g(X, \varphi Y)g(Z, \varphi W) - g(X, W)\eta(Y)\eta(Z) + g(X, Z)\eta)Y)\eta(W)$$

$$-g(Y, Z)\eta(X)\eta(W) + g(Y, W)\eta(X)\eta(Z)]$$

for any $X, Y, Z \in T(M)$.

Let L^m be an m-dimensional submanifold of a Kenmotsu space form $M^{2m+1}(c)$. If the one-form η constrained in M is zero, then we say M is a Legendrian submanifold. It is well-known that for such a submanifold φ maps any tangent vector to M at any $p \in M$ into the normal vector space $T_p^{\perp}(M)$ i.e. $\varphi(T_pM) \subset T_p^{\perp}M$. Actually, a Legendrian submanifold is a special C-*totally real submanifold* (i.e. the unit vector field ξ is orthonormal to M) [12] . Therefore we obtain from (2.6) and (2.7) that for any $X, Y(M)$,

$$g(\varphi X, \varphi Y) = g(X, Y), \quad \eta(X) = g(X, \xi) = 0. \tag{3.2}$$

4 r-almost Newton-Yamabe soliton

Let $\varphi : L^m \longrightarrow M^{m+p}$ be an Legendrian immersion into an $(m+1)$-dimensional Kenmotsu manifold M^{n+1}. We call L^m is an r-almost Newton-Yamabe soliton, for some $0 \leq r \leq m$, if there exist a smooth function $\psi : L^m \longrightarrow \mathbb{R}$ such that [6]

$$P_r \circ Hess\psi = (\rho - \lambda)g, \tag{4.1}$$

where λ is a smooth function on L^m and $P_r \circ Hess\psi$ stands for tensor given by

$$P_r \circ Hess\psi(X, Y) = g(P_r \nabla_X \nabla_\psi, Y), \tag{4.2}$$

for tangent vectors fields $X, Y \in \chi(M)$. For $r = 0$, equation (4.1) reduces to the definition of a gradient almost Yamabe soliton.

Example 4.1. For the case of minimal Legendrian submanifolds in $\mathbb{S}^{2m+1}(1)$. Let us consider the standard immersion of $\mathbb{L}^m$ in $\mathbb{S}^{2m+1}(1)$, which we know that its is totally geodesic. In particular, $P_r = 0$ for all $1 \leq r \leq m$, and choosing $\lambda = \frac{(n-1)}{n}$, we obtain that the immersion satisfies equation (4.1).

Example 4.2. Let $\mathbb{S}^{2m+1}(1)$ be the unit sphere in the Euclidean space $\mathbb{R}^{m+1}$ and $\psi : \mathbb{S}^{2m+1}(1) \hookrightarrow \mathbb{R}^{m+1}$ the natural embedding with induced metric g on $\mathbb{S}^{2m+1}(1)$, then $(\mathbb{S}^{2m+1}(1), \varphi, \xi, \eta, g)$ is a contact metric manifold.It is well known that this contact metric structure gives a Sasakian structure on $\mathbb{S}^{2m+1}(1)$ and its a Kenmotsu space form with constant φ-sectional curvature $c = -1$.

Let $i : L^m \longrightarrow \mathbb{S}^{2m+1}(1) \subset \mathbb{R}^{m+1} \cong \mathbb{R}^{2n+2}$ an immersion of a smooth m-dimensional manifold L^m in to unit sphere.

For a constant $t \in \mathbb{R}^{2m+2}$, according to [3], by choosing the functions $\bar{f}_t$ on $\mathbb{R}^{2m+2}$ such that

$$\bar{f}_l(t) = -g(t, l) + 2m - 1 \quad and \quad \psi_l(t) = -\bar{f}_l + c, \quad \bar{f}_l := i * \tilde{f}_l \in C^\infty(S^{2m+1})$$

where $l \in \mathbb{S}^{2m+1}(1)$, $t \neq 0$, $c \in \mathbb{R}^{2m+2}$ and $t = (t_1, ...t_{2m+1}) \in \mathbb{S}^{2m+1}$ is the position vector, we have that $(\mathbb{S}^{2m+1}, g, \nabla\psi_l, \lambda_l)$ satisfies equation (1.3) On the other hand, it is well know that $\mathbb{S}^{m+1}$ is totally umbilical with r-th mean curvature $H_r = 1$ and second fundamental form $B = I$. In particular, for every $0 \leq m$ the Newton tensor are given by

$$P_r = \alpha I, \tag{4.3}$$

where $\alpha = \sum_{j=0}^{r} (-1)^{r-j} \binom{m}{j}$. Hence, taking smooth function $\psi = \alpha^{-1}\psi_l$ we get that subamnifold satisfied equation (4.4).

Example 4.3. We recall the *Gaussian soliton* is the Euclidean space $\mathbb{R}^m$ endowed with its standard metric $|.|$ admits the standard Sasakian structure and the potential function $\psi(x) = \frac{\lambda}{4} |x|^2$. It is well know that the horospheres of the hyperbolic space $\mathbb{H}^{m+1}$ are totally umbilical hypersurface isometric to $\mathbb{R}^m$, having r-th mean curvature $H_r = 1$ and second fundamental form $B = I$. Hence, we can reason as in example (4.2) to verify that the horospheres $\mathbb{R}^m \hookrightarrow \mathbb{H}^{m+1}$ satisfies equation (4.4).

The Gauss equation implies that

$$R(X, Y, Z, W) = (\bar{R}(X, Y)Z)^T) + g(BX, Z)BY - g(AY, Z)BX \tag{4.4}$$

for every tangent vector fields $X, Y, Z \in \chi(L)$, where $()^T$ denotes the tangential components of a vector field in $\chi(L)$ along L^m. Here B stands for second fundamental form (or shape operator) of L^m in M^{m+1} with respect to a fixed orientation related to the second fundamental form h by

$$g(h(X, Y), \alpha) = g(B_\alpha X, Y), \tag{4.5}$$

where α is a normal vector field on L^m.

$\bar{R}$ and R denotes the curvature tensors of $M^{m+1}(c)$ and L^m, respectively.
In particular, the scalar curvature ρ of the submanifold L^m satisfies

$$2\rho = \sum_{i,j}^{n} g(\bar{R}(E_i, E_j)E_j, E_i) + n^2 \|H\|^2 - \|B\|^2, \tag{4.6}$$

where $\{E_1, ...E_m\}$ is an orthonormal frame on TM and $|.|$ denotes the Hilbert-Schmidt norm. When $M^{m+1}(c)$ is a Kenmotsu space form of constant sectional curvature c, we have the following equation

$$\rho = \frac{m(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4} + \frac{n^2 \|H\|^2}{2} - \frac{\|B\|^2}{2}. \tag{4.7}$$

Associated to second fundamental form B of the submanifold L^m there are m algebraic invariants, which are the elementary symmetric functions ρ_r of its principal curvatures $k_1, ...k_m$, given by

$$\rho_0 = 1, \quad \rho_r = \sum_{i_1 < ... < i_r} k_1, ...k_m. \tag{4.8}$$

The r-th mean curvature H_r of the immersion is define by $\binom{m}{r} H_r = \rho_r$.
If $r = 0$, we have $H_1 = \frac{1}{m} Tr(A) = H$ the mean curvature of L^m.

For each $0 \leq r \leq m$, we defines the Newton transformation $P_r : \chi(L) \longrightarrow \chi(L)$ of the submanifold M^n be setting $P_0 = I$ (the identity operator) and for $0 \leq r \leq m$, by the recurrence relation

$$P_r = \sum_{j=0}^{r} (-1)^{r-j} \binom{m}{j} H_j A^{r-j}, \tag{4.9}$$

where B^j denotes the composition of B with itself, j times ($B^0 = I$). Let us recall that associated to each Newton transformation P_r one has the second order linear differential operator $\mathcal{L}_r : C^\infty(L) \longrightarrow C^\infty(L)$ defined by

$$\mathcal{L}_r u = Tr(P_r \circ Hessu). \tag{4.10}$$

When $r = 0$, w note that $\mathcal{L}_0$ is just the Laplacian operator. Moreover, it is not difficult to see that

$$div_M(P_r \nabla u) = \sum_{i=1}^{m} g(\nabla_{E_i} P_r)\nabla_u, E_i) + \sum_{i=1}^{m} g(P_r(\nabla_{E_i} \nabla_u), E_i) \tag{4.11}$$

$$= g(div_M P_r, \nabla_u) + \mathcal{L}_r u,$$

where the divergence of P_r on L^m is given by

$$div_M P_r = Tr(\nabla P_r) = \sum_{i=1}^{m} (\nabla_{E_i} P_r) E_i. \tag{4.12}$$

In particular, when the ambient space has constant sectional curvature equation (4.11) reduces to

$$\mathcal{L}_r u = div_M(P_r \nabla u), \tag{4.13}$$

because $div_M P_r = 0$ (see [15] for more details).

Our aim, it also will be appropriate to deal with the so called traceless second fundamental form of the submanifold, which is is given by

$$\Phi = BHI, \qquad Tr(\Phi) = 0. \tag{4.14}$$

and

$$|\Phi|^2 = Tr(\Phi^2) = \frac{\|B\|^2}{2} - \frac{n^2\,\|H\|^2}{2} \geq 0. \tag{4.15}$$

with equality if and only if M^n is totally umbilical.

In order to establish our results let us mention the following maximum principle due to Caminha et al. for more details see [10]. We follows that, for each $p \geq 1$ use the notation

$$\mathcal{L}^p(L) = \left\{ u : L^m \longrightarrow \mathbb{R};\ \int_L |u|^p\, dL < +\infty \right\}. \tag{4.16}$$

Also, we have the following lemma:

Lemma 4.4. *Let X be a smooth vector field on the n-dimensional, complete, non compact, oriented Riemannian manifold M^n, such that $div_M X$ does not change sign on M^n. If $|X| \in \mathcal{L}^1(M)$, then $div_M X = 0$.*

The following results further generalized Theorem 1.2 in [2].

Theorem 4.5. *If the data (g, ψ, λ, r) be complete r-almost Newton-Yamabe soliton on Legendrian submanifold L^m in Kenmotsu space from $M^{m+1}(c)$ of constant sectional curvature c, with bounded second fundamental form and potential function $\psi : L^m \longrightarrow \mathbb{R}$ such that $|\nabla\psi| \in \mathcal{L}^1(L)$. Then we have*

(i) If $(c-3) \leq 0, (c+1) \leq 0$, $\lambda > 0$ and then L^m can not be L-minimal,

(ii) If $(c-3) < 0, (c+1) < 0$, $\lambda \geq 0$ and then L^m can not be L-minimal.

(iii) If $c = 3, c = -1$, $\lambda \geq 0$ and L^m is L-minimal, then L^m is isometric to the $\mathbb{R}^m$.

Proof. We know that the ambient space has constant sectional curvature, by equation (4.13) the operator $\mathcal{L}_r$ is a divergent type operator. On the other side, since L^m has bounded second fundamental form it follows from (4.9) that the Newton transformation P_r has bounded norm. In particular,

$$|P_r \nabla\psi| \leq |P_r|\,|\nabla\psi| \in \mathcal{L}^1(L), \tag{4.17}$$

Using (1) and (2), let us consider by contradiction that L^m is minimal. Then, equation (4.7) jointly with the considering $(c+3) \leq 0$ $(c+3 < 0)$ imply that the scalar curvature of L^m satisfies $\rho \leq 0 (\rho < 0)$. Hence, contracting (4.1) we have $\mathcal{L}_r\psi = m\lambda - \rho > 0$ in both case, which contradicts Lemma (4.4), since the fact after mentioned. This completes the proof of the first two assertions.
For the (3) assertion, since the ambient space has constant sectional curvature $c = 3, c = -1$ and L^m is minimal, then the equation (4.7) becomes as

$$\rho = -\frac{\|B\|^2}{2} \leq 0. \tag{4.18}$$

So, since $\lambda \geq 0$ we have that $\mathcal{L}_r(\psi) = m(\rho - \lambda) \geq 0$. Now, using the fact that $\mathcal{L}_r u = div_M(P_r \nabla u)$ and $|P_r\nabla\psi| \in \mathcal{L}^1(L)$, we have again from Lemma (1) that $\mathcal{L}_r\psi = 0$ on L^m. Hence, we conclude that $0 \geq m\rho = m\lambda \geq 0$, that is, $\rho = \lambda = 0$. This implies that $\frac{\|B\|^2}{2} = 0$. Therefore, the r-almost Newton-Yamabe soliton L^m must be geodesic and flat.

□

In order to prove our next theorems we will need the following lemmas, which corresponds to Theorem 3 [2].

Lemma 4.6. *Let u be a non-negative smooth subharmonic function on a complete Riemannian manifold M^n. If $u \in \mathcal{L}^p(M)$, for some $p > 1$, the u is constant.*

Further, we are in condition to establish the following result, which holds when the ambient space is an arbitrary Riemannian manifold.

Theorem 4.7. *Let the data (g, ψ, λ, r) be complete r-almost Newton-Yamabe soliton on Legendrian submanifold L^m in a Kenmotsu space form $M^{m+1}(c)$ of sectional curvature K, such that P_r is bounded from above (in the sense of quadratic forms) and its potential function $\psi : L^m \longrightarrow \mathbb{R}$ is non-negative and $\psi \in \mathcal{L}^p(L)$ for some $p > 1$. Then we have*

(i) If $K \leq 3, K \leq -1$, $\lambda > 0$ then L^m can not be L-minimal,

(ii) If $K < 3, K < -1$, $\lambda \geq 0$ then L^m can not be L-minimal,

(iii) If $K = 3, K = -1$, $\lambda \geq 0$ and L^m is L-minimal, then L^m is flat and totally geodesic.

Proof. For proving (1), we begin with a contradiction that L^m is minimal our assumption on the sectional curvature of the ambient space and equation (4.6) imply that $\tau \leq 0$. Hence, contracting equation (4.1) we have

$$\mathcal{L}_r\psi = m(\rho - \lambda) > 0. \tag{4.19}$$

Thus, since we are considering that P_r is bounded from above, there exists a positive constant ω such that

$$\omega\Delta\psi \geq \mathcal{L}_r\psi > 0. \tag{4.20}$$

In particular, from Lemma (4.6) we get that ψ must be constant, which gives a contradiction. Finally, reasoning as in the proof of Theorem (4.5) we can easily obtain (2) and (3).

In our next results we generalized Theorem 1.5 of [2] for the case when $X = \nabla\psi$, giving conditions for an r- almost Newton-Yamabe soliton on Legendrian submanifold in a Kenmotsu space form to be totally umbilical since it has bounded second fundamental form. Therefore, we prove the following theorem:

Theorem 4.8. *If the data (g, ψ, λ, r) be complete r-almost Newton-Yamabe soliton on Legendrian submanifold L^m in Kenmotsu space $M^{m+1}(c)$ of constant sectional curvature c, with bounded second fundamental form and potential function $\psi : L^m \longrightarrow \mathbb{R}$ such that $|\nabla\psi| \in \mathcal{L}^1(L)$. Then we have*

(i) $\lambda \geq \frac{m(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4} + \frac{m^2\|H\|^2}{2}$, then L^m is totally geodesic, with $\lambda = \frac{m(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4}$, and scalar curvature $\rho = \frac{m(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4}$,

(ii) If L^m is compact and $\lambda \geq \frac{m(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4} + \frac{n^2\|H\|^2}{2}$, then L^m is isometric to a Euclidean sphere,

(iii) If $\lambda \geq -\frac{(m-1)}{2}\left[\frac{m(c-3)}{4} + \frac{(c+1)}{2} + \frac{m^2\|H\|^2}{2}\right]$, then L^m is totally umbilical. In particular, the scalar curvature $\rho = m(m-1)2K_L$ is constant, where $K_L = \frac{8\lambda}{m(m-1)}$ is the sectional curvature of L^m.

Proof. To prove (1), using the equations (4.1) and (4.7), we obtain

$$\mathcal{L}_r\psi = m[\lambda + \frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4m} + \frac{m\|H\|^2}{2}] + \|B\|^2. \tag{4.21}$$

□

Then, for our consideration on λ, we get that $\mathcal{L}_r\psi$ is non-negative function on L^m. By Lemma (4.4) we find that $\mathcal{L}_r\psi$ vanishes identically. Hence, from equation (4.21) we arrive at that L^m is totally geodesic and $\lambda = \frac{m(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4}$. Moreover, it is clear form (4.7) that $\rho = \frac{m(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4}$, which complete the proof of (1).

If L^m is compact, as it is totally geodesic, then the ambient space must be necessarily a sphere $\mathbb{S}^{n+1}$ and M^n is isometric to the Euclidean sphere $\mathbb{S}^n$, proving (2).

For the assertion (3), we start with equation (4.21) that can be written in terms of the traceless second fundamental form Φ as

$$\mathcal{L}_r\psi = \frac{m}{2}[2\lambda - (m-1)\frac{(c+5+4H^2)}{4}] + \|\Phi\|^2 . \tag{4.22}$$

Therefore, our assumption on λ gives $\mathcal{L}_r\psi \geq 0$. Then by applying Lemma (4.4) once again we have $\mathcal{L}_r\psi = 0$. This implies that $|\Phi|^2$, that is, L^m is a totally umbilical. In particular κ of L^m is constant and L^m has constant sectional curvature given by $K_M = \frac{c+5+\kappa^2}{8}$. This combined with (4.22) , we obtain that

$$\lambda = \frac{(m-1)(c+5+4H^2)}{8} = \frac{(m-1)(c+5+\kappa^2)}{8} \tag{4.23}$$

$$= (m-1)K_L,$$

which implies that $\rho = m(m-1)K_L$, as desired. □

Another application of Theorem (4.7), we can also obtain the following theorem:

Theorem 4.9. *Let the data (g, ψ, λ, r) be complete r-almost Newton-Yamabe soliton on Legendrian submanifold L^m in Kenmotsu space form $M^{m+1}(c)$ with constant sectional curvature c, such that P_r is bounded from above and its potential function $\psi : L^m \longrightarrow \mathbb{R}$ is non-negative and $\psi \in \mathcal{L}^p(L)$ for some $p > 1$. Then we have*

(i) $\lambda \geq \frac{(m-1)(c-3)}{8} + \frac{(m-1)(c+1)}{4m} + mH^2$, then L^m is totally geodesic, with $\lambda = \frac{(m-1)(c-3)}{8} + \frac{(m-1)(c+1)}{4m}$, and scalar curvature $\rho = \frac{(m-1)(c-3)}{8} + \frac{(m-1)(c+1)}{4m}$.

(ii) If $\lambda \geq (m-1)\frac{(c+5+4H^2)}{8}$, then L^m is totally umbilical. In particular, the scalar curvature $\rho = m(m-1)K_M$ is constant, where $K_M = \frac{2\lambda}{m(m-1)}$ is the sectional curvature of L^m.

Proof. Let us begin observing that by equation (4.21) and assumption on λ we get

$$\mathcal{L}_r\psi = \frac{m}{2}[2\lambda - (m-1)\frac{(c+5+4H^2)}{4}] + \|\Phi\|^2 \geq 0. \tag{4.24}$$

Since we are assuming that P_r is bounded from above, there is a positive constant ω such that

$$\omega\Delta\psi \geq \mathcal{L}_r\psi \geq 0. \tag{4.25}$$

Using Lemma (4.6), we have that ψ must constant. Therefore $\mathcal{L}_r\psi = 0$, and equation (4.24) we conclude that L^m is totally geodesic, $\lambda = \frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4}$ and $\rho = \frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4}$, proving assertion (1), reasoning as in Theorem (4.8), it is easy to prove assertion (2). □

5 1-Almost Newton-Yamabe Solitons in locally symmetric Kenmotsu space form $M^{m+1}(c)$

This section based on the study of 1-almost Newton-Yamabe solitons on Legendrain submanifold of a locally symmetric Kenmotsu space form.

We know that a Riemannian manifold is called locally symmetric if all the covariant derivative components of its curvature tensor vanishes identically. In this aspect, such spaces exhibit an specific extension of constant curvature spaces.

Let L^m be a hypersurface immersed into a locally symmetric Kenmotsu space form $M^{m+1}(c)$. In what follows we initiated our curvature constraint, which will be consider in the prime results of this segment. More precisely, we will consider that there is a constant μ such that the sectional curvature C_K of the ambient space M^{m+1} satisfies the following equality:

$$C_K(\eta, t) = \frac{\mu}{m}, \tag{5.1}$$

where the vectors $\eta \in T^{\perp}(L)$ and $t \in T(L)$.

A locally symmetric Kenmotsu space form $M^{m+1}(c)$ of constant sectional curvature c is a locally symmetric space and it is easy to observe that the curvature condition (5.1) is satisfies for every hypersurface $\mathbb{H}^m$ immersed into $M^{m+1}(c)$, with $\frac{\mu}{m} = c$. Therefore in some extent our consideration is natural generalization of the case where the ambient space has constant sectional curvature. Moreover, when the ambient manifold is a Riemannian product of two Riemannian manifolds of constant sectional curvature, say $L = L_1(k_1) \times L_2(k_2)$, then L is locally symmetric and, if $k_1 = 0$ and $k_2 \geq 0$, then every hyperplane of the type $\mathbb{H} = \mathbb{H}_1 \times M_2(k_2)$, where $\mathbb{H}_1$ is an orientable and connected hypersurafce immersed in $M_1(k_2)$, satisfied the curvature constraint (5.1) with $\mu = 0$.

Let $M^{m+1}(c)$ be a locally symmetric Kenmotsu space form $M^{m+1}(c)$ satisfying condition (5.1) and let $\{E_1, ...E_{m+1}\}$ be an orthonormal frame on $T(M)$. Then, its scalar curvature $\bar{\rho}$ is given by

$$\bar{\rho} = \sum_{i=1}^{m+1} Ric(E_i, E_i) \tag{5.2}$$

$$=\sum_{i,j=1}^{m} g(Ric(E_i, E_j)E_j, E_j) + 2\sum_{i=1}^{m} g(Ric(E_{m+1}, E_i)E_{m+1}, E_i)$$

$$\bar{\tau} = \sum_{i,j=1}^{m} g(Ric(E_i, E_j)E_j, E_j) + 2\mu. \tag{5.2}$$

Moreover, it is well know fact that scalar curvature of a locally symmetric Kenmotsu space form $M^{m+1}(c)$ is constant. Thus $= \sum_{i,j=1}^{m} g(Ric(E_i, E_j)E_j, E_j)$ is a constant naturally attached to a locally symmetric Riemannian manifold satisfying (5.1). Therefore, for the sake of simplicity, we choose the following notation $\bar{\rho_S} := \frac{1}{m(m-1)} \sum_{i,j=1}^{m} g(Ric(E_i, E_j)E_j, E_j)$. It is worth pointing out that when M^{m+1} is a space of constant sectional curvature, the the constant $\bar{\rho_S}$ agrees with its sectional curvature.

The following results are the generalization of Theorem (4.5) for the context of 1-almost Newton-Yamabe soliton immersed in a locally symmetric Einstein manifold.

Theorem 5.1. *Let M^{m+1} be a locally symmetric Einstein manifolds satisfying the curvature condition (5.1). Let the data $(g, \psi, \lambda, 1)$ be complete 1-almost Newton-Yamabe soliton on Legendrian submanifold of Kenmotsu space form $M^{m+1}(c)$ of constant sectional curvature c, with bounded second fundamental form and potential function $\psi : L^m \longrightarrow \mathbb{R}$ such that $|\nabla\psi| \in \mathcal{L}^1(L)$ and let M^{m+1} be a locally symmetric . Then we have*

(i) If $\bar{\rho_S} \leq 0$, $\lambda > 0$ then L^m can not be L-minimal.

(ii) If $\bar{\rho_S} < 0$, $\lambda \geq 0$ then L^n can not be L-minimal.

(iii) If $\bar{\rho_S} = 0$, $\lambda \geq 0$ and L^m is L-minimal, then L^m is totally geodesic.

Proof. For the proof of (1) considering the proof of theorem (4.5) by contradiction that M^n is L-minimal. Then by our assumption on the constant we get from the equation (4.6) that the scalar curvature of L^m satisfies $\rho \leq 0$, which implies that $\mathcal{L}_r(\psi) = m\lambda - \rho \geq 0$.

On the other side, we have the differential operator $\mathcal{L}_1$ satisfies

$$\mathcal{L}_1\psi = div_L(P_1\nabla\psi) - g(div_L P_1, \nabla\psi). \tag{5.3}$$

In particular, taking an orthonormal frame $\{E_1, ..., E_m\}$ in $T(L)$ and denoting by N the orientation of L^m, it follows from Lemma 25 of [2] that

$$g(div_L P_1, \nabla\psi) = \sum_{i=1}^{m} g(R(N, E_i)\nabla\psi, E_i) = Ric(N, \nabla\psi). \tag{5.4}$$

Since $M^{m+1}(c)$ is consider to be Einstein we conclude by equation (5.3) combined with the equation (5.4), we arrive at

$$\mathcal{L}_1\psi = div_L(P_1\nabla\psi). \tag{5.5}$$

Moreover, as we have observed from Theorem (4.5) we obtain from our consideration on second fundamental form that $|\nabla\psi| \in \mathcal{L}^1(L)$. Therefore, we are in position to apply Lemma (4.4) to conclude that $\mathcal{L}_r\psi = 0$, which gives a contradiction. Finally, reasoning as above it is easy to prove (2) and (3).

□

Now, we obtaining the analogous results to Theorem (4.8) in the case where $r = 1$ and the ambient space is locally symmetric. Particularly, we obtain the following theorem:

Theorem 5.2. *Let the data $(g, \psi, \lambda, 1)$ be a complete 1-almost Newton-Yamabe soliton on Legendrian submanifold L^m of Kenmotsu space form $M^{m+1}(c)$ of constant sectional curvature c, with bounded second fundamental form and potential function $\psi : L^m \longrightarrow \mathbb{R}$ such that $|\nabla\psi| \in \mathcal{L}^1(L)$ and let M^{m+1} be a locally symmetric Einstein manifolds satisfying the curvature condition (5.1). Then we have*

(i) $\lambda \geq \left[\frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4m}\right]\bar{\rho_S} + mH^2$*, then L^m is totally geodesic, with* $\lambda = \left[\frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4m}\right]\bar{\rho_S}$ *and scalar curvature* $\rho = \left[\frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4m}\right]\bar{\rho_S}$,

(ii) If $\lambda \geq \left[\frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4m}\right](\bar{\rho_S} + H^2)$*, then L^m is totally umbilical. In particular, the scalar curvature* $\rho = \left[\frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4m}\right](\bar{\rho_S} + \kappa^2)$ *is constant, where κ is the principal curvature of L^m.*

Proof. The proof is similar as in the proof of Theorem (4.8). For the sake of completeness, we give the following argument that proves (1). Taking trace in (4.1) and using definition of the constant $\bar{\rho_S}$, we obtain equation (4.6) that

$$\mathcal{L}_r\psi = m[\lambda - \left[\frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4m}\right]\bar{\rho_S} - mH^2] + \|B\|^2, \tag{5.6}$$

which implies that $\mathcal{L}_1\psi \geq 0$ because our assumption on λ. Then from Lemma (4.4) we obtain that $\mathcal{L}_r\psi = 0$. Therefore, we conclude from equation (5.6) that L^m is totally geodesic with $\lambda = (m-1)\bar{\rho_S}$ and scalar curvature $\rho = m(m-1)\bar{\rho_S}$. This complete the prove of the result.

□

We completing our paper mentioning the following theorem, which can be furnish from the similar manner used in the proof of Theorem (4.9) and (5.2).

Theorem 5.3. *Let the data $(g, \psi, \lambda, 1)$ be a complete 1-almost Newton-Yamabe soliton on Legendrian submanifold of Kenmotsu space form $M^{m+1}(c)$ such that P_r is bounded from above, its potential function $\psi : L^m \longrightarrow \mathbb{R}$ is non-negative and such that $|\nabla\psi| \in \mathcal{L}^1(L)$ for some $p > 1$ and let M^{n+1} be a locally symmetric Einstein manifolds satisfying the curvature condition (5.1). Then we have* *(i)* $\lambda \geq \left[\frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4m}\right]\bar{\rho_S} + mH^2$, *then* L^m *is totally geodesic, with* $\lambda = \left[\frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4m}\right]\bar{\rho_S}$ *and scalar curvature* $\rho = \left[\frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4m}\right]\bar{\rho_S}$.

(ii) If $\lambda \geq \left[\frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4m}\right](\bar{\tau_S} + H^2)$, *then* L^m *is totally umbilical. In particular, the scalar curvature* $\rho = \left[\frac{(m-1)(c-3)}{8} + \frac{(1-m)(c+1)}{4m}\right](\bar{\tau_S} + \kappa^2)$ *is constant, where* κ *is the principal curvature of* L^m.

We observe that from Theorems (5.1), (5.2) and (5.3) we can replace that the hypothesis that the ambient space M^{n+1} is Einstein under the condition, contant curvature tensor identically vanishes.

Example 5.4. Since the canonical immersion $\mathbb{S}^m \hookrightarrow \mathbb{S}^{m+1} \times \mathbb{R}$ is totally geodesic, proceeding as in Example (4.1) we see that this immersion satisfies equation (4.1) for all $1 \leq r \leq n$ and $\lambda = \frac{(m-1)}{m}$.

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Received: March 25, 2020.
Accepted: June 7, 2020