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Characteristic of Conformal Ricci soliton and Conforma gradient Ricci soliton in LP-Sasakian Manifold

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Abstract In this paper we have studied Lorentzian Para - Sasakian Manifold(LP - Sasakain) and Lorentizian Special Para Sasakian manifold(LSP - Sasakian) manifold admitting conformal Ricci soliton and conformal gradient Ricci soliton equation.

1 Introduction

Matsumoto [1] has initiated the study of Lorentzian almost Paracontact manifold. After him many scientists studied Lorentizian almost paracontact manifolds and their different classes namely LP- Sasakian and manifolds [2],[3],[4].

An $n-$dimensional Lorentzian manifold M is a smooth connected para-contact Housdroff manifold with a Lorentizian metric g, i.e. M admits a smooth symmetric second order tensor field g such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \longrightarrow R$ is a non-degenerate inner product of signature $(-, +, +, ..., +)$ where T_pM denotes the tangent vector space of M at p.

In Riemannian manifold (M, g), the Ricci soliton[5] equation is given by

$$Ric + \frac{1}{2}\pounds_X g + \lambda g = 0, \tag{1.1}$$

where $\pounds$ is the Lie derivative, S is the Ricci tensor, V is a vector field on M and λ is a constant. Ricci solitons are the natural generalization of Einstein metrics and are self-similar solutions to the Ricci flow. Ricci solitons have been used as a popular tool in Physics and called quasi-Einstein metric [6],[7].

A.E. Fischer [8] has introduced the concept of conformal Ricci flow during 2004-2005. Lu, Qing and Zheng[9] used DeTurck's trick to rewrite conformal Ricci flow as a strong parabolic-elliptic partial differential equations. Then they proved short time existences for conformal Ricci flow on compact manifolds as well as on asymptotically flat manifolds. The concept of conformal Ricci soliton [10] was introduced by N.Basu and A.Bhattacharyya in 2015. The conformal Ricci soliton equation is given by

$$\pounds_V g + 2S = [2\lambda - (p + \frac{2}{n})]g, \tag{1.2}$$

where p is the non-dynamical scalar field and the scalar curvature $R(g) = -1$. The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation and it also satisfies the conformal Ricci flow equation.

If in the conformal Ricci soliton equation the tangent vector V is gradient of a smooth funtion f the equation will be called conformal gradient Ricci soliton equation and written as

$$\nabla\nabla f + 2S = [2\lambda - (p + \frac{2}{n})]g. \tag{1.3}$$

We shall study conformal Ricci soliton and conformal gradient Ricci soliton in LP- Sasakian and LSP- Sasakain manifold.

2 Preliminaries:

Let M be an $n-$dimensional Lorientzian para Sasakian Manifold (LP - Sasakian)[11] manifold with structure (ϕ, ξ, η, g) where ϕ is a (1,1) tensor field, ξ is a contravariant vector field, η is a 1-form and g is a Lorentzian metric if

$$\eta(\xi) = -1, \phi^2 = I + \eta \otimes \xi. \tag{2.1}$$

$$\phi(\xi) = 0; \;\; \eta\phi = 0; \;\; \nabla_X \xi = \phi X; \;\; rank(\phi) = n-1. \tag{2.2}$$

$$\eta(X) = g(\xi, X); \;\; g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{2.3}$$

$$(\nabla_X \eta)(Y) = \Omega(X,Y), \;\; \Omega(X,Y) = \Omega(Y,X), \tag{2.4}$$

where $\Omega(X,Y) = g(\phi Y, X)$.

An $n-$ dimensional Lorentzian manifold (M, g) is said to be Lorentzin special para Sasakian (LSP- Sasakian) manifold if M admits a timelike unit vector field ξ with associated 1-form η satisfies

$$\Omega(X,Y) = (\nabla_X \eta)(Y) = \epsilon g(X,Y) + \eta(X)\eta(Y), \;\; \epsilon^2 = 1. \tag{2.5}$$

An LSP Sasakian manifold is always LP Sasakian manifold but the converse is not true.

On the other hand, the eigenvalues of ϕ are $-1, 0, 1$ and the multiplicity of 0 is 1 by (2.2). Let k and l be the multiplicities of -1 and 1 respectively. Then $tr\phi = l - k$. So, if $(tr\phi)^2 = (n-1)^2$, then either $l = 0$ or $k = 0$. In this case, we call our structure is a trivial LP- Sasakian structure.

In an $n-$ dimensional LP- Sasakian manifold with structure (ϕ, ξ, η, g), we know the following relations hold

$$\begin{aligned} \eta(R(X,Y)Z) &= g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \\ S(X,\xi) &= (n-1)\eta(X). \end{aligned} \tag{2.6}$$

$$\begin{aligned} R(\xi,Y)X &= g(Y,X)\xi - \eta(X)Y \\ R(Y,X)\xi &= \eta(X)Y - \eta(Y)X. \end{aligned} \tag{2.7}$$

$$\begin{aligned} \phi(R(X,\phi Y)Z) = R(X,Y)Z &+ 2[\eta(Y)X - \eta(X)Y]\eta(Z) + 2[g(X,Z)\eta(y) \\ &- g(Y,Z)\eta(X)]\xi + \Omega(X,Z)\phi Y - \Omega(Y,Z)\phi X \\ &+ g(Y,Z)X - g(X,Z)Y. \end{aligned} \tag{2.8}$$

Here R is the curvature tensor and S is the Ricci tensor.

An $n-$ dimensional LP-Sasakian manifold is said to be η- Einstein if the Ricci tensor S satisfies

$$S = ag + b\eta \otimes \eta, \tag{2.9}$$

where a, b are smooth functions on the manifold.

In η-Einstein LP-Sasakian manifold, the Ricci tensor S is of the form

$$S(X,Y) = \frac{n}{1-n} g(X,Y) + \frac{n-1-n^2}{n-1} \eta(X)\eta(Y). \tag{2.10}$$

We have considered scalar curvature is -1, as it is so, for conformal Ricci flow Ricci operator is of the form

$$QX = \frac{n}{1-n} X + \frac{n-n^2-1}{n-1} \eta(X)\xi. \tag{2.11}$$

3 Conformal Ricci soliton:

Theorem 3.1. *If an LP Sasakian manifold admits conformal Ricci soliton and a symmetric parallel* $(0, 2)$ *tensor field, then the soliton is steady, shirking or expanding according as* $\lambda = \frac{1}{2}(p + \frac{2}{n}),\ \lambda > \frac{1}{2}(p + \frac{2}{n})\ or\ \lambda < \frac{1}{2}(p + \frac{2}{n})$.

Proof: We consider an LP- Sasakian manifold which admits conformal Ricci soliton equation. Now as $[2\lambda - (p + \frac{2}{n})]$ is constant, we have $\nabla[2\lambda - (p + \frac{2}{n})]g = 0$. From [11], we know that if an LP- Sasakian manifold admits a symmetric parallel (0,2) tensor, then the tensor is a constant multiple of the metric tensor. Hence we can write $\pounds_V g + 2S = cg$, where c is a constant. Now from (1.2) we get $\frac{1}{2}[c + (p + \frac{2}{n}] = \lambda$. The soliton is steady, shrinking or expanding according as $c = 0,\ c > 0\ or\ c < 0$. So, we can say the soliton is steady, Shirking or expanding according as $\lambda = \frac{1}{2}(p + \frac{2}{n}),\ \lambda > \frac{1}{2}(p + \frac{2}{n})\ or\ \lambda < \frac{1}{2}(p + \frac{2}{n})$.□

Theorem 3.2. *If LP-Sasakian manifold admits conformal Ricci soliton and V is pointwise collinear with* ξ*, then* V *is a constant multiple of* ξ *provided* $\lambda = \frac{p}{2} + 1 + \frac{1-n^2}{n}$.

Proof: Let V be point-wise collinear with ξ, i.e. $V = b\xi$, where b is a function on the LP-Sasakian manifold. Then we have

$$\pounds_V g + 2S = [2\lambda - (p + \frac{2}{n})]g.$$

Putting $V = b\xi$, we have

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X, Y) + [(p + \frac{2}{n}) - 2\lambda]g(X, Y) = 0.$$

Using (2.2), we get

$$2bg(\phi X, Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X, Y) + [(p + \frac{2}{n}) - 2\lambda]g(X, Y) = 0. \quad (3.1)$$

Putting $Y = \xi$ in (3.1), we have

$$(Xb) + (\xi b)\eta(X) + 2(n - 1)\eta(X) + [(p + \frac{2}{n}) - 2\lambda]\eta(X) = 0. \quad (3.2)$$

If we put $X = \xi$ in (3.2), the equation reduces into

$$\xi b = \frac{1}{2}(p + \frac{2}{n}) - \lambda - (n - 1). \quad (3.3)$$

Now putting the value of (3.3) in (3.2), we get

$$Xb = [\frac{1}{2}(p + \frac{2}{n}) - \lambda - (n - 1)]\eta(X). \quad (3.4)$$

If we consider $Xb = 0$, we get

$$\frac{1}{2}(p + \frac{2}{n}) - \lambda - (n - 1) = 0$$

$$\implies \lambda = \frac{p}{2} + 1 + \frac{1 - n^2}{n}.$$

So we can conclude that V is constant multiple of ξ, provided $\lambda = \frac{p}{2} + 1 + \frac{1-n^2}{n}$.□

Corollary 3.3. *If a LP- Sasakian manifold admits conformal Ricci soliton and the vector field V is collinear with* ξ*, then the constant* $\lambda = \frac{n^2-1}{n} + \frac{p}{2}$.

Proof. Putting $V=\xi$ in the conformal Ricci soliton equation (1.2), we get

$$2g(\phi X,Y)+2S(X,Y)=[2\lambda-(p+\frac{2}{n})]g(X,Y). \tag{3.5}$$

Now putting $X=\xi$ in the above equation and using (2.6), we get $\lambda=\frac{n^2-1}{n}+\frac{p}{2}$.□

Theorem 3.4. *If a LSP- Sasakian manifold admits conformal Ricci soliton, then the manifold must be an Einstein manifold and moreover if V is collinear with ξ then the metric g can be expressed as a product of two one-forms.*

Proof. For LSP- Sasakian manifold

$$(\nabla_X\eta)Y=g(\phi X,Y)=\epsilon[g(X,Y)+\eta(X)\eta(Y)];\epsilon^2=1. \tag{3.6}$$

From (3.1) we have

$$2bg(\phi X,Y)+(Xb)\eta(Y)+(Yb)\eta(X)+2S(X,Y)+[(p+\frac{2}{n})-2\lambda]g(X,Y)=0. \tag{3.7}$$

Now using (3.6) and (3.4) in (3.1), we get

$$2S(X,Y)=[2(\lambda-n+1)-2b\epsilon-(p+\frac{2}{n})]\eta(X)\eta(Y)+[2\lambda-(p+\frac{2}{n})-2b\epsilon]g(X,Y), \tag{3.8}$$

which shows that the manifold is η-Einstein manifold.

Lets us put $[2(\lambda-n+1)-2b\epsilon-(p+\frac{2}{n})]=\gamma$ and $[2\lambda-(p+\frac{2}{n})-2b\epsilon]=\delta$.
Here γ and δ are constants.
Putting $V=\xi$ in (1.2), we have

$$g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X)+2S(X,Y)+[(p+\frac{2}{n})-2\lambda]g(X,Y)=0.$$

Using (3.7), we get

$$g(\phi X,Y)+g(\phi Y,X)+[(p+\frac{2}{n})-2\lambda+2\gamma]g(X,Y)+2\gamma\eta(X)\eta(Y)=0. \tag{3.9}$$

From (2.4), we have

$$2\Omega(X,Y)+[(p+\frac{2}{n})-2\lambda+2\gamma]g(X,Y)+2\delta\eta(X)\eta(Y)=0.$$

Using (2.5), we get

$$[(p+\frac{2}{n})-2\lambda+2\delta+2\epsilon]g(X,Y)+[2\delta+2\epsilon]\eta(X)\eta(Y)=0 \tag{3.10}$$

So we can conclude that g can be expressed as product of two one forms.□

Example: Consider the 3− dimensional manifold $M=(x,y,z)\in R^3$ where (x,y,z) are the standard notation of R^3 and lets us consider the vector fields

$$e_1=e^z\frac{\partial}{\partial y},\ \ e_2=e^z(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}),\ \ e_3=\frac{\partial}{\partial z}$$

where e_1,e_2,e_3 are linearly independent at each point of M.
Let g be the metric defined by

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=-1$$

$$\text{and}\quad g(e_1,e_2)=g(e_1,e_3)=g(e_2,e_3)=0.$$

Let η be the one form defined by $\eta(x) = g(x, e_3)$ for any vector field $X \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1) = -e_1,\ \phi(e_2) = -e_2, \phi(e_3) = 0.$$

Then using the linearity of ϕ and g we have

$$\eta(e_3) = -1, \quad \phi^2 X = x + \eta(X)e_3, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for all vector fields $X, Y \in \chi(M)$. Hence for $e_3 = \xi$, the structure defined a Lorentzian para contact structure on M. Let ∇ be the Levi-Civita connection with respect to the Lorentizian metric g, then we obtain

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2.$$

Taking $e_3 = \xi$ and using Koszul's formula for the Lorentizian metric g, we obtain

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = -e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

with the help of above results we get

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1$$

$$R(e_1, e_2)e_2 = e_1, \quad R(e_2, e_3)e_2 = -e_3, \quad R(e_1, e_3)e_2 = 0$$

$$R(e_1, e_2)e_1 = -e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_3 = -e_1.$$

From the above expression the Ricci tensor is given by

$$S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) - g(R(e_1, e_3)e_3, e_1) = 2.$$

Similarly, we get, $\quad S(e_2, e_2) = 2, \quad S(e_3, e_3) = 2$

and $S(e_1, e_2) = S(e_2, e_3) = S(e_1, e_3) = 0.$

Now from (3.5), we obtain

$$S(e_i, e_i) + g(\phi e_i, e_i) = [\lambda - \frac{1}{2}(p + \frac{2}{3})]g(e_i, e_i) \tag{3.11}$$

where $i = 1, 2, 3$
we get $\lambda = \frac{p}{2} + \frac{4}{3}$.
Therefore $\lambda = \frac{p}{2} + \frac{4}{3} > 0$, so we can conclude that the Lorentizian metric satisfies conformal Ricci Soliton equation on (M, ϕ, ξ, g) and the conformal Ricci soliton is expanding as λ is positive.□

4 Conformal gradient Ricci soliton

Theorem 4.1. *If a LP- Sasakian manifold admits conformal gradient Ricci soliton then the manifold must be Einstein manifold.*

Proof. Conformal gradient Ricci soliton equation (1.3) can be written as

$$\nabla_Y Df = QY + [2\lambda - (p + \frac{2}{n})]Y, \tag{4.1}$$

where D is gradient operator of g. So, we can write

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X, \tag{4.2}$$

$$\implies g(R(\xi,Y)Df,\xi) = g((\nabla_\xi Q)Y,\xi) - g((\nabla_Y Q)\xi,\xi). \tag{4.3}$$

Now using (2.11) and (2.2), we have

$$(\nabla_Y Q)X = \frac{1}{n-1}(X + \eta(X)\xi) + (\frac{n - n^2 - 1}{n-1})[g(Y,\phi X)\xi + \eta(X)\phi Y], \tag{4.4}$$

$$\implies g((\nabla_X Q)\xi - (\nabla_\xi Q)X, \xi) = 0. \tag{4.5}$$

Then from (4.3), we have

$$g(R(\xi,X)Df,\xi) = 0. \tag{4.6}$$

From (2.7) and (4.6), we get

$$g(R(\xi,Y)Df,\xi) = -g(Y,Df) - \eta(Df)\eta(Y) = 0.$$

Hence

$$Df = -\eta(Df)\xi = -g(Df,\xi)\xi = -(\xi f)\xi. \tag{4.7}$$

Using (4.7) in (1.3), we have

$$S(X,Y) + [2\lambda - (p + \frac{2}{n})]g = -Y(\xi f)\eta(X) - \xi f g(X,\phi Y). \tag{4.8}$$

Now if we put $X = \xi$, we get

$$S(\xi,Y) + [2\lambda - (p + \frac{2}{n})]g(\xi,Y) = -Y(\xi f)\eta(\xi) - \xi f g(\xi,\phi Y).$$

Using (2.7) and (3.5), we have

$$[2\lambda + n - 1 - (p + \frac{2}{n})]\eta(Y) = Y(\xi f). \tag{4.9}$$

From this it is clear that if $2\lambda + n - 1 - (p + \frac{2}{n}) = 0$, i.e. $\lambda = \frac{1}{2} + \frac{p}{2} + \frac{1}{n} - \frac{n}{2}$, then $\xi f =$ constant.

From (4.7), we get $Df = -(\xi f)\xi = cf$. In particular taking a frame field $\xi f = 0$, we get $f =$ constant.

Also from (1.3), we have

$$2S = [2\lambda - (p + \frac{2}{n})]g. \tag{4.10}$$

Now putting the value of $\lambda = \frac{1}{2} + \frac{p}{2} + \frac{1}{n} - \frac{n}{2}$, we get

$$S(X,Y) = \frac{1-n}{2} g(X,Y). \tag{4.11}$$

Which shows that the manifold is Einstein.□

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