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DERIVATION OF THE HALF-SIDE TANGENT FORMULAS VIA LAGRANGE'S THEOREM (FOR POLAR SPHERICAL TRIANGLES)

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Abstract Spherical Geometry is a non-Euclidean geometry that provides good tools for terrestrial location and has great applicability in Astronomy. For this reason, trigonometry in spherical triangles is an interesting subject to study. Because it is a well-developed field, there are many widely known theorems and formulas with classical demonstrations. This article aims to present an alternative proof for the Half-Side Tangent Formulas, using the Langrange's Theorem concerning polar triangles.

1 Introduction

Spherical geometry was essential for the development of location geography and also for navigation. Interestingly, even before the new (non-Euclidean) geometries were officialized, the Greeks in ancient times already used Spherical Geometry intuitively to study astronomy. It was this civilization that proposed the concept of the celestial sphere, beginning the first studies in Position Astronomy (also known as Spherical Astronomy), which fundamentally concerns the directions in which celestial bodies are viewed, without regard for their distance.

This geometry provides good tools for terrestrial location and has great applicability in Astronomy. For this reason, trigonometry in spherical triangles is an interesting subject to study. As we studied spherical trigonometry theorems we were inspired to present a different proof for The Half-Side Tangent Formulas by using the Langrange's Theorem (concerning polar triangles).

Therefore, in this article we will present this proof.

2 Preliminary Concepts and Classical Proof for The Half-Side Tangent Formulas

Before presenting the main proof this article deals with, we need to present some basic concepts and classic results of Spherical Geometry and Trigonometry.

Definition 2.1. On a spherical surface, three points not belonging to the same great circle, connected by arcs of great circles, determine a spherical triangle. Such points are called vertices of the triangle and the corresponding arcs are called sides of the triangle.

Figure 1 exemplifies a spherical triangle with vertices A, B and C in a spherical surface of center O.

Definition 2.2. The axis of a great circle is the diameter of the sphere perpendicular to the plane of the circle. The poles of a great circle are the two points in which its axis meets the surface of the sphere. These two points also are called diametrically opposite.

We should not consider sides of spherical triangles measuring more than 180°, since any two points on the spherical surface can be joined by two arcs (which are equal only if these two points are diametrically opposite). Therefore we may always replace the side greater than 180° by the remaining arc of the great circle to which it belongs.

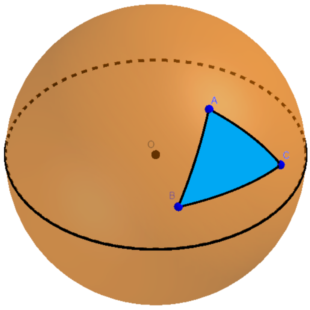

Figure 1. Spherical triangle ABC.

Remark 2.3. Note that one consequence of Definition 2.2 is that the distance from a pole to any point of its great circle is always equal to 90°.

Definition 2.4. One spherical triangle is called the polar triangle of a second spherical triangle when the sides of the first triangle has their poles at the vertices of the second.

Figure 2 shows an illustration of a polar triangle $A'B'C'$ of a first given tiangle ABC.

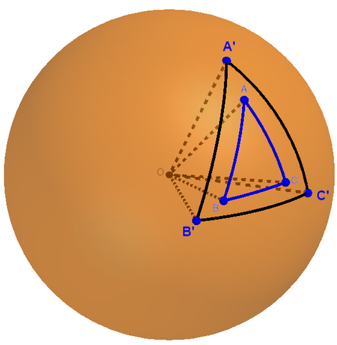

Figure 2. Polar triangles $A'B'C'$ and ABC.

For polar triangles, the following result is valid:

Proposition 2.5. *Let be the spherical triangles ABC and $A'B'C'$. If $A'B'C'$ is polar of ABC, so ABC is polar of $A'B'C'$.*

You can read a proof for Proposition 2.5 in [1]. Finally, we present below the Lagrange's Theorem, which we will use to support our proof for the Half-Side Tangent Formulas.

Theorem 2.6. *(The Lagrange's Theorem) - In two polar triangles, each angle of one is measured by the supplement of the corresponding side of the other. That is, in polar triangles arranged as in Figure 3, it is true that:*

$$\widehat{A} = 180° - a'$$
$$\widehat{B} = 180° - b'$$
$$\widehat{C} = 180° - c'$$

Proof. Let's use Figure 3 as a support. First we extend sides b and c until we determine the points E and F in the second triangle. Then we have:

$$\overset{\frown}{\mathrm{B'F}} = 90° \tag{2.1}$$

$$\overset{\frown}{\mathrm{C'E}} = 90° \tag{2.2}$$

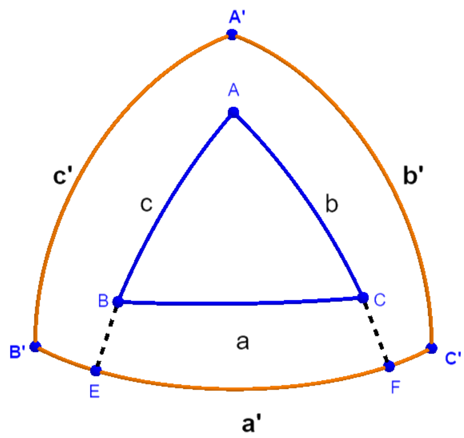

Figure 3. Figure support for Theorem 2.6.

because B' is pole of $\overset{\frown}{\mathrm{AC}}$ and C' is pole of $\overset{\frown}{\mathrm{AB}}$.
Adding the Equations (2.1) and (2.2) we have:

$$\overset{\frown}{\mathrm{B'F}} + \overset{\frown}{\mathrm{C'E}} = 180^\circ \tag{2.3}$$

We know that

$$\overset{\frown}{\mathrm{B'F}} = \overset{\frown}{\mathrm{B'E}} + \overset{\frown}{\mathrm{EF}} \tag{2.4}$$

$$\overset{\frown}{\mathrm{C'E}} = \overset{\frown}{\mathrm{EF}} + \overset{\frown}{\mathrm{C'F}} \tag{2.5}$$

Substituting Equations (2.4) and (2.5) in (2.3):

$$\overset{\frown}{\mathrm{B'E}} + \overset{\frown}{\mathrm{EF}} + \overset{\frown}{\mathrm{EF}} + \overset{\frown}{\mathrm{C'F}} = 180^\circ$$

$$a' + \overset{\frown}{\mathrm{EF}} = 180^\circ$$

Since A is pole of $\overset{\frown}{\mathrm{EF}}$, we have $\overset{\frown}{\mathrm{EF}} = \widehat{A}$. So

$$\widehat{A} = 180^\circ - a'$$

The other equalities can be proved analogously. □

Following are some important spherical trigonometry results for this paper. For these results the spherical triangle ABC will be arranged as in Figure 4.

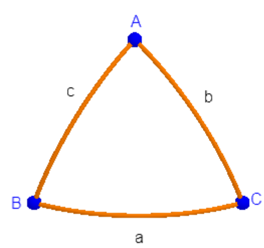

Figure 4. Spherical triangle ABC.

Theorem 2.7. *(The Side Cosine Formulas) - Let be a spherical triangle of vertices A, B and C, with internal angles measuring $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ whose opposite sides measure a, b and c, respectively, as in the Figure 4. So:*

$$\cos(a) = \cos(b).\cos(c) + \sin(b).\sin(c).\cos(\widehat{A})$$

$$\cos(b) = \cos(a).\cos(c) + \sin(a).\sin(c).\cos(\widehat{B})$$

$$\cos(c) = \cos(a).\cos(b) + \sin(a).\sin(b).\cos(\widehat{C})$$

Theorem 2.8. *(The Angle Cosine Formulas) - Let be a spherical triangle of vertices A, B and C, with internal angles measuring $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ whose opposite sides measure a, b and c, respectively, as in the Figure 4. So:*

$$\cos(\widehat{A}) = -\cos(\widehat{B}).\cos(\widehat{C}) + \sin(\widehat{B}).\sin(\widehat{C}).\cos(a)$$

$$\cos(\widehat{B}) = -\cos(\widehat{A}).\cos(\widehat{C}) + \sin(\widehat{A}).\sin(\widehat{C}).\cos(b)$$

$$\cos(\widehat{C}) = -\cos(\widehat{A}).\cos(\widehat{B}) + \sin(\widehat{A}).\sin(\widehat{B}).\cos(c)$$

The readers can find proofs for Theorem 2.7 and Theorem 2.8 in [1, 2, 3, 4].

Considering that Figure 4 shows a oblique[1] triangle, we can use it as support for the next Theorems 2.9 and 2.10.

Theorem 2.9. *(The Half-Angle Tangent Formulas) - Let be a spherical oblique triangle of vertices A, B and C, with internal angles measuring $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ whose opposite sides measure a, b and c, respectively, as in the Figure 4. Considering s half of the triangle perimeter, it is true:*

$$\tan^2(\tfrac{\widehat{A}}{2}) = \frac{\sin(s-b).\sin(s-c)}{\sin(s).\sin(s-a)}$$

$$\tan^2(\tfrac{\widehat{B}}{2}) = \frac{\sin(s-a).\sin(s-c)}{\sin(s).\sin(s-b)}$$

$$\tan^2(\tfrac{\widehat{C}}{2}) = \frac{\sin(s-a).\sin(s-b)}{\sin(s).\sin(s-c)}$$

Proof. From Theorem 2.8 follows:

$$\cos(a) = \cos(b).\cos(c) + \sin(b).\sin(c).\cos(\widehat{A}) \implies$$

$$\cos(\widehat{A}) = \frac{\cos(a) - \cos(b).\cos(c)}{\sin(b).\sin(c)} \tag{2.6}$$

Adding 1 to both members of the Equation (2.6):

$$1 + \cos(\widehat{A}) = 1 + \left(\frac{\cos(a) - \cos(b).\cos(c)}{\sin(b).\sin(c)}\right) \tag{2.7}$$

Developing the Equation (2.7) we have:

$$1 + \cos(\widehat{A}) = \frac{\sin(b).\sin(c) + \cos(a) - \cos(b).\cos(c)}{\sin(b).\sin(c)} \implies$$

$$1 + \cos(\widehat{A}) = \frac{\cos(a) - (\cos(b).\cos(c) - \sin(b).\sin(c))}{\sin(b).\sin(c)}$$

Using the equality (A.1) listed in the Appendix A:

$$1 + \cos(\widehat{A}) = \frac{\cos(a) - \cos(b+c)}{\sin(b).\sin(c)} \tag{2.8}$$

[1] Any triangle that is not a right triangle.

And applying the equality (A.3) listed in the Appendix A in the Equation (2.8), we have:

$$1+\cos(\widehat{A}) = 2\cos^2(\tfrac{\widehat{A}}{2}) = \frac{\cos(a)-\cos(b+c)}{\sin(b).\sin(c)} \tag{2.9}$$

Now, using the equality (A.4) listed in the Appendix A in the Equation (2.9):

$$2\cos^2(\tfrac{\widehat{A}}{2}) = \frac{-2\sin(\frac{a+b+c}{2}).\sin(\frac{a-b-c}{2})}{\sin(b).\sin(c)} \tag{2.10}$$

And applying the equality (A.5) listed in the Appendix A, we have:

$$2\cos^2(\tfrac{\widehat{A}}{2}) = \frac{-2\sin(\frac{a+b+c}{2}).(-\sin(\frac{b+c-a}{2}))}{\sin(b).\sin(c)} = \frac{2\sin(\frac{a+b+c}{2}).\sin(\frac{b+c-a}{2})}{\sin(b).\sin(c)} \tag{2.11}$$

On the other hand, subtracting 1 from the Equation (2.6) and rearranging it properly, we have:

$$1-\cos(\widehat{A}) = 1-\left(\frac{\cos(a)-\cos(b).\cos(c)}{\sin(b).\sin(c)}\right) \tag{2.12}$$

Developing the Equation (2.12) we have:

$$1-\cos(\widehat{A}) = \frac{\sin(b).\sin(c)-\cos(a)+\cos(b).\cos(c)}{\sin(b).\sin(c)} \implies$$

$$1-\cos(\widehat{A}) = \frac{-\cos(a)+(\sin(b).\sin(c)+\cos(b).\cos(c))}{\sin(b).\sin(c)}$$

Using the equality (A.1) listed in the Appendix A:

$$1-\cos(\widehat{A}) = \frac{-\cos(a)+\cos(b-c)}{\sin(b).\sin(c)} \tag{2.13}$$

And applying the equality (A.2) listed in the Appendix A in the Equation (2.13), we have:

$$2\sin^2(\tfrac{\widehat{A}}{2}) = \frac{\cos(b-c)-\cos(a)}{\sin(b).\sin(c)} \tag{2.14}$$

Applying the equalities (A.4) and (A.5) listed in the Appendix A in the Equation (2.14), we have:

$$2\sin^2(\tfrac{\widehat{A}}{2}) = \frac{-2\sin(\frac{a+b-c}{2}).(-\sin(\frac{a+c-b}{2}))}{\sin(b).\sin(c)} = \frac{2\sin(\frac{a+b-c}{2}).\sin(\frac{a+c-b}{2})}{\sin(b).\sin(c)} \tag{2.15}$$

Thus, from the Equations (2.11) and (2.15), we deduce:

$$\tan^2(\tfrac{\widehat{A}}{2}) = \frac{\sin(\frac{b+a-c}{2}).\sin(\frac{a+c-b}{2})}{\sin(\frac{a+b+c}{2}).\sin(\frac{b+c-a}{2})} \tag{2.16}$$

Since s is half of the perimeter of the spherical triangle, we have:

$$a+b+c=2s \implies -a+a+a+b+c=2s \implies b+c-a=2s-2a=2(s-a)$$

This implies

$$\frac{1}{2}(b+c-a) = s-a$$

Likewise, we have:

$$a+b+c=2s \implies a-b+b+b+c=2s \implies a-b+c=2s-2b=2(s-b)$$

This implies

$$\frac{1}{2}(a-b+c)=s-b$$

Similarly, there are also:

$$a+b+c=2s \implies a+b+c+c-c=2s \implies b+a-c=2s-2c=2(s-c)$$

This implies

$$\frac{1}{2}(b+a-c)=s-c$$

Substituting the above considerations in Equation (2.16), we obtain:

$$\tan^2(\tfrac{\widehat{A}}{2})=\frac{\sin(s-b).\sin(s-c)}{\sin(s).\sin(s-a)} \tag{2.17}$$

This proves one of the proposed formulas. Analogously, it is possible to prove:

$$\tan^2(\tfrac{\widehat{B}}{2})=\frac{\sin(s-a).\sin(s-c)}{\sin(s).\sin(s-b)}$$

$$\tan^2(\tfrac{\widehat{C}}{2})=\frac{\sin(s-a).\sin(s-b)}{\sin(s).\sin(s-c)}$$

□

The following Theorem 2.10 is what the main result of this study is about. This Theorem, for obvious reasons, presents formulas known as Half-Side Tangent Formulas. However, for less direct reasons, its formulas are also called by many "Sailor's Formulas", because former Portuguese sailors, power of navigation at the time, used these formulas to calculate the hours in the ocean [4].

At this point of the text, we will present below the classical proof of this famous Theorem. Then, in the Section 3, we will present the proof we obtained in our study using the Lagrange's Theorem 2.6 as a tool, completing the main purpose of this article.

Theorem 2.10. *(The Half-Side Tangent Formulas) - Let be a spherical oblique triangle of vertices A, B and C, with internal angles measuring $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ whose opposite sides measure a, b and c, respectively, as in the Figure 4. Considering S half of the sum of the internal angles of the triangle, it is true:*

$$\tan^2(\tfrac{a}{2})=-\frac{\cos(S).\cos(S-\widehat{A})}{\cos(S-\widehat{B}).\cos(S-\widehat{C})}$$

$$\tan^2(\tfrac{b}{2})=-\frac{\cos(S).\cos(S-\widehat{B})}{\cos(S-\widehat{A}).\cos(S-\widehat{C})}$$

$$\tan^2(\tfrac{c}{2})=-\frac{\cos(S).\cos(S-\widehat{C})}{\cos(S-\widehat{A}).\cos(S-\widehat{B})}$$

Proof. From Theorem 2.8 follows:

$$\cos(\widehat{A})=-\cos(\widehat{B}).\cos(\widehat{C})+\sin(\widehat{B}).\sin(\widehat{C}).\cos(a)$$

That is

$$\cos(a)=\frac{\cos(\widehat{A})+\cos(\widehat{B}).\cos(\widehat{C})}{\sin(\widehat{B}).\sin(\widehat{C})} \tag{2.18}$$

Subtracting Equation (2.18) from 1, we have:

$$1-\cos(a) = 1-\left(\frac{\cos(\widehat{A})+\cos(\widehat{B}).\cos(\widehat{C})}{\sin(\widehat{B}).\sin(\widehat{C})}\right) \Longrightarrow$$

$$1-\cos(a) = \frac{-\cos(\widehat{A})+(\sin(\widehat{B}).\sin(\widehat{C})-\cos(\widehat{B}).\cos(\widehat{C}))}{\sin(\widehat{B}).\sin(\widehat{C})} \Longrightarrow$$

$$1-\cos(a) = \frac{-\cos(\widehat{A})-(\cos(\widehat{B}).\cos(\widehat{C})-\sin(\widehat{B}).\sin(\widehat{C}))}{\sin(\widehat{B}).\sin(\widehat{C})}$$

Using the equality (A.1) listed in the Appendix A:

$$1-\cos(a) = \frac{-\cos(\widehat{A})-\cos(\widehat{B}+\widehat{C})}{\sin(\widehat{B}).\sin(\widehat{C})} \Longrightarrow$$

$$1-\cos(a) = -\left(\frac{\cos(\widehat{A})+\cos(\widehat{B}+\widehat{C})}{\sin(\widehat{B}).\sin(\widehat{C})}\right)$$

Using the equality (A.2) listed in the Appendix A, we have:

$$2\sin^2\left(\tfrac{a}{2}\right) = -\left(\frac{\cos(\widehat{B}+\widehat{C})+\cos(\widehat{A})}{\sin(\widehat{B}).\sin(\widehat{C})}\right)$$

And using the equality (A.6) listed in the Appendix A, follows:

$$2\sin^2\left(\tfrac{a}{2}\right) = \frac{-2\cos\left(\frac{\widehat{A}+\widehat{B}+\widehat{C}}{2}\right).\cos\left(\frac{\widehat{B}+\widehat{C}-\widehat{A}}{2}\right)}{\sin(\widehat{B}).\sin(\widehat{C})} \tag{2.19}$$

On the other hand, adding 1 to the Equation (2.18), we have:

$$1+\cos(a) = 1+\left(\frac{\cos(\widehat{A})+\cos(\widehat{B}).\cos(\widehat{C})}{\sin(\widehat{B}).\sin(\widehat{C})}\right) \Longrightarrow$$

$$1+\cos(a) = \frac{\cos(\widehat{A})+(\sin(\widehat{B}).\sin(\widehat{C})+\cos(\widehat{B}).\cos(\widehat{C}))}{\sin(\widehat{B}).\sin(\widehat{C})}$$

Using the equality (A.1) listed in the Appendix A:

$$1+\cos(a) = \frac{\cos(\widehat{B}-\widehat{C})+\cos(\widehat{A})}{\sin(\widehat{B}).\sin(\widehat{C})}$$

Applying the equality (A.3) listed in the Appendix A, we have:

$$2\cos^2\left(\tfrac{a}{2}\right) = \frac{\cos(\widehat{B}-\widehat{C})+\cos(\widehat{A})}{\sin(\widehat{B}).\sin(\widehat{C})}$$

And applying the equality (A.6) listed in the Appendix A, follows:

$$2\cos^2\left(\tfrac{a}{2}\right) = \frac{2\cos\left(\frac{\widehat{A}+\widehat{B}-\widehat{C}}{2}\right).\cos\left(\frac{\widehat{B}-\widehat{C}-\widehat{A}}{2}\right)}{\sin(\widehat{B}).\sin(\widehat{C})} \tag{2.20}$$

From Equations (2.19) and (2.20), we have:

$$\tan^2\left(\tfrac{a}{2}\right) = -\frac{\cos\left(\frac{\widehat{A}+\widehat{B}+\widehat{C}}{2}\right).\cos\left(\frac{\widehat{B}+\widehat{C}-\widehat{A}}{2}\right)}{\cos\left(\frac{\widehat{A}+\widehat{B}-\widehat{C}}{2}\right).\cos\left(\frac{\widehat{B}-\widehat{C}-\widehat{A}}{2}\right)} \tag{2.21}$$

Now, Using the equality (A.7) listed in the Appendix A, we have $\cos\left(\frac{\widehat{B}-\widehat{C}-\widehat{A}}{2}\right) = \cos\left(\frac{\widehat{A}+\widehat{C}-\widehat{B}}{2}\right)$ and Equation (2.21) becomes:

$$\tan^2\left(\tfrac{a}{2}\right) = -\frac{\cos\left(\frac{\widehat{B}+\widehat{C}+\widehat{A}}{2}\right).\cos\left(\frac{\widehat{B}+\widehat{C}-\widehat{A}}{2}\right)}{\cos\left(\frac{\widehat{A}+\widehat{B}-\widehat{C}}{2}\right).\cos\left(\frac{\widehat{A}+\widehat{C}-\widehat{B}}{2}\right)} \tag{2.22}$$

Considering $\widehat{A}+\widehat{B}+\widehat{C}=2S$, we have:

$$\widehat{A}+\widehat{B}+\widehat{C}=2S \implies \widehat{A}+\widehat{C}-\widehat{B}=2(S-\widehat{B}) \implies \frac{\widehat{A}+\widehat{C}-\widehat{B}}{2}=S-\widehat{B}$$

$$\widehat{A}+\widehat{B}+\widehat{C}=2S \implies \widehat{B}+\widehat{C}-\widehat{A}=2(S-\widehat{A}) \implies \frac{\widehat{B}+\widehat{C}-\widehat{A}}{2}=S-\widehat{A}$$

$$\widehat{A}+\widehat{B}+\widehat{C}=2S \implies \widehat{A}+\widehat{B}-\widehat{C}=2(S-\widehat{C}) \implies \frac{\widehat{A}+\widehat{B}-\widehat{C}}{2}=S-\widehat{C}$$

With the previous considerations, from Equation (2.22) it is concluded that:

$$\tan^2\left(\tfrac{a}{2}\right)=-\frac{\cos(S).\cos(S-\widehat{A})}{\cos(S-\widehat{B}).\cos(S-\widehat{C})}$$

The other formulas can be proved analogously. □

3 The Half-Side Tangent Formulas via Lagrange's Theorem

Now comes the part where we will present the main result of this study. Below we will present an alternative of proof for Theorem 2.10, different from the classical demonstration, via Lagrange's Theorem 2.6.

Theorem 3.1. *(The Half-Side Tangent Formulas) - Let be a spherical oblique triangle of vertices A, B and C, with internal angles measuring $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ whose opposite sides measure a, b and c, respectively, as in the Figure 4. Considering S half of the sum of the internal angles of the triangle, it is true:*

$$\tan^2\left(\tfrac{a}{2}\right)=-\frac{\cos(S).\cos(S-\widehat{A})}{\cos(S-\widehat{B}).\cos(S-\widehat{C})}$$

$$\tan^2\left(\tfrac{b}{2}\right)=-\frac{\cos(S).\cos(S-\widehat{B})}{\cos(S-\widehat{A}).\cos(S-\widehat{C})}$$

$$\tan^2\left(\tfrac{c}{2}\right)=-\frac{\cos(S).\cos(S-\widehat{C})}{\cos(S-\widehat{A}).\cos(S-\widehat{B})}$$

Proof. Consider Theorem 2.9 applied to polar triangle $A'B'C'$ of the spherical triangle ABC (See Figure 3 if necessary):

$$\tan^2\left(\tfrac{\widehat{A'}}{2}\right)=\frac{\sin\left(\frac{b'+a'-c'}{2}\right).\sin\left(\frac{a'+c'-b'}{2}\right)}{\sin\left(\frac{a'+b'+c'}{2}\right).\sin\left(\frac{b'+c'-a'}{2}\right)} \tag{3.1}$$

From Theorem 2.6 we have:

$$a'=180^\circ-\widehat{A}$$
$$b'=180^\circ-\widehat{B}$$
$$c'=180^\circ-\widehat{C}$$
$$\widehat{A'}=180^\circ-a$$
$$\widehat{B'}=180^\circ-b$$
$$\widehat{C'}=180^\circ-c$$

Replacing the previous considerations in Equation (3.1):

$$\tan^2\left(\tfrac{180^\circ-a}{2}\right)=\frac{\sin\left(\frac{180^\circ-\widehat{A}+180^\circ-\widehat{B}-(180^\circ-\widehat{C})}{2}\right).\sin\left(\frac{180^\circ-\widehat{A}+180^\circ-\widehat{C}-(180^\circ-\widehat{B})}{2}\right)}{\sin\left(\frac{180^\circ-\widehat{A}+180^\circ-\widehat{B}+180^\circ-\widehat{C}}{2}\right).\sin\left(\frac{180^\circ-\widehat{B}+180^\circ-\widehat{C}-(180^\circ-\widehat{A})}{2}\right)} \implies$$

$$\tan^2(90^\circ - \tfrac{a}{2}) = \frac{\sin(90^\circ + \frac{-\widehat{A}-\widehat{B}+\widehat{C}}{2}).\sin(90^\circ + \frac{-\widehat{A}-\widehat{C}+\widehat{B}}{2})}{\sin(270^\circ + \frac{-\widehat{A}-\widehat{B}-\widehat{C}}{2}).\sin(90^\circ + \frac{-\widehat{B}-\widehat{C}+\widehat{A}}{2})} \Longrightarrow$$

$$\tan^2(90^\circ - \tfrac{a}{2}) = \frac{\sin(90^\circ - \frac{\widehat{A}+\widehat{B}-\widehat{C}}{2}).\sin(90^\circ - \frac{\widehat{A}+\widehat{C}-\widehat{B}}{2})}{\sin(270^\circ - \frac{\widehat{A}+\widehat{B}+\widehat{C}}{2}).\sin(90^\circ - \frac{\widehat{B}+\widehat{C}-\widehat{A}}{2})} \tag{3.2}$$

From equalities (A.8), (A.9) and (A.10) listed in the Appendix A follows $\tan(90^\circ - x) = \frac{1}{\tan(x)}$, $\sin(90^\circ - x) = \cos(x)$ e $\sin(270^\circ - x) = -\cos(x)$, with that Equation (3.2) becomes:

$$\frac{1}{\tan^2(\frac{a}{2})} = \frac{\cos(\frac{\widehat{A}+\widehat{B}-\widehat{C}}{2}).\cos(\frac{\widehat{A}+\widehat{C}-\widehat{B}}{2})}{-\cos(\frac{\widehat{A}+\widehat{B}+\widehat{C}}{2}).\cos(\frac{\widehat{B}+\widehat{C}-\widehat{A}}{2})}$$

That is

$$\tan^2(\tfrac{a}{2}) = -\frac{\cos(\frac{\widehat{A}+\widehat{B}+\widehat{C}}{2}).\cos(\frac{\widehat{B}+\widehat{C}-\widehat{A}}{2})}{\cos(\frac{\widehat{A}+\widehat{B}-\widehat{C}}{2}).\cos(\frac{\widehat{A}+\widehat{C}-\widehat{B}}{2})} \tag{3.3}$$

Considering $\widehat{A} + \widehat{B} + \widehat{C} = 2S$, we have:

$$\widehat{A} + \widehat{B} + \widehat{C} = 2S \implies \widehat{A} + \widehat{C} - \widehat{B} = 2(S - \widehat{B}) \implies \frac{\widehat{A} + \widehat{C} - \widehat{B}}{2} = S - \widehat{B}$$

$$\widehat{A} + \widehat{B} + \widehat{C} = 2S \implies \widehat{B} + \widehat{C} - \widehat{A} = 2(S - \widehat{A}) \implies \frac{\widehat{B} + \widehat{C} - \widehat{A}}{2} = S - \widehat{A}$$

$$\widehat{A} + \widehat{B} + \widehat{C} = 2S \implies \widehat{A} + \widehat{B} - \widehat{C} = 2(S - \widehat{C}) \implies \frac{\widehat{A} + \widehat{B} - \widehat{C}}{2} = S - \widehat{C}$$

With the previous considerations, from Equation (3.3) it is concluded that:

$$\tan^2(\tfrac{a}{2}) = -\frac{\cos(S).\cos(S - \widehat{A})}{\cos(S - \widehat{B}).\cos(S - \widehat{C})}$$

The other formulas can be proved analogously.

□

4 Final Remarks

It is very important and interesting to study Spherical Geometry and Trigonometry due to its great applicability in Astronomy and terrestrial location. While studying this beautiful geometry, we worked out a proof for an important Theorem (different from its classical proof).

This Theorem presents formulas commonly known as The Half-Side Tangent Formulas (or Sailor's Formulas). In this article we presented the classical proof, which is shown in the books, and then we presented our proposal of proof using the Lagrange's Theorem as a support.

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A Well-Known Trigonometric Identities for Plane Geometry

This Appendix presents some well-known trigonometric identities for plane geometry. Such equalities will be constantly requested throughout the text of this study.

$$\cos(a \pm b) = \cos(a).\cos(b) \mp \sin a.\sin(b) \tag{A.1}$$

$$1 - \cos(a) = 2\sin^2\left(\tfrac{a}{2}\right) \tag{A.2}$$

$$1 + \cos(a) = 2\cos^2\left(\tfrac{a}{2}\right) \tag{A.3}$$

$$\cos(a) - \cos(b) = -2\sin\left(\tfrac{a+b}{2}\right).\sin\left(\tfrac{a-b}{2}\right) \tag{A.4}$$

$$\sin(-a) = -\sin(a) \tag{A.5}$$

$$\cos(a) + \cos(b) = 2\cos\left(\tfrac{a+b}{2}\right).\cos\left(\tfrac{a-b}{2}\right) \tag{A.6}$$

$$\cos(-a) = \cos(a) \tag{A.7}$$

$$\tan(90° - a) = \frac{1}{\tan(a)} \tag{A.8}$$

$$\sin(90° - a) = \cos(a) \tag{A.9}$$

$$\sin(270° - a) = -\cos(a) \tag{A.10}$$

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