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SOME CHARACTERIZATIONS OF TRANSLATION SURFACES GENERATED BY SPHERICAL INDICATRICES OF SPACE CURVES

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Abstract In this study, we determine translation surfaces generated by spherical indicatrices of space curves in $\mathbb{E}^3$ and obtain some characterizations based on the fact that such surfaces are flat or minimal. Also, we give some examples of such surfaces by using Mathematica.

1 Introduction

The parameterization of a translation surface is in 3-dimensional Euclidean space is given by :

$$X(u,v) = (u, 0, f(u)) + (0, v, g(v))$$

where f and g are real valued differentiable functions on the open interval. In 1835, H. Scherk was proved that the minimal translation surface, excluding planes, is only the Scherk surface given by the parameterization:

$$X(u,v) = \left(u, v, \frac{1}{c}\log\left|\frac{\cos(cu)}{\cos(cv)}\right|\right), \text{where c is non-zero real constant}$$

The generalized type of a translation surface is the surface shaped by moving α parallel to itself in such a way that a point of the curve moves along β [6]. Therefore, the parameterization of the surface is determined as:

$$X(u,v) = \alpha(u) + \beta(v), \tag{1.1}$$

where α and β are curves given by the parameters u and v, respectively. There are many papers on translation surfaces. Vestraelen et al. studied minimal translation surfaces in n-dimensional Euclidean space [16]. Liu obtained some characterizations about the translation surfaces with constant mean curvature or constant Gauss curvature in 3-dimensional Euclidean space $\mathbb{E}^3$ and 3-dimensional Minkowski space $\mathbb{E}^3_1$ [9]. Muntenau and Nistor study the second fundamental form of the translation surfaces in 3-dimensional Euclidean space $\mathbb{E}^3$ [10] and obtained some characterizations by using the second Gaussian curvature K_{II} of the translation surfaces . Cetin et al. expressed some computations about the translation surface in terms of Frenet vector fields and the curvatures of generator curves of the surface [3, 4]. Cetin et al. studied on parallel surface to translation surfaces in $\mathbb{E}^3$ [5]. Ali et al. gave some results on some special points of the translation surfaces in $\mathbb{E}^3$ [1]. Since the translation surfaces are surfaces produced by two space curves, some basic calculations of the surface can be stated in terms of Frenet vectors and curvatures of the curve. There is a different version of the relation between curve and surface in the studies [11], [12], [13], [14]. In these studies, some special curves lying on the surface are studied and Frenet vectors and curvatures of the curve are expressed in terms of some basic calculations of the surface.

In this paper, we determine translation surfaces generated by tangent, normal and binormal indicatrices of space curves in $\mathbb{E}^3$, respectively and obtain some characterizations based on the fact that such surfaces are flat or minimal. Also we give some examples of such surfaces by using Mathematica.

2 Preliminaries

Let $\varphi : I \subset \mathbb{R} \to \mathbb{E}^3 : s \to \varphi(s)$ be an arbitrary curve given by the arc-length parameter s in $\mathbb{E}^3$. Let $\{t, n, b\}$ and κ, τ be the Frenet vector fields and curvature functions of the curve φ, respectively. There is a relation between the derivatives of Frenet vector fields with respect to arc-length parameter s and themselves as follows:

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix},$$

Definition 2.1. A curve $\varphi : I \subset \mathbb{R} \to \mathbb{E}^3$, with unit speed, is a general helix if there is a constant vector u, so that $\langle t, u\rangle = \cos\theta$ is constant along the curve, where $t(s) = \varphi'(s)$ is a unit tangent vector of φ at s [7].

Theorem 2.2. *A curve $\varphi : I \subset \mathbb{R} \to \mathbb{E}^3$, with unit speed, is a general helix if and only if $\left(\frac{\tau}{\kappa}\right)(s) = constant$. If both of $\kappa(s) \neq 0$ and $\tau(s)$ are constant, then it is called a circular helix [7].*

Definition 2.3. Let φ be a unit speed regular curve in Euclidean 3-space with Frenet vectors t, n and b. The unit tangent vectors along the curve φ generate a curve (t) on the sphere of radius 1 about the origin. The curve (t) is called the spherical indicatrix of t or more commonly, (t) is called tangent indicatrix of the curve φ. If $\varphi = \varphi(s)$ is a natural representation of φ, then $(t) = t(s)$ will be a representation of (t). Similarly one considers the principal normal indicatrix $(n) = n(s)$ and binormal indicatrix $(b) = b(s)$ [15].

Let $M : X = X(u, v) \subset \mathbb{E}^3$ be a regular surface. Then the unit normal vector field of the surface M is determined by

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|},$$

where $X_u = \dfrac{\partial X(u,v)}{\partial u}$, $X_v = \dfrac{\partial X(u,v)}{\partial v}$ are the parameter curves of M and $\times$ denotes the vector product of $\mathbb{E}^3$. The coefficients of the first fundamental form and second fundamental form are given by, respectively as follows:

$$E = \langle X_u, X_u\rangle,\ F = \langle X_u, X_v\rangle,\ G = \langle X_v, X_v\rangle$$

and

$$l = \langle X_{uu}, N\rangle,\ m = \langle X_{uv}, N\rangle,\ n = \langle X_{vv}, N\rangle.$$

Gauss and mean curvatures of the surface M are expressed as follows:

$$K = \frac{ln - m^2}{EG - F^2}, \tag{2.1}$$

$$H = \frac{1}{2}\frac{En + Gl - 2Fm}{EG - F^2}. \tag{2.2}$$

Definition 2.4. If Gauss curvature of a regular surface in $\mathbb{E}^3$ vanishes, the surface is called flat and if its mean curvature vanishes, then the surface is called minimal surface [8].

Definition 2.5. A constant angle surface in $\mathbb{E}^3$ is a surface whose unit normal vector makes a constant angle with an assigned direction field [2].

3 Translation Surfaces Generated by Spherical Indicatrices of Space Curves in Euclidean 3-Space

Let $\alpha : I \to \mathbb{E}^3$ and $\beta : J \to \mathbb{E}^3$ be non-degenerate curves given by arc-length parameters u and v, respectively. Let $\{t_\alpha, n_\alpha, b_\alpha, \kappa_\alpha, \tau_\alpha\}$ and $\{t_\beta, n_\beta, b_\beta, \kappa_\beta, \tau_\beta\}$ be Frenet Apparatus of the curves α and β, respectively. In this section, we investigate the translation surfaces generated by tangent indicatrices, principal normal indicatrices and binormal indicatrices of the curves α and β and obtain some characterizations for such surfaces.

3.1 Translation Surfaces Generated by Tangent Indicatrices of Space Curves in Euclidean 3-Space

Translation surface generated by tangent indicatrices of space curves in $\mathbb{E}^3$ is defined by :

$$M_1 : X(u,v) = t_\alpha(u) + t_\beta(v). \tag{3.1}$$

Calculating the partial derivative with respect to u and v of the translation surface is given by the parametrization (3.1), we obtain

$$X_u = \kappa_\alpha n_\alpha, \quad X_v = \kappa_\beta n_\beta.$$

Hence, the components of the first fundamental form of the surface M_1 are obtained as:

$$E = \kappa_\alpha^2, \tag{3.2}$$
$$F = \kappa_\alpha \kappa_\beta \cos[\phi(u,v)], \tag{3.3}$$
$$G = \kappa_\beta^2. \tag{3.4}$$

Note that $\phi = \phi(u,v)$ is the smooth angle function between n_α and n_β.
In that case, the unit normal vector of the translation surface M_1 is obtained as:

$$N(u,v) = \frac{n_\alpha \times n_\beta}{\sin[\phi(u,v)]}, \tag{3.5}$$

Since the surface M_1 is a regular surface, $\sin[\phi(u,v)] \neq 0$. The principal normal vector of the curve α can be expressed as a linear combination of $\{t_\beta, n_\beta, b_\beta\}$ as:

$$n_\alpha = \mu_1 t_\beta + \mu_2 n_\beta + \mu_3 b_\beta, \tag{3.6}$$

where

$$\begin{aligned} \mu_1 &= \langle n_\alpha, t_\beta\rangle = \sin[\phi(u,v)]\cos[\gamma(u,v)], \\ \mu_2 &= \langle n_\alpha, n_\beta\rangle = \cos[\phi(u,v)], \\ \mu_3 &= \langle n_\alpha, b_\beta\rangle = \sin[\phi(u,v)]\sin[\gamma(u,v)]. \end{aligned} \tag{3.7}$$

Similarly, the principal normal vector of the curve β can be expressed as a linear combination of $\{t_\alpha, n_\alpha, b_\alpha\}$ as:

$$n_\beta = \lambda_1 t_\alpha + \lambda_2 n_\alpha + \lambda_3 b_\alpha, \tag{3.8}$$

where

$$\begin{aligned} \lambda_1 &= \langle n_\beta, t_\alpha\rangle = \sin[\phi(u,v)]\cos[\theta(u,v)], \\ \lambda_2 &= \langle n_\beta, n_\alpha\rangle = \cos[\phi(u,v)], \\ \lambda_3 &= \langle n_\beta, b_\alpha\rangle = \sin[\phi(u,v)]\sin[\theta(u,v)]. \end{aligned} \tag{3.9}$$

We can write the unit normal vector of surface M_1 in two different ways:
By using (3.5) and (3.8), it is determined by

$$N_1 = \sin[\theta(u,v)]t_\alpha - \cos[\theta(u,v)]b_\alpha \tag{3.10}$$

or by combining (3.5) and (3.6), it is given as:

$$N_2 = -\sin[\gamma(u,v)]t_\beta + \cos[\gamma(u,v)]b_\beta. \tag{3.11}$$

Also, the components of the second fundamental form of the surface M_1 are computed as:

$$l = -{\kappa_\alpha}^2 \left[\cos[\theta(u,v)]\frac{\tau_\alpha}{\kappa_\alpha} + \sin[\theta(u,v)]\right], \tag{3.12}$$
$$m = 0, \tag{3.13}$$
$$n = {\kappa_\beta}^2 \left[\cos[\gamma(u,v)]\frac{\tau_\beta}{\kappa_\beta} + \sin[\gamma(u,v)]\right]. \tag{3.14}$$

Proposition 3.1. *The Gaussian curvature K and the mean curvature H of the translation surface M_1 are found as the follows, respectively:*

$$K = -\frac{\Big[\cos[\theta(u,v)]\frac{\tau_\alpha}{\kappa_\alpha} + \sin[\theta(u,v)]\Big]\Big[\cos[\gamma(u,v)]\frac{\tau_\beta}{\kappa_\beta} + \sin[\gamma(u,v)]\Big]}{\sin^2[\phi(u,v)]}, \tag{3.15}$$

$$H = \frac{-\Big[\cos[\theta(u,v)]\frac{\tau_\alpha}{\kappa_\alpha} + \sin[\theta(u,v)]\Big] + \Big[\cos[\gamma(u,v)]\frac{\tau_\beta}{\kappa_\beta} + \sin[\gamma(u,v)]\Big]}{2\sin^2[\phi(u,v)]}. \tag{3.16}$$

Proof. By substituting (3.2), (3.3), (3.4), (3.12), (3.13), (3.14) in (2.1) and (2.2), we obtain (3.15) and (3.16), respectively. □

Theorem 3.2. *If the surface M_1 is flat then*

$$\cos[\theta(u,v)]\frac{\tau_\alpha}{\kappa_\alpha} + \sin[\theta(u,v)] = 0 \ \textit{or} \ \cos[\gamma(u,v)]\frac{\tau_\beta}{\kappa_\beta} + \sin[\gamma(u,v)] = 0. \tag{3.17}$$

Proof. It is obvious from Definition 2.4 and (3.15). □

Theorem 3.3. *If the surface M_1 is flat, then the angle θ is a function that depends only on u or the angle γ is a function that depends only on v.*

Proof. Let the surface M_1 be flat. Then (3.17) holds.
If $\cos[\theta(u,v)]\frac{\tau_\alpha}{\kappa_\alpha} + \sin[\theta(u,v)] = 0$, then $\frac{\tau_\alpha}{\kappa_\alpha} = -\tan[\theta(u,v)]$. Hence the angle θ becomes only a function of u. Similarly, if $\cos[\gamma(u,v)]\frac{\tau_\beta}{\kappa_\beta} + \sin[\gamma(u,v)] = 0$, then $\frac{\tau_\beta}{\kappa_\beta} = -\tan[\gamma(u,v)]$. So, the angle γ becomes only a function of v. □

Theorem 3.4. *Let the surface M_1 be flat. If the curves α and β are helices then the angles θ or γ are constant.*

Proof. We assume that the surface M_1 is flat. In that case the equation (3.17) is satisfied. If $\cos[\theta(u,v)]\frac{\tau_\alpha}{\kappa_\alpha} + \sin[\theta(u,v)] = 0$, then $\frac{\tau_\alpha}{\kappa_\alpha} = -\tan[\theta(u,v)]$. Since α is a helix curve, $\tan[\theta(u,v)]$ becomes constant and it is implies that θ is constant. If $\cos[\gamma(u,v)]\frac{\tau_\beta}{\kappa_\beta} + \sin[\gamma(u,v)] = 0$, then $\frac{\tau_\beta}{\kappa_\beta} = -\tan[\gamma(u,v)]$. Since β is a helix curve, $\tan[\gamma(u,v)]$ is constant. Hence γ becomes a constant angle. □

Theorem 3.5. *Let the surface M_1 be flat. If the curves α and β are planar curves then the angles $\theta = \pi k$ or $\gamma = \pi k$ $(k \in \mathbb{Z})$.*

Proof. Let α and β be planar curves, then $\tau_\alpha = 0$ and $\tau_\beta = 0$. Since the surface M_1 is flat, from (3.17) we obtain that $\sin[\theta(u,v)] = 0$ or $\sin[\gamma(u,v)] = 0$. If $\sin[\theta(u,v)] = 0$, then $\theta = \pi k$, $k \in \mathbb{Z}$. Similarly, if $\sin[\gamma(u,v)] = 0$, then $\gamma = \pi k$, $k \in \mathbb{Z}$. □

Theorem 3.6. *Let the surface M_1 be flat. If the curves α and β are helices, then the surface M_1 is a constant angle surface.*

Proof. We suppose that the surface M_1 is flat and the curves α and β are helices. From Theorem 3.4, $\theta = \theta_0$ or $\gamma = \gamma_0$ are constant angles. Without loss of generality, we suppose that θ is constant. Since α is helix, then there exists a unit constant direction u_α which makes a constant angle with unit tangent vector t_α of the curve α. Then $\langle t_\alpha, u_\alpha\rangle = \cos\delta_0 = constant$. Hence we can define u_α as:

$$u_\alpha = \cos\delta_0 t_\alpha + \sin\delta_0 b_\alpha. \tag{3.18}$$

By using (3.10) and (3.18), we get

$$\begin{aligned}\langle N_1, u_\alpha\rangle &= \sin\theta_0\cos\delta_0 - \cos\theta_0\sin\delta_0 \\ &= constant.\end{aligned}$$

From Definition 2.5 it completes the proof. □

Theorem 3.7. *If the surface M_1 is minimal then*

$$\cos[\theta(u,v)]\frac{\tau_\alpha}{\kappa_\alpha} + \sin[\theta(u,v)] = \cos[\gamma(u,v)]\frac{\tau_\beta}{\kappa_\beta} + \sin[\gamma(u,v)]. \tag{3.19}$$

Proof. It is obvious from Definition 2.4 and (3.16). □

Theorem 3.8. *Let the surface M_1 be minimal. If the curves α and β are planar curves then the angle between n_α and b_β and the angle between b_α and n_β are the same.*

Proof. Let α and β be planar curves, then $\tau_\alpha = 0$ and $\tau_\beta = 0$. Since the surface M_1 is minimal, from (3.19) we have that $\sin[\theta(u,v)] = \sin[\gamma(u,v)]$. In that case (3.7) and (3.9) are equal to each other. □

Example 3.9. Let α and β be curves in $\mathbb{E}^3$ given by

$$\alpha(u) = \left(\cos\left[\frac{u}{\sqrt{5}}\right], \frac{2u}{\sqrt{5}}, \sin\left[\frac{u}{\sqrt{5}}\right]\right),$$

$$\beta(v) = \frac{1}{2}\left(v + \sqrt{1+v^2}, (v+\sqrt{1+v^2})^{-1}, \sqrt{2}\ln(v+\sqrt{1+v^2})\right)$$

where α and β are curves given by the arc-length parameters u and v, respectively. The tangent indicatrices of the curve α and β are as:

$$t_\alpha(u) = \frac{1}{\sqrt{5}}\left(-\sin\left[\frac{u}{\sqrt{5}}\right], 2, \cos\left[\frac{u}{\sqrt{5}}\right]\right)$$

and

$$t_\beta(v) = \frac{1}{2}\left(\frac{v+\sqrt{1+v^2}}{\sqrt{1+v^2}}, -\frac{1}{\sqrt{1+v^2}(v+\sqrt{1+v^2})}, \frac{\sqrt{2}}{\sqrt{1+v^2}}\right).$$

Then the translation surface generated by t_α and t_β tangent indicatrices of space curves is as:

$$M_1(u,v) = \left(-\frac{\sin\left[\frac{u}{\sqrt{5}}\right]}{\sqrt{5}} + \frac{v+\sqrt{1+v^2}}{2\sqrt{1+v^2}}, \frac{2}{\sqrt{5}} - \frac{1}{2\sqrt{1+v^2}(v+\sqrt{1+v^2})}, \frac{\cos\left[\frac{u}{\sqrt{5}}\right]}{\sqrt{5}} + \frac{\sqrt{2}}{2\sqrt{1+v^2}}\right).$$

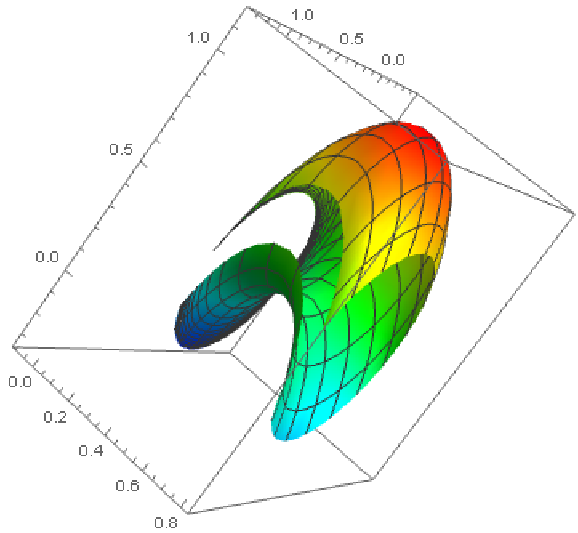

Figure 1. Translation surface generated by tangent indicatrices of space curves

3.2 Translation Surfaces Generated by Principal Normal Indicatrices of Space Curves in Euclidean 3-Space

Translation surface generated by principal normal indicatrices of space curves in $\mathbb{E}^3$ is determined by:

$$M_2 : X(u,v) = n_\alpha(u) + n_\beta(v). \tag{3.20}$$

By calculating the partial derivative with respect to u and v of the translation surface is given by the parametrization (3.20), we obtain

$$X_u = -\kappa_\alpha t_\alpha + \tau_\alpha b_\alpha, \;\; X_v = -\kappa_\beta t_\beta + \tau_\beta b_\beta.$$

The Frenet vector fields of the curve α can be written as a linear combination of $\{t_\beta, n_\beta, b_\beta\}$ as:

$$t_\alpha = \lambda_1 t_\beta + \lambda_2 n_\beta + \lambda_3 b_\beta, \tag{3.21}$$
$$n_\alpha = \lambda_4 t_\beta + \lambda_5 n_\beta + \lambda_6 b_\beta, \tag{3.22}$$
$$b_\alpha = \lambda_7 t_\beta + \lambda_8 n_\beta + \lambda_9 b_\beta. \tag{3.23}$$

Similarly, the Frenet vector fields of the curve β can be written as a linear combination of $\{t_\alpha, n_\alpha, b_\alpha\}$ as:

$$t_\beta = \lambda_1 t_\alpha + \lambda_4 n_\alpha + \lambda_7 b_\alpha, \tag{3.24}$$
$$n_\beta = \lambda_2 t_\alpha + \lambda_5 n_\alpha + \lambda_8 b_\alpha, \tag{3.25}$$
$$b_\beta = \lambda_3 t_\alpha + \lambda_6 n_\alpha + \lambda_9 b_\alpha, \tag{3.26}$$

where

$$\begin{array}{lll} \langle t_\alpha, t_\beta\rangle = \lambda_1, & \langle t_\alpha, n_\beta\rangle = \lambda_2, & \langle t_\alpha, b_\beta\rangle = \lambda_3, \\ \langle n_\alpha, t_\beta\rangle = \lambda_4, & \langle n_\alpha, n_\beta\rangle = \lambda_5, & \langle n_\alpha, b_\beta\rangle = \lambda_6, \\ \langle b_\alpha, t_\beta\rangle = \lambda_7, & \langle b_\alpha, n_\beta\rangle = \lambda_8, & \langle b_\alpha, b_\beta\rangle = \lambda_9. \end{array} \tag{3.27}$$

Hence, the components of the first fundamental form of the surface M_2 are obtained as the following:

$$E = \kappa_\alpha^2 + \tau_\alpha^2, \tag{3.28}$$

$$F = \kappa_\alpha\kappa_\beta\left(\lambda_1 - \frac{\tau_\beta}{\kappa_\beta}\lambda_3\right) - \tau_\alpha\kappa_\beta\left(\lambda_7 - \frac{\tau_\beta}{\kappa_\beta}\lambda_9\right), \tag{3.29}$$

$$G = \kappa_\beta^2 + \tau_\beta^2. \tag{3.30}$$

Then, the unit normal vector of the translation surface M_2 is found as:

$$N(u,v) = \frac{\kappa_\alpha\kappa_\beta\left[(t_\alpha \times t_\beta) - \frac{\tau_\beta}{\kappa_\beta}(t_\alpha \times b_\beta)\right] - \tau_\alpha\kappa_\beta\left[(b_\alpha \times t_\beta) - \frac{\tau_\beta}{\kappa_\beta}(b_\alpha \times b_\beta)\right]}{\sqrt{EG - F^2}}, \tag{3.31}$$

where

$$EG - F^2 = (\kappa_\alpha^2 + \tau_\alpha^2)(\kappa_\beta^2 + \tau_\beta^2) - \left[\kappa_\alpha\kappa_\beta\big(\lambda_1 - \frac{\tau_\beta}{\kappa_\beta}\lambda_3\big) - \tau_\alpha\kappa_\beta\big(\lambda_7 - \frac{\tau_\beta}{\kappa_\beta}\lambda_9\big)\right]^2.$$

By using (3.24), (3.26) and (3.31), the unit normal vector of the surface M_2 is written as:

$$N_1 = \frac{\kappa_\alpha\kappa_\beta\left[\frac{\tau_\alpha}{\kappa_\alpha}\big(\lambda_4 - \frac{\tau_\beta}{\kappa_\beta}\lambda_6\big)t_\alpha - \left[\big(\lambda_7 - \frac{\tau_\beta}{\kappa_\beta}\lambda_9\big) + \frac{\tau_\alpha}{\kappa_\alpha}\big(\lambda_1 - \frac{\tau_\beta}{\kappa_\beta}\lambda_3\big)\right]n_\alpha + \big(\lambda_4 - \frac{\tau_\beta}{\kappa_\beta}\lambda_6\big)b_\alpha\right]}{\sqrt{EG - F^2}}$$

or by using (3.21), (3.23) and (3.31), the unit normal vector of the surface M_2 can be expressed as:

$$N_2 = \frac{\kappa_\alpha\kappa_\beta\left[\frac{\tau_\beta}{\kappa_\beta}\big(\frac{\tau_\alpha}{\kappa_\alpha}\lambda_8 - \lambda_2\big)t_\beta + \left[\big(\lambda_3 + \frac{\tau_\beta}{\kappa_\beta}\lambda_1\big) - \frac{\tau_\alpha}{\kappa_\alpha}\big(\lambda_9 + \frac{\tau_\beta}{\kappa_\beta}\lambda_7\big)\right]n_\beta + \big(\frac{\tau_\alpha}{\kappa_\alpha}\lambda_8 - \lambda_2\big)b_\beta\right]}{\sqrt{EG - F^2}},$$

Also, the components of the second fundamental form of the surface M_2 are computed as:

$$l = \frac{\kappa_\alpha\kappa_\beta}{\sqrt{EG - F^2}}\left[\kappa_\alpha\Big(\frac{\tau_\alpha}{\kappa_\alpha}\Big)'\big(\lambda_4 - \frac{\tau_\beta}{\kappa_\beta}\lambda_6\big) - \big(\kappa_\alpha{}^2 + \tau_\alpha{}^2\big)\Big[\big(\frac{\tau_\beta}{\kappa_\beta}\lambda_9 - \lambda_7\big) + \frac{\tau_\alpha}{\kappa_\alpha}\big(\frac{\tau_\beta}{\kappa_\beta}\lambda_3 - \lambda_1\big)\Big]\right] \tag{3.32}$$

$$m = 0, \tag{3.33}$$

$$n = -\frac{\kappa_\alpha\kappa_\beta}{\sqrt{EG - F^2}}\left[\kappa_\beta\Big(\frac{\tau_\beta}{\kappa_\beta}\Big)'\big(\lambda_2 - \frac{\tau_\alpha}{\kappa_\alpha}\lambda_8\big) + \big(\kappa_\beta{}^2 + \tau_\beta{}^2\big)\Big[\big(\lambda_3 + \frac{\tau_\beta}{\kappa_\beta}\lambda_1\big) - \frac{\tau_\alpha}{\kappa_\alpha}\big(\lambda_9 + \frac{\tau_\beta}{\kappa_\beta}\lambda_7\big)\Big]\right] \tag{3.34}$$

Proposition 3.10. *The Gaussian curvature K and the mean curvature H of the translation surface M_2 are obtained as follows, respectively:*

$$K = -\frac{\kappa_\alpha{}^2\kappa_\beta{}^2}{(\sqrt{EG - F^2})^4}\begin{pmatrix} \kappa_\alpha\Big(\frac{\tau_\alpha}{\kappa_\alpha}\Big)'\big(\lambda_4 - \frac{\tau_\beta}{\kappa_\beta}\lambda_6\big) \\ -\big(\kappa_\alpha{}^2 + \tau_\alpha{}^2\big)\big(\frac{\tau_\beta}{\kappa_\beta}\lambda_9 - \lambda_7\big) \\ -\big(\kappa_\alpha{}^2 + \tau_\alpha{}^2\big)\frac{\tau_\alpha}{\kappa_\alpha}\big(\frac{\tau_\beta}{\kappa_\beta}\lambda_3 - \lambda_1\big) \end{pmatrix}\begin{pmatrix} \kappa_\beta\Big(\frac{\tau_\beta}{\kappa_\beta}\Big)'\big(\lambda_2 - \frac{\tau_\alpha}{\kappa_\alpha}\lambda_8\big) \\ +\big(\kappa_\beta{}^2 + \tau_\beta{}^2\big)\big(\lambda_3 + \frac{\tau_\beta}{\kappa_\beta}\lambda_1\big) \\ -\big(\kappa_\beta{}^2 + \tau_\beta{}^2\big)\frac{\tau_\alpha}{\kappa_\alpha}\big(\lambda_9 + \frac{\tau_\beta}{\kappa_\beta}\lambda_7\big) \end{pmatrix}, \tag{3.35}$$

$$H = \frac{\kappa_\alpha\kappa_\beta}{2(\sqrt{EG - F^2})^3}\begin{pmatrix} \kappa_\alpha\Big(\frac{\tau_\alpha}{\kappa_\alpha}\Big)'\big(\kappa_\beta{}^2 + \tau_\beta{}^2\big)\big(\lambda_4 - \frac{\tau_\beta}{\kappa_\beta}\lambda_6\big) \\ -\big(\kappa_\alpha{}^2 + \tau_\alpha{}^2\big)\big(\kappa_\beta{}^2 + \tau_\beta{}^2\big)\big(\frac{\tau_\beta}{\kappa_\beta}\lambda_9 - \lambda_7\big) \\ -\big(\kappa_\alpha{}^2 + \tau_\alpha{}^2\big)\big(\kappa_\beta{}^2 + \tau_\beta{}^2\big)\frac{\tau_\alpha}{\kappa_\alpha}\big(\frac{\tau_\beta}{\kappa_\beta}\lambda_3 - \lambda_1\big) \\ -\kappa_\beta\Big(\frac{\tau_\beta}{\kappa_\beta}\Big)'\big(\kappa_\alpha{}^2 + \tau_\alpha{}^2\big)\big(\lambda_2 - \frac{\tau_\alpha}{\kappa_\alpha}\lambda_8\big) \\ -\big(\kappa_\alpha{}^2 + \tau_\alpha{}^2\big)\big(\kappa_\beta{}^2 + \tau_\beta{}^2\big)\big(\lambda_3 + \frac{\tau_\beta}{\kappa_\beta}\lambda_1\big) \\ +\big(\kappa_\alpha{}^2 + \tau_\alpha{}^2\big)\big(\kappa_\beta{}^2 + \tau_\beta{}^2\big)\frac{\tau_\alpha}{\kappa_\alpha}\big(\lambda_9 + \frac{\tau_\beta}{\kappa_\beta}\lambda_7\big) \end{pmatrix}. \tag{3.36}$$

Proof. By substituting (3.28), (3.29), (3.30), (3.32), (3.33), (3.34) in (2.1) and (2.2), we obtain (3.35) and (3.36), respectively. □

Theorem 3.11. *If the surface M_2 is flat then*

$$\kappa_\alpha\Big(\frac{\tau_\alpha}{\kappa_\alpha}\Big)'\big(\lambda_4-\frac{\tau_\beta}{\kappa_\beta}\lambda_6\big)-\big({\kappa_\alpha}^2+{\tau_\alpha}^2\big)\Big[\big(\frac{\tau_\beta}{\kappa_\beta}\lambda_9-\lambda_7\big)+\frac{\tau_\alpha}{\kappa_\alpha}\big(\frac{\tau_\beta}{\kappa_\beta}\lambda_3-\lambda_1\big)\Big]=0$$

or

$$\kappa_\beta\Big(\frac{\tau_\beta}{\kappa_\beta}\Big)'\big(\lambda_2-\frac{\tau_\alpha}{\kappa_\alpha}\lambda_8\big)+\big({\kappa_\beta}^2+{\tau_\beta}^2\big)\Big[\big(\lambda_3+\frac{\tau_\beta}{\kappa_\beta}\lambda_1\big)-\frac{\tau_\alpha}{\kappa_\alpha}\big(\lambda_9+\frac{\tau_\beta}{\kappa_\beta}\lambda_7\big)\Big]=0. \tag{3.37}$$

Proof. It is obvious from Definition 2.4 and (3.35). □

Theorem 3.12. *Let the surface M_2 be flat. If the curves α and β are planar curves then t_α and b_β are orthogonal or b_α and t_β are orthogonal.*

Proof. Let α and β be planar curves. Then $\tau_\alpha=0$ and $\tau_\beta=0$. Since the surface M_2 is flat from (3.37) we obtain that $\lambda_3=0$ or $\lambda_7=0$. If $\lambda_3=0$, from (3.27) $\langle t_\alpha,b_\beta\rangle=0$. Then it implies that t_α and b_β are orthogonal. Similarly, if $\lambda_7=0$, then from (3.27) $\langle b_\alpha,t_\beta\rangle=0$. Then it implies that b_α and t_β are orthogonal. □

Theorem 3.13. *If the surface M_2 is minimal then*

$$\begin{pmatrix} \kappa_\alpha\Big(\frac{\tau_\alpha}{\kappa_\alpha}\Big)'\big({\kappa_\beta}^2+{\tau_\beta}^2\big)\big(\lambda_4-\frac{\tau_\beta}{\kappa_\beta}\lambda_6\big)\\ -\big({\kappa_\alpha}^2+{\tau_\alpha}^2\big)\big({\kappa_\beta}^2+{\tau_\beta}^2\big)\big(\frac{\tau_\beta}{\kappa_\beta}\lambda_9-\lambda_7\big)\\ -\big({\kappa_\alpha}^2+{\tau_\alpha}^2\big)\big({\kappa_\beta}^2+{\tau_\beta}^2\big)\frac{\tau_\alpha}{\kappa_\alpha}\big(\frac{\tau_\beta}{\kappa_\beta}\lambda_3-\lambda_1\big)\\ -\kappa_\beta\Big(\frac{\tau_\beta}{\kappa_\beta}\Big)'\big({\kappa_\alpha}^2+{\tau_\alpha}^2\big)\big(\lambda_2-\frac{\tau_\alpha}{\kappa_\alpha}\lambda_8\big)\\ -\big({\kappa_\alpha}^2+{\tau_\alpha}^2\big)\big({\kappa_\beta}^2+{\tau_\beta}^2\big)\big(\lambda_3+\frac{\tau_\beta}{\kappa_\beta}\lambda_1\big)\\ +\big({\kappa_\alpha}^2+{\tau_\alpha}^2\big)\big({\kappa_\beta}^2+{\tau_\beta}^2\big)\frac{\tau_\alpha}{\kappa_\alpha}\big(\lambda_9+\frac{\tau_\beta}{\kappa_\beta}\lambda_7\big) \end{pmatrix}=0. \tag{3.38}$$

Proof. It is obvious from Definition 2.4 and (3.36). □

Theorem 3.14. *If the surface M_2 is minimal and the curves α and β are planar curves then the angle between t_α and b_β and the angle between b_α and t_β are the same.*

Proof. Let α and β be planar curves, then $\tau_\alpha=0$ and $\tau_\beta=0$. Since the surface M_2 is minimal from (3.38) we obtain $\lambda_3=\lambda_7$. Hence, from (3.27) $\langle t_\alpha,b_\beta\rangle=\langle b_\alpha,t_\beta\rangle$. □

Example 3.15. Let α and β be curves in $\mathbb{E}^3$ given by

$$\alpha(u)=\frac{1}{\sqrt{5}}\Big(\sqrt{1+u^2},2u,\ln(u+\sqrt{1+u^2})\Big),$$

$$\beta(v)=\Big(\frac{5}{13}\cos[v],\frac{8}{13}-\sin[v],-\frac{12}{13}\cos[v]\Big)$$

where α and β are curves given by the arc-length parameters u and v, respectively. The principal normal indicatrices of the curve α and β are as follows:

$$n_\alpha(u)=\Big(\frac{1}{\sqrt{1+u^2}},0,-\frac{u}{\sqrt{1+u^2}}\Big)$$

and

$$n_\beta(v) = \Big(-\frac{5}{13}\cos[v], \sin[v], \frac{12}{13}\cos[v] \Big).$$

Then the translation surface generated by n_α and n_β principal normal indicatrices of space curves is as:

$$M_2(u,v) = \Big(\frac{1}{\sqrt{1+u^2}} - \frac{5}{13}\cos[v], \sin[v], -\frac{u}{\sqrt{1+u^2}} + \frac{12}{13}\cos[v] \Big).$$

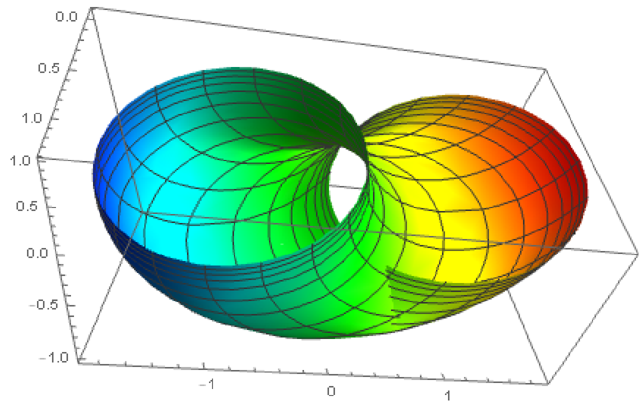

Figure 2. Translation surface generated by principal normal indicatrices of space curves

3.3 Translation Surfaces Generated by Binormal Indicatrices of Space Curves in Euclidean 3-Space

Translation surface generated by binormal indicatrices of non-planar space curves in $\mathbb{E}^3$ is determined by :

$$M_3 : X(u,v) = b_\alpha(u) + b_\beta(v). \tag{3.39}$$

Calculating the partial derivative with respect to u and v of the translation surface is given by the parametrization (3.39), we obtain

$$X_u = -\tau_\alpha n_\alpha, \quad X_v = -\tau_\beta n_\beta.$$

Hence, the components of the first fundamental form of the surface M_3 are obtained as the following:

$$E = \tau_\alpha^2, \tag{3.40}$$

$$F = \tau_\alpha \tau_\beta \cos[\theta(u,v)], \tag{3.41}$$

$$G = \tau_\beta^2. \tag{3.42}$$

Note that $\theta = \theta(u,v)$ is the smooth angle function between n_α and n_β.
Then, the unit normal vector of the translation surface M_3 is given by the parametrization (3.39) is as:

$$N(u,v) = \frac{n_\alpha \times n_\beta}{\sin[\theta(u,v)]}, \tag{3.43}$$

Since the surface M_3 is a regular surface, $\sin[\theta(u,v)] \neq 0$. Principal normal vector of the curve α can be expressed as a linear combination of $\{t_\beta, n_\beta, b_\beta\}$ as:

$$n_\alpha = \mu_1 t_\beta + \mu_2 n_\beta + \mu_3 b_\beta, \tag{3.44}$$

where

$$\begin{aligned}\mu_1 &= \langle n_\alpha, t_\beta \rangle = \sin[\theta(u,v)]\cos[\phi(u,v)],\\ \mu_2 &= \langle n_\alpha, n_\beta \rangle = \cos[\theta(u,v)],\\ \mu_3 &= \langle n_\alpha, b_\beta \rangle = \sin[\theta(u,v)]\sin[\phi(u,v)].\end{aligned} \tag{3.45}$$

Similarly, the principal normal vector of the curve β can be expressed as a linear combination of $\{t_\alpha, n_\alpha, b_\alpha\}$ as:

$$n_\beta = \lambda_1 t_\alpha + \lambda_2 n_\alpha + \lambda_3 b_\alpha, \tag{3.46}$$

where

$$\begin{aligned}\lambda_1 &= \langle n_\beta, t_\alpha \rangle = \sin[\theta(u,v)]\cos[\gamma(u,v)],\\ \lambda_2 &= \langle n_\beta, n_\alpha \rangle = \cos[\theta(u,v)],\\ \lambda_3 &= \langle n_\beta, b_\alpha \rangle = \sin[\theta(u,v)]\sin[\gamma(u,v)].\end{aligned} \tag{3.47}$$

We can have the unit normal vector of the surface M_3 in two different ways:
By using (3.43) and (3.46), it is obtained by

$$N_1 = \sin[\gamma(u,v)]t_\alpha - \cos[\gamma(u,v)]b_\alpha \tag{3.48}$$

or by combining (3.43) and (3.44), it is found as:

$$N_2 = -\sin[\phi(u,v)]t_\beta + \cos[\phi(u,v)]b_\beta \tag{3.49}$$

Also, the components of the second fundamental form of the surface M_3 are computed as follows:

$$l = \kappa_\alpha \tau_\alpha \left[\cos[\gamma(u,v)]\frac{\tau_\alpha}{\kappa_\alpha} + \sin[\gamma(u,v)]\right], \tag{3.50}$$

$$m = 0, \tag{3.51}$$

$$n = -\kappa_\beta \tau_\beta \left[\cos[\phi(u,v)]\frac{\tau_\beta}{\kappa_\beta} + \sin[\phi(u,v)]\right]. \tag{3.52}$$

Proposition 3.16. *The Gaussian curvature K and the mean curvature H of the translation surface M_3 are obtained as, respectively:*

$$K = -\frac{\kappa_\alpha\kappa_\beta\left[\cos[\gamma(u,v)]\frac{\tau_\alpha}{\kappa_\alpha} + \sin[\gamma(u,v)]\right]\left[\cos[\phi(u,v)]\frac{\tau_\beta}{\kappa_\beta} + \sin[\phi(u,v)]\right]}{\tau_\alpha\tau_\beta\sin^2\theta}, \tag{3.53}$$

$$H = \frac{\kappa_\alpha\kappa_\beta\left[\frac{\tau_\beta}{\kappa_\beta}\left[\cos[\gamma(u,v)]\frac{\tau_\alpha}{\kappa_\alpha} + \sin[\gamma(u,v)]\right] - \frac{\tau_\alpha}{\kappa_\alpha}\left[\cos[\phi(u,v)]\frac{\tau_\beta}{\kappa_\beta} + \sin[\phi(u,v)]\right]\right]}{2\tau_\alpha\tau_\beta\sin^2\theta}. \tag{3.54}$$

Proof. By substituting (3.40), (3.41), (3.42), (3.50), (3.51), (3.52) in (2.1) and (2.2), we get (3.53) and (3.54), respectively. □

Theorem 3.17. *If the surface M_3 is flat then*

$$\cos[\gamma(u,v)]\frac{\tau_\alpha}{\kappa_\alpha} + \sin[\gamma(u,v)] = 0 \;\; \textit{or} \; \cos[\phi(u,v)]\frac{\tau_\beta}{\kappa_\beta} + \sin[\phi(u,v)] = 0. \tag{3.55}$$

Proof. It is obvious from Definition 2.4 and (3.53). □

Theorem 3.18. *If the surface M_3 is flat, then the angle γ is a function that depends only on u or the angle ϕ is a function that depends only on v.*

Proof. Let the surface M_3 be flat. Then (3.55) holds. If $\cos[\gamma(u,v)]\frac{\tau_\alpha}{\kappa_\alpha}+\sin[\gamma(u,v)]=0,$ then $\frac{\tau_\alpha}{\kappa_\alpha}=-\tan[\gamma(u,v)]$. Hence the angle γ is only a function of u. Similarly, if $\cos[\phi(u,v)]\frac{\tau_\beta}{\kappa_\beta}+\sin[\phi(u,v)]=0$, then $\frac{\tau_\beta}{\kappa_\beta}=-\tan[\phi(u,v)]$. So, the angle ϕ depends only on v. □

Theorem 3.19. *Let the surface M_3 be flat. If the curves α and β are helices, then the angles γ or ϕ are constant.*

Proof. We assume that the surface M_3 is flat. In that case the equation (3.55) is satisfied. If $\cos[\gamma(u,v)]\frac{\tau_\alpha}{\kappa_\alpha}+\sin[\gamma(u,v)]=0$, then $\frac{\tau_\alpha}{\kappa_\alpha}=-\tan[\gamma(u,v)]$. Since α is a helix curve, $\tan[\gamma(u,v)]$ becomes constant and it is implies that γ is constant. If $\cos[\phi(u,v)]\frac{\tau_\beta}{\kappa_\beta}+\sin[\phi(u,v)]=0$, then $\frac{\tau_\beta}{\kappa_\beta}=-\tan[\phi(u,v)]$. Since β is a helix curve, $\tan[\phi(u,v)]$ is constant. Hence ϕ becomes a constant angle. □

Theorem 3.20. *Let the surface M_3 be flat. If the curves α and β are helices, then the surface M_3 is a constant angle surface.*

Proof. We suppose that the surface M_3 is flat and the curves α and β are helices. From Theorem 3.19, $\gamma=\gamma_0$ or $\phi=\phi_0$ are constant angles. Without loss of generality, we assume that γ is constant. Since α is helix, then there exists a unit constant direction u_α which makes a constant angle with unit tangent vector t_α of the curve α. Then $\langle t_\alpha,u_\alpha\rangle=\cos\psi_0=constant$. Hence we can define u_α as:

$$u_\alpha=\cos\psi_0 t_\alpha+\sin\psi_0 b_\alpha. \tag{3.56}$$

By using (3.48) and (3.56), we get

$$\begin{aligned}\langle N_1,u_\alpha\rangle &= \sin\gamma_0\cos\psi_0-\cos\gamma_0\sin\psi_0\\ &= constant.\end{aligned}$$

From Definition 2.5 it completes the proof. □

Theorem 3.21. *If the surface M_3 is minimal then*

$$\frac{\tau_\beta}{\kappa_\beta}\left[\cos[\gamma(u,v)]\frac{\tau_\alpha}{\kappa_\alpha}+\sin[\gamma(u,v)]\right]=\frac{\tau_\alpha}{\kappa_\alpha}\left[\cos[\phi(u,v)]\frac{\tau_\beta}{\kappa_\beta}+\sin[\phi(u,v)]\right]. \tag{3.57}$$

Proof. It is obvious from Definition 2.4 and (3.54). □

Example 3.22. Let α and β be curves in $\mathbb{E}^3$ given by

$$\alpha(u)=\left(1+\frac{u}{\sqrt{3}}\right)\left(\cos\left[\ln\left(1+\frac{u}{\sqrt{3}}\right)\right],\sin\left[\ln\left(1+\frac{u}{\sqrt{3}}\right)\right],1\right),$$

$$\beta(v)=\frac{1}{2}\left(v+\sqrt{1+v^2},(v+\sqrt{1+v^2})^{-1},\sqrt{2}\ln(v+\sqrt{1+v^2})\right)$$

where α and β are curves given by the arc-length parameters u and v, respectively. The binormal indicatrices of the curve α and β are as follows:

$$b_\alpha(u)=\frac{1}{\sqrt{6}}\left(\sin\left[\ln\left(1+\frac{u}{\sqrt{3}}\right)\right]-\cos\left[\ln\left(1+\frac{u}{\sqrt{3}}\right)\right],-\sin\left[\ln\left(1+\frac{u}{\sqrt{3}}\right)\right]-\cos\left[\ln\left(1+\frac{u}{\sqrt{3}}\right)\right],2\right)$$

and

$$b_\beta(v) = \frac{1}{2}\left(-\frac{1}{\sqrt{1+v^2}(v+\sqrt{1+v^2})}, \frac{v+\sqrt{1+v^2}}{\sqrt{1+v^2}}, \frac{\sqrt{2}}{\sqrt{1+v^2}} \right).$$

Then the translation surface generated by b_α and b_β binormal indicatrices of space curves is as:

$$M_3(u,v) = \begin{pmatrix} \frac{\sin\left[\ln\left(1+\frac{u}{\sqrt{3}}\right)\right] - \cos\left[\ln\left(1+\frac{u}{\sqrt{3}}\right)\right]}{\sqrt{6}} - \frac{1}{2\sqrt{1+v^2}(v+\sqrt{1+v^2})}, \\ \frac{-\sin\left[\ln\left(1+\frac{u}{\sqrt{3}}\right)\right] - \cos\left[\ln\left(1+\frac{u}{\sqrt{3}}\right)\right]}{\sqrt{6}} + \frac{v+\sqrt{1+v^2}}{2\sqrt{1+v^2}}, \\ \frac{2}{\sqrt{6}} + \frac{\sqrt{2}}{2\sqrt{1+v^2}} \end{pmatrix}.$$

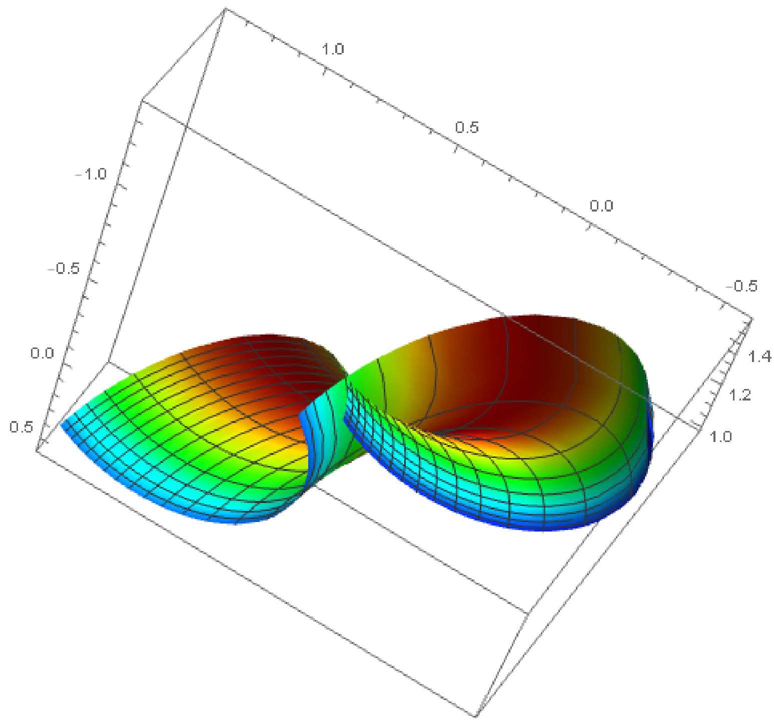

Figure 3. Translation surface generated by binormal indicatrices of space curves

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