Palestine Journal of Mathematics

Vol. 11(1)(2022) , 47–56

PARTIAL WEAK STABILIZATION OF AN UNBOUNDED DISTRIBUTED BILINEAR SYSTEM

R. El Ayadi

Communicated by Najib Mahdou

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.

Keywords and phrases: Partial weak stabilization, feedback stabilization, relatively bounded operator, distributed bilinear systems.

Abstract In this paper, we study the weak stabilization of distributed homogeneous unbounded bilinear system $\frac{\partial z(t)}{\partial t} = Az(t) + v(t)Bz(t)$; where the operator A is the infinitesimal generator of a linear semigroup of contractions on real Hilbert space. The control operator B is supposed linear bounded with respect to the graph norm of A. We propose a family of feedback controls that ensure the partial weak stabilization of parabolic and hyperbolic systems. Illustrating examples are provided.

1 Introduction

In this paper, we deal with the following infinite-dimensional bilinear system:

$$\frac{\partial z(t)}{\partial t} = Az(t) + v(t)Bz(t),\ z(0) = z_0 \in H, \tag{1.1}$$

The real-valued function $v(\cdot)$ is the control and $z(t)$ is the corresponding mild solution of (1.1). The unbounded operator A is the infinitesimal generator of a linear C_0-semigroup of contractions $(S(t))_{t\geq 0}$ on a real Hilbert space H whose norm and inner product are denoted respectively by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. The unbounded linear control operator $B : D(B) \to H$ is A-bounded (a.k.a. relatively bounded w.r.t. A) (see e.g. [16, 8, 5]); in the sense that $D(A) \subset D(B)$ and there exist constants $\alpha,\ \beta \in \mathbb{R}^+$ such that

$$\|Bz\| \leq \alpha\|Az\| + \beta\|z\|,\ \forall z \in D(A)\cdot \tag{1.2}$$

Many authors treated the stabilization of the unbounded bilinear system by nonlinear feedback control (see, e.g., [12, 1, 7, 2, 6]). However, in [6], the authors consider the case when the control operator B is A-bounded. They have provided sufficient conditions for strong and weak stabilizations of the system (1.1) in parabolic and hyperbolic cases by the following bounded control:

$$v(t) = -\frac{\varrho\langle Bz(t), z(t)\rangle}{1 + |\langle Bz(t), z(t)\rangle|},\quad \text{where}\ \ \varrho > 0. \tag{1.3}$$

The concept of partial stability, that is, stability with respect to a part of the system's states arises in the study of many engineering systems, such as flexible structures with elastic beams and plates, satellite with moving masses, combustion systems, biocenology (see, eg., [22, 21, 18]). For instance, using the predators-prey model of Lotka-Volterra, the authors in [18] showed that, if a part of prey is isolated, then the corresponding population increases without bound; in contrast, a subset of the prey species remains stable.

The problem of partial stabilization of finite and infinite systems has been studied by many authors (see, e.g., [14, 19, 21, 22, 15]). The authors used different methods, e.g., LaSalle's invariance principle, Hamilton's principle, and Lyapunov's method to establish sufficient conditions ensuring partial stabilization. In the study [9], the authors have considered the bilinear system (1.1), where the linear operator A generates a C_0-semigroup of contractions on H, the control operator B is bounded linear compact and the output operator C is bounded linear from a real

Hilbert space H to a Banach space Y. The authors showed only that the feedback (1.3), partially weakly stabilizes the system (1.1) provided that the following assumption holds:

$$\langle BS(t)y, S(t)y\rangle = 0, \quad \text{for all } \ t \geq 0 \Longrightarrow Cy = 0. \tag{1.4}$$

In this work, we extend the results of stabilization developed in [9, 6] to address the problem of the unbounded partial weak stabilization of the system (1.1) in parabolic and hyperbolic cases. Precisely we consider the case where the unbounded linear operator A generates a C_0-semigroup of contractions, the unbounded linear control operators B is A-bounded, the nonlinear output operator $C : D(C) \subset H \to Y$ is unbounded from its domain $D(C)$ to a Banach space Y. We propose a family of controls that ensure the well-posedness and the partial stabilization of the bilinear system (1.1).

The plan of this paper is as follows. The next section provides the basic material for unbounded operators, nonlinear semigroups, and the definition of partial weak stabilization. The third section focuses on the well-posedness of the closed-loop systems. In the fourth section, we present our main results, and we study the problem of partial weak stabilization. In the last section, we give illustrating examples covering the parabolic and hyperbolic cases.

2 Review on nonlinear semigroups and unbounded operators

Let a linear operator $A : D(A) \subset H \to H$ be the infinitesimal generator of a C_0-semigroup. If B is A-bounded, then the operator $B|_{D(A)}$ (restriction of B to $D(A)$) admits a $A-$extension denoted by $\tilde{B}$ defined in the following Definition.

Definition 2.1. [11, 20] Let a linear operator $A : D(A) \subset H \to H$ be the infinitesimal generator of a C_0-semigroup $(S(t))$, and let the operator $D : D(A) \to H$. The operator $\widetilde{D}$ defined by:

$$\widetilde{D}x = \lim_{\lambda \to +\infty} \lambda DR(\lambda, A)x, \ \forall x \in D(\widetilde{D}) := \{x \in H \ / \ \lim_{\lambda \to +\infty} \lambda DR(\lambda, A)x \text{ exists}\},$$

where $R(\lambda, A)$ is the resolvent operator of A, is called the Yosida extension (a.k.a. the A-extension) of D.

Remark 2.2. It is clear that, $\widetilde{B}$ is an extension of $B|_{D(A)}$, indeed $D(A) \subset D\left(\widetilde{B}\right)$ and $\widetilde{B} = B|_{D(A)}$ on $D(A)$.

Now let us recall the following technical Theorem, which are useful to establish some results of partial weak stabilization.

Theorem 2.3. *[11, 6] Let a linear operator $A : D(A) \subset H \to H$ be the infinitesimal generator of a C_0-semigroup $(S(t))$. If B is a A-bounded operator such that $\|BS(t_0)\|$ is bounded on $D(A)$ for some $t_0 > 0$, then the A-extension $\widetilde{B}$ of $B|_{D(A)}$ satisfies $S(t)H \subset D(\widetilde{B})$, for all $t \geq t_0$.*

Let us now recall the notion of nonlinear semigroups.

Definition 2.4. [17] Let H be a Hilbert space. A strongly continuous semigroup $(T(t))_{t\geq 0}$ (eventually nonlinear) on H is a family of continuous maps $T(t) : H \longrightarrow H$, satisfying

(i) $T(0) =$ identity,

(ii) $T(t + s) = T(t)T(s)$, for all $t, s \in \mathbb{R}^+$,

(iii) the function $t \to S(t)x$ is continuous in $t \geq 0$ for each $x \in H$.

If in addition $\|T(t)y_1 - T(t)y_2\| \leq \|y_1 - y_2\|$, for every $t \geq 0$ and $y_1, y_2 \in H$, then $T(t)$ is said to be a contraction semigroup on H.

Remark 2.5. the infinitesimal generator of the contraction semigroup $T(t)$ defined by:

$$\mathcal{A}y = \lim_{h \to 0^+} \frac{T(h)y - y}{h}, \ \text{for all } y \in D(\mathcal{A}) := \{y \in H \ / \ \lim_{h \to 0^+} \frac{T(h)y - y}{h} \text{ exists in } H\},$$

is dissipative, i.e, $\langle \mathcal{A}y_1 - \mathcal{A}y_2, y_1 - y_2\rangle \leq 0$, for all $y_1, y_2 \in D(\mathcal{A})$ (see [17]).

For $\phi \in H$, the weak $\omega-$limit set of ϕ is the (possibly empty) set defined by $\omega_w(\phi) = \{\psi \in H;\ \text{there exists a sequence } t_n \to +\infty,\ \text{such that } T(t_n)\phi \rightharpoonup \psi,\ \text{as } n \to +\infty\}$. Recall that the set $\omega_w(\phi)$ isinvariant under the action of any contraction semigroup $(T(t))_{t\geq 0}$ (see [17]).

Let $C : D(C) \to Y$ be an unbounded (eventually nonlinear) operator with domain $D(C) \supset D(A)$, where the output state-space Y is a Banach space equipped with the norm $\|.\|_Y$.
Let the following closed-loop system:

$$\frac{\partial z(t)}{\partial t} = Az(t) + v(t)Bz(t),\ z(0) = z_0 \in H, \tag{2.1}$$

where the feedback control $v(t) = h(z(t))$, with $h : D(A) \subset H \to \mathbb{R}$. We are now ready to present the definition of weak partial stability.

Definition 2.6. An equilibrium point α of the closed-loop system (2.1) is said to be partially weakly stable with respect to C if the following properties are satisfied:

(i) for each $z_0 \in H$ there exists a unique mild solution $z(t)$ defined for all $t \geq 0$ such that $z(0) = z_0$,

(ii) the equilibrium point α of (1.1) is Lyapunov stable:
for any $\varepsilon > 0$, there is a number $\delta(\varepsilon) > 0$ such that for any $z_0 \in H$, $\|z_0 - \alpha\| \leq \delta(\varepsilon)$, we have $\|z(t) - \alpha\|_Y \leq \varepsilon$, for all $t \geq 0$,

(iii) there is a number $\eta > 0$, such that, for any $z_0 \in D(A)$ with $\|z_0 - \alpha\| \leq \eta$, we have $Cz(t) \rightharpoonup C\alpha$, as $t \to +\infty$.

In this case, we also say that the feedback $v(t) = h(z(t))$ partially weakly stabilizes the equilibrium point α.

Remark 2.7. Let ω be a nonempty subregion of Ω if, in the above definition, we consider the output operator $C = \chi_\omega$. We retrieve the notion of regional stability of the closed-loop systems.

3 Considered systems and well-posedness

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a nonnegative nondecreasing continuous function. The purpose of this section is to study the partial stabilization of the system (1.1) using the control:

$$v(t) = -f(\langle Bz(t), z(t)\rangle), \tag{3.1}$$

where z is the solution of the corresponding closed-loop system, i.e.,

$$\frac{\partial z(t)}{\partial t} = \mathcal{A}z(t), \tag{3.2}$$

where $\mathcal{A}y = Ay - f(\langle By, y\rangle)By,\ \forall y \in D(\mathcal{A}) = D(A)\cdot$

In the sequel, we will analyze the well-posedness of the system (3.2).

Theorem 3.1. *Let A generate a semigroup $S(t)$ of contractions on H, and let $B : D(B) \longrightarrow H$ be a linear A-bounded operator such that:*

(i) $\langle B\xi_1, \xi_2\rangle = \langle \xi_1, B\xi_2\rangle$, $\forall\ \xi_1, \xi_2 \in D(A)$,

(ii) $\langle B\xi_1, \xi_1\rangle \geq 0$, $\forall\ \xi_1 \in D(A)$,

(iii) the function $f : \mathbb{R}^+ \to \mathbb{R}$ is nonnegative nondecreasing continuous,

(iv) the operator $V : D(A) \to \mathbb{R}$ defined by $V(\xi) = f(\langle B\xi, \xi\rangle)$ is bounded by $K \in [0, \frac{1}{\alpha})$ (where α is the constant given in (1.2))$\cdot$

Then for all $z_0 \in H$, the system (3.2) admits a unique solution $z \in \mathcal{C}([0, +\infty[; H)$ given by $z(t) = e^{t\mathcal{A}}z_0\cdot$. Furthermore, $\mathcal{A}$ generates a contraction semigroup $e^{t\mathcal{A}}$ on H.

Proof. Let us set $\varphi(\xi) = \langle B\xi, \xi\rangle$, $\forall \xi \in D(A)$ and let us consider the map :

$$\phi = g(\varphi), \text{ with } g(z) = \frac{1}{2}\int_0^z f(w)dw.$$

Let ϕ', $\varphi' : D(A) \to H$ are respectively the Gâteaux derivatives of ϕ and φ. Since B is self-adjoint, then for all $\xi \in D(A)$, we have

$$\phi'(\xi) = f(\langle B\xi, \xi\rangle)B\xi, \text{ for all } \xi \in D(A)\cdot \tag{3.3}$$

Since $B = B^* \geq 0$ on $D(A)$, then for all $(\xi_1, \xi_2) \in D(A)^2$, we have:

$$\mu^2\varphi(\xi_2) + 2\mu < B\xi_1, \xi_2 > +\varphi(\xi_1) = \langle B(\xi_1 + \mu\xi_2), \xi_1 + \mu\xi_2\rangle \geq 0, \ \forall \mu \in \mathbb{R}\cdot$$

Thus $|\langle B\xi_1, \xi_2\rangle| \leq \sqrt{\varphi(\xi_1)}\sqrt{\varphi(\xi_2)}$, which implies that, we have:

$$\varphi(\lambda\xi_1 + (1-\lambda)\xi_2) \leq \left(\lambda\sqrt{\varphi(\xi_1)} + (1-\lambda)\sqrt{\varphi(\xi_2)}\right)^2, \ \forall \lambda \in [0,1].$$

Therefore φ is convex, since f is nonnegative nondecreasing then so is ϕ. It follows that ϕ' is monotone. On the other hand, (3.3) combined with the continuity of f implies that ϕ' is hemicontinuous.
On the other hand, it comes from (1.2) that, we have:

$$\|V(\xi)B\xi\| \leq K\ (\alpha\|A\xi\| + \beta\|\xi\|), \ \forall \xi \in D(A).$$

Since A generates a semigroup of contractions and $\phi'(\cdot)$ is monotone hemicontinuous and $K\alpha < 1$, then the operator $-\mathcal{A}$ is maximal monotone, and hence $\mathcal{A}$ generate a semigroup of contractions $e^{t\mathcal{A}}z_0$, and the function $z(t) = e^{t\mathcal{A}}z_0$ is a solution of (3.2) (see [3]). □

Remark 3.2. For all $z_0 \in D(A)$, we have

$$\|z(t)\| \leq \|z_0\|, \ \forall t \geq 0 \tag{3.4}$$

and $z(t) \in D(A)$ admits a right derivative at t (see [13]), which is such that:

$$\frac{d^+z(t)}{dt} = \mathcal{A}z(t)\cdot \tag{3.5}$$

and we have

$$\|\mathcal{A}z(t)\| \leq \|\mathcal{A}z_0\|\cdot \tag{3.6}$$

Remark 3.3. (i) As examples of a function verifying the assumptions (iii) and (iv), one can take $f(s) = \varrho\frac{s}{1+s}$ or $f(s) = c$ (where, $\varrho, c \geq 0$).

(ii) It is easy to verify that if the operator B is bounded from H to itself, then the condition (iv) is superfluous.

4 Partial weak stability

In this section, we give sufficient conditions to obtain partial weak stabilization of (1.1) utilizing the control (3.1). Along this section, let $C : D(C) \subset H \to Y$ be an unbounded (eventually nonlinear) operator with domain $D(C) \supset D(A)$, where Y is a Banach space. In the sequel, we assume that B is A-bounded, then from Theorem (2.3), $B|_{D(A)}$ admits a $A-$extension denoted by $\tilde{B}$.
Let us now introduce the following sets:

$$\mathcal{M} = \{\varphi \in D(A)\ /\ f(\langle Be^{tA}\varphi, e^{tA}\varphi\rangle)\langle Be^{tA}\varphi, e^{tA}\varphi\rangle = 0, \ \forall t \geq 0\}$$

and

$$\widetilde{\mathcal{M}} = \{\varphi \in H\ /\ f(\langle \widetilde{B}e^{tA}\varphi, e^{tA}\varphi\rangle)\langle \widetilde{B}e^{tA}\varphi, e^{tA}\varphi\rangle = 0, \ \forall t \geq 0\}$$

and let us consider the following hypotheses:

(C*$_1$*) : For all sequence $(y_j) \subset D(A)$, such that the sequence $(\|y_j\|)$ is decreasing and $y_j \rightharpoonup y$ in H, we have:

$$f(\langle BS(t)y_j, S(t)y_j\rangle)\langle BS(t)y_j, S(t)y_j\rangle \to 0, \text{ as } j \to +\infty, \ \forall t \geq 0 \Longrightarrow \begin{array}{l}\text{there exists a subsequence } (y_{\gamma(j)})\\ \text{such that } Cy_{\gamma(j)} \rightharpoonup 0 \text{ as } j \to +\infty\cdot\end{array}$$

(C*$_2$*) : For all sequence $(y_n) \subset D(A)$, such that $y_n \rightharpoonup y$ in H, we have the following implication:

$$\left.\begin{array}{l}\text{1. The sequence } (\|y_n\|) \text{ is decreasing}\\ \text{2. } (\|Ay_n\|) \text{ is bounded,}\\ \text{3. } f(\langle By_n, y_n\rangle)\langle By_n, y_n\rangle \to 0.\end{array}\right\} \Longrightarrow \begin{array}{l}\text{there exists a subsequence } (y_{\gamma(n)}) \text{ of } (y_n) \text{ such}\\ \text{that } Cy_{\gamma(n)} \rightharpoonup 0 \text{ as } n \to +\infty.\end{array}$$

Remark 4.1. By taking the null sequence in the conditions ***(C*$_1$*)*** and ***(C*$_2$*)*** , we obtain $C0 = 0$.

The following Theorem concerns the partial weak stability of the system (3.2).

Theorem 4.2. *Suppose that*

(i) the hypotheses of Theorem 3.1 are verified,

(ii) for some $\delta > 0$; $f(s) \leq \delta s$,

(iii) one of the conditions **(C$_1$)**, **(C$_2$)** *fulfilled.*

Then for all $z_0 \in D(A)$, we have, $Cz(t) \rightharpoonup 0$, as $t \to +\infty$. Therefore the control (3.1) partially weakly stabilizes the equilibrium point 0 of the system (3.2).

Proof. Let $z_0 \in D(A)$. According to Remark 3.2, the function $\tau \to z(\tau)$ admits a right derivative at all time, then we have

$$\frac{d^+\|z(\tau)\|^2}{d\tau} = 2\langle \mathcal{A}z(\tau), z(\tau)\rangle, \quad \forall \tau \geq 0.$$

Moreover, A is dissipative, so that $\langle Ay, y\rangle \leq 0$, $\forall y \in D(A)$. Then, we get

$$\int_{s_1}^{s_2} f\Big(\langle Bz(\tau), z(\tau)\rangle\Big)\langle Bz(\tau), z(\tau)\rangle d\tau \leq \frac{1}{2}(\|z(s_1)\|^2 - \|z(s_2)\|^2), \ 0 \leq s_1 \leq s_2\cdot$$

Thus, from (3.4), we deduce that

$$\int_0^{+\infty} f\Big(\langle Bz(\tau), z(\tau)\rangle\Big)\langle Bz(\tau), z(\tau)\rangle \, d\tau < +\infty, \tag{4.1}$$

and $\omega_w(z_0) \neq \emptyset$. Let $\varphi_0 \in \omega_w(z_0)$ and let $t_j \to +\infty$ such that $z(t_j) = e^{t_j\mathcal{A}}z_0 \rightharpoonup \varphi_0$, as $j \to +\infty$. Let $t \geq 0$, from (4.1), we get

$$\lim_{j\to+\infty}\int_{t_j}^{t+t_j} f\Big(\langle Bz(\tau), z(\tau)\rangle\Big)\langle Bz(\tau), z(\tau)\rangle \, d\tau = 0\cdot \tag{4.2}$$

Using (1.2), (3.4) and (3.6), we deduce that

$$\|Bz(\tau)\| \leq \frac{\alpha}{1-\alpha K}\|\mathcal{A}z_0\| + \frac{\beta}{1-\alpha K}\|z_0\| =: M_{z_0}, \ \forall \tau \geq 0\cdot \tag{4.3}$$

Since f is nonnegative, it follows from the dominated convergence theorem that, we have

$$f(\langle Bz(t+t_j), z(t+t_j)\rangle)\langle Bz(t+t_j), z(t+t_j)\rangle \to 0, \text{ as } j \to +\infty. \tag{4.4}$$

Moreover, the variation of constants formula gives

$$z(t+t_j) = S(t)z(t_j) + \int_{t_j}^{t+t_j} V(z(\tau))S(t+t_j-\tau)(Bz(\tau))d\tau\cdot$$

Thus, since $S(t)$ is of contraction then

$$\|z(t+t_j)-S(t)z(t_j)\| \leq M_{z_0}\int_{t_j}^{t+t_j}\left|f\Big(\langle Bz(\tau),z(\tau)\rangle\Big)\right|d\tau\cdot$$

From Schwarz's inequality, we obtain

$$\|z(t+t_j)-S(t)z(t_j)\| \leq \sqrt{t}\,M_{z_0}\sqrt{\int_{t_j}^{t+s_j}\Big(f(\langle Bz(\tau),z(\tau)\rangle)\Big)^2 d\tau}\cdot$$

Taking into account that, for all $s \geq 0$, we have $f(s) \leq \delta s$, we derive

$$\|z(t+t_j)-S(t)z(t_j)\| \leq \sqrt{\delta t}\,M_{z_0}\sqrt{\int_{t_j}^{t+t_j} f(\langle Bz(\tau),z(\tau)\rangle)\langle Bz(\tau),z(\tau)\rangle d\tau}\cdot \tag{4.5}$$

Which by (4.1), gives

$$\lim_{j\to+\infty}[z(t+t_j)-S(t)z(t_j)] = 0\cdot \tag{4.6}$$

Furthermore, from (1.2), (3.6) and the fact that $S(t)$ is of contractions, we get

$$\|BS(t)z(t_j)\| \leq \frac{\alpha}{1-K\alpha}\|\mathcal{A}z_0\| + \frac{K\alpha\beta+\beta}{1-K\alpha}\|z_0\|\cdot \tag{4.7}$$

Since B is self-adjoint, then the above inequality combined with (4.3), (4.6), gives

$$\langle BS(t)z(t_j),S(t)z(t_j)\rangle - \langle Bz(t+t_j),z(t+t_j)\rangle \to 0, \text{ as } j\to+\infty\cdot$$

Based on this, (4.7) and the continuity of f, we obtain

$$f(\langle BS(t)z(t_j),S(t)z(t_j)\rangle)\Big(\langle BS(t)z(t_j),S(t)z(t_j)\rangle - \langle Bz(t+t_j),z(t+t_j)\rangle\Big) \to 0, \text{ as } j\to+\infty\cdot$$

Then using (4.4) and Heine-Cantor theorem, we deduce that

$$f(\langle BS(t)z(t_j),S(t)z(t_j)\rangle)\langle BS(t)z(t_j),S(t)z(t_j)\rangle \to 0, \text{ as } j\to+\infty\cdot \tag{4.8}$$

If ***(C***$_1$***)*** is verified, then we conclude that there exists a subsequence $(z(t_{\gamma(j)}))$ of $(z(t_j))$ such that $Cz(t_{\gamma(j)}) \rightharpoonup 0$ as $j\to+\infty$. Following the same techniques and using (3.4) and condition ***(C***$_1$***)***, we deduce that 0 is the unique limit for $Cz(t)$. Thus we conclude that $Cz(t) \rightharpoonup 0$ as $t\to+\infty$. Furthermore, from (4.4), we have $f\Big(\langle Bz(t_j),z(t_j)\rangle\Big)\langle Bz(t_j),z(t_j)\rangle \to 0$ as $j\to+\infty$. Moreover, according to the Remark 3.2, we have $(\|Az(t_j)\|)_{j\geq 1}$ is bounded. This if **(C**$_2$**)** is verified, then there exists a subsequence $(z(t_{\gamma(j)}))$ of $(z(t_j))$ such that $Cz(t_{\gamma(j)}) \rightharpoonup 0$ as $j\to+\infty$. Following similar procedures we show that $Cz(t)$ has a unique limit point. We conclude that $Cz(t) \rightharpoonup 0$ as $t\to+\infty$. Which completes the proof of the Theorem. □

On the other hand, under the following hypothesis:

(C$_3$***) :*** There exists $t_0 > 0$ such that $\|BS(t_0)\|$ is bounded on $D(A)$ and for all sequence $(y_n) \subset D(A)$ and $y \in H$ such that the sequence $(\|y_n\|)$ is decreasing and $y_n \rightharpoonup y$ in H; there exists a subsequence $(y_{\gamma(n)})$ such that for all $t \geq t_0$, we have $BS(t)y_{\gamma(n)} \to \widetilde{B}S(t)y$ in H, as $n\to+\infty$.

We have the following partial weak stabilization result:

Theorem 4.3. *Suppose that:*

(i) The hypotheses of Theorem 3.1 are verified,

(ii) for some $\delta > 0$; $f(s) \leq \delta s$,

(iii) the condition **(C**$_3$**)** *holds,*

(iv) the operator C sends every bounded subsets of $D(A)$ into weakly compact subsets of Y.

Then

1. *If the assumption $\widetilde{\mathcal{M}} \subset \ker(C)$ holds, then for all $z_0 \in D(A)$, we have $Cz(t) \rightharpoonup 0$, as $t \to +\infty$. Therefore the control (3.1) partially weakly stabilizes an equilibrium point φ_0 of the system (3.2).*
2. *If the condition $\widetilde{\mathcal{M}} = \{0\}$ holds, then for all $z_0 \in D(A)$, we have $Cz(t) \rightharpoonup C(0)$, as $t \to +\infty$. Therefore the control (3.1) partially weakly stabilizes the equilibrium point 0 of the system (3.2).*

Proof. 1. Let $z_0 \in D(A)$, and let $t_j \to +\infty$ such that $z(t_j) = e^{t_j \mathcal{A}} z_0 \rightharpoonup \varphi_0 \in \omega_w(z_0)$, as $j \to +\infty$. From the condition *(C_3)*, there exists a subsequence of (t_j), still denoted by (t_j), such that for all $t \geq t_0$, we have

$$f(\langle BS(t)z(t_j), S(t)z(t_j)\rangle)\langle BS(t)z(t_j), S(t)z(t_j)\rangle \to f(\langle \widetilde{B}S(t)\varphi_0, S(t)\varphi_0\rangle)\langle \widetilde{B}S(t)\varphi_0, S(t)\varphi_0\rangle, \text{ as } j \to +\infty\cdot$$

According to (4.8), we conclude that

$$f(\langle \widetilde{B}S(t)\varphi_0, S(t)\varphi_0\rangle)\langle \widetilde{B}S(t)\varphi_0, S(t)\varphi_0\rangle = 0. \tag{4.9}$$

Thus $\varphi_0 \in \widetilde{\mathcal{M}}$. Since $\widetilde{\mathcal{M}} \subset \ker(C)$, then $C\varphi_0 = 0$. According to the conditions (iv) on the operator C, then there exists a subsequence of (t_j), still denoted by (t_j), such that $Cz(t_j) \rightharpoonup C\varphi_0 = 0$, as $j \to +\infty$. Using the same methods as in the proof of Theorem 4.2, we show that $Cz(t)$ has a unique limit point. We conclude that $Cz(t) \rightharpoonup 0$ as $t \to +\infty$.

2. Let $z_0 \in D(A)$, and let $t_j \to +\infty$ such that $z(t_j) = e^{t_j \mathcal{A}} z_0 \rightharpoonup \varphi_0 \in \omega_w(z_0)$, as $j \to +\infty$. From (4.9), $\varphi_0 \in \widetilde{\mathcal{M}}$, then $\varphi_0 = 0$. Using the same techniques as in 1. we conclude that $Cz(t) \rightharpoonup C(0)$ as $t \to +\infty$. □

Remark 4.4. (i) The condition (iv) of Theorem 4.3 is verified for example for any linear bounded operator $C \in L(X, Y)$.

(ii) In the particular case that $B \in L(H)$, $C \in L(H, Y)$ and $f(s) = \dfrac{\varrho s}{s+1}$ (where ϱ is the gain control), the condition $\widetilde{\mathcal{M}} \subset \ker(C)$ is equivalent to the assumption (1.4). Then the first result of the Theorem (4.3) is the unbounded generalization of the result in [9].

(iii) As a class of operators that satisfy the assumption *(C_3)*, the linear compact operators B.

5 Applications

5.1 Heat equation

Let $\Omega = (0, 1)$ and let us consider the bilinear system given by the following heat equation:

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) = \Delta z(t,x) - v(t)\Delta z(t,x), \text{ on } (0,+\infty) \times \Omega \\ z'(t,0) = z'(t,1) = 0, \ \forall\, t > 0 \\ z(0,x) = z_0(x), \text{ on } \Omega\cdot \end{cases} \tag{5.1}$$

Let us consider the following conventional state-space $H = L^2(\Omega)$ and $Az = \Delta z$, for all $z \in D(A) = \{z \in L^2(\Omega) \,/\, \Delta z \in L^2(\Omega),\ z'(0) = z'(1) = 0\}$, and let $\lambda_j = -\pi^2(j-1)^2$, $j = 1, ...$ is the eigenvalues of A associated to the eigenvector $\psi_1(x) = 1$ and $\psi_j = \sqrt{2}\cos((j-1)\pi x)$, $j = 2, ...$ The operator $Bz = -\Delta z$ with domain $D(B) := D(A)$ is self adjoint, positive and it is A-bounded with $\alpha = 1$ and $\beta = 0$.
Then we can state the following result concerning the weak stability of (5.1).

Proposition 5.1. *For $0 < \varrho < 1$ and for any $z_0 \in L^2(\Omega)$, the system (5.1), controlled by the feedback $v(t) = -\varrho \dfrac{\int_\Omega |\nabla z(t,x)|^2\, dx}{1 + \left|\int_\Omega |\nabla z(t,x)|^2 dx\right|}$ admits a unique mild solution $z \in \mathcal{C}([0, +\infty[; H)$. Furthermore, for all $z_0 \in D(A)$, we have the following results:*

(i) there exists $c \in \mathbb{R}$ such that $z(t) \rightharpoonup c1_\Omega$ in $L^2(\Omega)$, as $t \longrightarrow +\infty$.

(ii) for a constant $\gamma \geq 1$ and for any $a \in L^2(\Omega)$ verifying $\int_0^1 a(x)dx = 0$, we have:

$$\int_0^1 z(t,x)^\gamma a(x)dx \to 0, \text{ as } t \longrightarrow +\infty.$$

Proof. Let $\varphi \in D(A)$ such that $\langle BS(t)\varphi, S(t)\varphi\rangle = 0$. Since, for all $t \geq 0$, $AS(t) = S(t)A$ then we have $BS(t)\varphi = -\sum_{j=1}^{+\infty} \lambda_j e^{t\lambda_j}\langle\varphi,\psi_j\rangle\psi_j$, which implies that

$$\langle BS(t)\varphi, S(t)\varphi\rangle = -\sum_{j=1}^{+\infty} \lambda_j e^{t\lambda_j}|\langle\varphi,\psi_j\rangle|^2.$$

Then, there exists $c_0 \in \mathbb{R}$, such that $\varphi = c_0 1_\Omega$. We deduce that $\tilde{\mathcal{M}} = \{c\ 1_\Omega\ /\ c \in \mathbb{R}\}$. On the other hand, the condition ***(C***$_3$***)*** holds (see [10]). Let $a \in L^2(\Omega)$ verifying $\int_0^1 a(x)dx = 0$ and a constant $\gamma \geq 1$, it is easy to see that, the operators $C\xi = \xi - c1_\Omega$ and $C\xi = \int_0^1 \xi(x)^\gamma a(x)dx$, with $D(C) = H^1(\Omega)$, verify the condition (iv) of Theorem 4.3. By using Theorem 4.3, we deduce that for all $z_0 \in D(A)$, we have: $z(t) \rightharpoonup c1_\Omega$ in $L^2(\Omega)$ and $\int_0^1 z(t,x)^\gamma a(x)dx \rightharpoonup 0$, as $t \longrightarrow +\infty$. □

5.2 Transport equation

Let $\Omega =]0,+\infty[$ and let us consider the bilinear system given by the following transport equation in the state-space $H = L^2(\Omega)$:

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) = -\dfrac{\partial z}{\partial x}(t,x) + v(t)\chi_\omega(x)a(x)z(t,x), \text{ on }]0,+\infty[\times\Omega \\ z(t,0) = 0, \text{ on }]0,+\infty[\\ z(0,x) = z_0(x) \text{ on } \Omega\cdot \end{cases} \tag{5.2}$$

where ω is a non empty subset of Ω, a is such that $a(x) > 0$, a.e $x \in \omega$, $\int_\omega x\ a^2(x)\ dx < +\infty$, $a \in L^2(\Omega)$ and $a \notin L^4(\Omega)$ (for example $a(x) = \dfrac{1}{\sqrt[4]{x(x^2+1)}}$). Let us defined the unbounded linear operator $Az = -\dfrac{\partial z}{\partial x}$, for all $z \in D(A) =: \{y \in H^1(\Omega)\ /\ y(0) = 0\}$. The control operator B is defined by $By = \chi_\omega ay$, for all $y \in D(A)$. It is easy to see that the operator B is unbounded from $L^2(\Omega)$ to $L^2(\Omega)$. Moreover by Morrey's inequality (see [4]), there exists $k > 0$ such that for all $z \in D(A)$ we have $|z(x)| \leq k\sqrt{x}\ \|\nabla z\|_{L^2(\Omega)}$, a.e $x \in \Omega$, then

$$\int_0^{+\infty} |\chi_\omega(x)a(x)z(x)|^2 dx \leq k^2\ \|\nabla z\|^2_{L^2(\Omega)} \int_\omega xa^2(x)\ dx,$$

which implies that B is $A-$bounded with $\alpha = k\sqrt{\int_\omega x\ a^2(x)\ dx}$.

We are ready to state the following result:

Proposition 5.2. *Suppose that $a(x) > 0$, a.e $x \in \omega \subset \Omega$ and $\int_\omega x\ a^2(x)\ dx < +\infty$. Then there exists $\rho_0 > 0$ such that for any $0 < \varrho < \rho_0$ and for all $z_0 \in L^2(\Omega)$, the system (5.2) controlled by the feedback $v(t) = -\varrho \dfrac{\int_\omega a(x)\ |z(x,t)|^2\ dx}{1 + \int_\omega a(x)\ |z(x,t)|^2 dx}$ admits a unique solution $z \in \mathcal{C}([0,+\infty[; H)$, and for all $z_0 \in H_0^1(\Omega)$, we have: $\chi_\omega z(t) \rightharpoonup 0$ in $L^2(\Omega)$, as $t \to +\infty$.*

Proof. Let $(y_n)_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ a sequence such that $y_n \rightharpoonup y$ in H and $< By_n, y_n > \to 0$, as $n \to +\infty$, there exists a subsequence of $(y_n)_{n\in\mathbb{N}}$ still denoted by $(y_n)_{n\in\mathbb{N}}$ such that $\chi_\omega a(x) y_n^2(x) \to 0$, a.e $x \in \Omega$ as $n \to +\infty$. Moreover by Morrey's inequality, there exists $C > 0$ such that for all test function φ, we have $|\chi_\omega y_n(x)\ \varphi(x)| \leq C\|\nabla y_n\|_{L^2(\Omega)}\sqrt{x}\ |\varphi(x)|$, a.e $x \in \Omega$. If in addition $(Ay_n)_{n\in\mathbb{N}}$ is bounded in H, then. This combined with the dominated convergence theorem, gives $\langle \chi_\omega y_n, \varphi\rangle \to 0$ as $n \to +\infty$, we conclude that $\chi_\omega y_n \rightharpoonup 0$ in H, as $n \to +\infty$. Then $y = 0$, thus the condition ***(C₂)*** is verified. We can conclude by using Theorem 4.2. □

6 Conclusion

To sum up, the study's objective was to examine the partial weak stabilization of unbounded bilinear systems in real Hilbert space. The well-posedness of our nonlinear closed-loop system was established. The partial weak stabilization results were developed; various examples illustrating the obtained results are given covering both parabolic and hyperbolic cases. This work opens up other questions, such as the partial strong stabilization, the robustness of the controls (3.1), and the asymptotic estimates of the system's output.

References

[1] Berrahmoune, L. (2010). Stabilization of unbounded bilinear control systems in Hilbert space. Journal of mathematical analysis and applications, 372(2), 645-655.

[2] Boutoulout, A., El Ayadi, R., & Ouzahra, M. (2014). An unbounded stabilization problem of bilinear systems. Mathematics and Computers in Simulation, 102, 39-50.

[3] Brezis, H. (1973). Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. Elsevier.

[4] Brezis, H. (2010). Functional analysis, Sobolev spaces and partial differential equations. Springer Science & Business Media.

[5] Desch, W., & Schappacher, W. (1984). On relatively bounded perturbations of linear C_0-semigroups. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 11(2), 327-341.

[6] El Ayadi, R., & Ouzahra, M. (2019). Feedback stabilization for unbounded bilinear systems using bounded control. IMA Journal of Mathematical Control and Information, 36(4), 1073-1087.

[7] El Ayadi, R. E., Ouzahra, M., & Boutoulout, A. (2012). Strong stabilisation and decay estimate for unbounded bilinear systems. International journal of control, 85(10), 1497-1505.

[8] Engel, K. J., & Nagel, R. (2001, June). One-parameter semigroups for linear evolution equations. In Semigroup forum (Vol. 63, No. 2, pp. 278-280). Springer-Verlag.

[9] Hamidi, Z., & Ouzahra, M. (2018). Partial stabilisation of non-homogeneous bilinear systems. International Journal of Control, 91(6), 1251-1258.

[10] Haraux, A. (2001). Some sharp estimates for parabolic equations. Journal of Functional Analysis, 187(1), 110-128.

[11] Hille, E., & Phillips, R. S. (1996). Functional analysis and semi-groups (Vol. 31). American Mathematical Soc.

[12] Idrissi, A. (2003). On the unboundedness of control operators for bilinear systems. Quaestiones Mathematicae, 26(1), 105-123.

[13] Komura, Y. (1967). Nonlinear semi-groups in Hilbert space. Journal of the Mathematical Society of Japan, 19(4), 493-507.

[14] Lyapunov, A. M. (1992). The general problem of the stability of motion. International journal of control, 55(3), 531-534.

[15] Nersesov, S. G., & Ashrafiuon, H. (2019, June). The Use of Partial Stability in the Analysis of Interconnected Systems. In 2019 18th European Control Conference (ECC) (pp. 155-159). IEEE.

[16] Pazy, A. (2012). Semigroups of linear operators and applications to partial differential equations (Vol. 44). Springer Science & Business Media.

[17] Pazy, A. (1978). On the asymptotic behavior of semigroups of nonlinear contractions in Hilbert space. Journal of Functional Analysis, 27(3), 292-307.

[18] Rouche, N., Habets, P., & Laloy, M. (1977). Stability theory by Liapunov's direct method (Vol. 4). New York: Springer-Verlag.

[19] Vorotnikov, V. I. (2012). Partial stability and control. Springer Science & Business Media.

[20] Weiss, G. (1989). Admissible observation operators for linear semigroups. Israel Journal of Mathematics, 65(1), 17-43.

[21] Zuyev, A. L. (2002). Partial stabilization of a rigid body with several elastic beams. IFAC Proceedings Volumes, 35(1), 139-144.

[22] Zuyev, A. L. (2015). Partial stabilization and control of distributed parameter systems with elastic elements. Springer International Publishing.

Author information

R. El Ayadi, Laboratory of Modeling and Mathematical Structures. Department of Mathematics. Faculty of Science and Technology of Fez, Box 2202. University Sidi Mohamed Ben Abdellah, Morocco.
E-mail: rachid.elayadi@usmba.ac.ma

Received: June 7, 2020.
Accepted: September 21, 2020.