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Regularization of a class of ill-posed Cauchy problems associated with generators of analytic semigroups

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Abstract This work is concerned the study of a class of ill-posed Cauchy problems associated with a densely defined linear operator A in a Banach space E. It is proved that if $-A$ is the generator of an analytic semigroup, then there exists a family of regularizing operators by using the quasi-reversibility method, fractional powers and semigroups of linear operators.

1 Analytic Semigroups

1.1 Semigroups

We are interested here in linear equations of state defined by semigroups. Consider an equation of state of the form :

$$\begin{cases} y'(t) &= Ay(t) \quad 0<t<T, \\ y(t_0) &= y_0, \end{cases} \tag{1.1}$$

this equation can be studied using an abstract approach depending on the properties of the operator A. We can also study it by considering the properties of its solution y. This second approach expresses, for an initial state y_0 at time t_0, the solution at time $t+s$, which can be obtained indifferently from :

- the state y_0, and its evolution up to time $t+s$, or
- the state y_0, and its evolution until time t, then from the state at time t to the state at time $t+s$.

This naturally leads to considering the semigroup approach. Given a Hilbert space E representing the state space, we consider the following definition.

Definition 1.1. We call a strongly continuous semigroup a family $(\Phi(t))_{t\geq 0}$ of operators of $\mathcal{L}(E)$ satisfying the following properties :

1. $\Phi(0)=I$.
2. $\Phi(t+s)=\Phi(t)\Phi(s)$, for all $t,s\geq 0$.
3. $\|\Phi(t)y-y\|\to 0$ when $t\to 0^+$, for all $y\in E$.

The family of operators $(\Phi(t))_{t\geq 0}$ obviously depends on the dynamics A of the system. In addition, we have the following definition.

Definition 1.2. The infinitesimal generator of the semigroup $(\Phi(t))_{t\geq 0}$ is the unbounded linear operator A defined by :

$$Ay=\lim_{t\to 0^+}\frac{\Phi(t)y-y}{t}, \tag{1.2}$$

when this limit exists.

The domain of A, denoted $\mathcal{D}(A)$, is the set of y in E such that this limit exists

$$\mathcal{D}(A)=\left\{y\in E/\lim_{t\to 0^+}\frac{\Phi(t)y-y}{t} \text{ exist}\right\}. \tag{1.3}$$

Immediate properties of the semigroups are given in the following proposition [5].

Proposition 1.3. *1.* $\forall y \in \mathcal{D}(A); \Phi(t)\, y \in \mathcal{D}(A)$ *and* $\frac{d}{dt}\Phi(t)y = A\Phi(t)\, y = \Phi(t)Ay$.

2. $\mathcal{D}(A)$ *is a subspace dense in* E $\left(\overline{\mathcal{D}(A)} = E\right)$.

3. The operator A *is closed.*

Property of exponential growth of semigroup

Lemma 1.4. *[2]*

1. Let $\Phi(t)$ *be a strongly continuous semigroup then there exists* $M \geq 1$ *and* $\omega \in \mathbb{R}$ *such that* $\|\Phi(t)\| \leq Me^{\omega t}; \forall t \geq 0$.

2. If $\Phi(t)$ *is a semigroup strongly continuous at the origin is the increase* $\|\Phi(t)\| \leq Me^{\omega t}$. *Then* $\Phi(t)$ *is strongly continuous at a point* $t > 0$.

Semigroup and Laplace transform

Let us define the application $R_\lambda : E \to E$ by $R_\lambda y = \int_0^\infty e^{-\lambda t}\Phi(t)ydt$. It is clear that R_λ is a linear operator. In addition we have

$$\|R_\lambda y\| \leq \int_0^\infty \left\|e^{-\lambda t}\Phi(t)y\right\| dt \leq \frac{M}{\operatorname{Re}\lambda - \omega}\|y\|; \forall y \in E.$$

From which it follows that R_λ is a bounded linear operator.

Definition 1.5. The operator $R(\lambda) = R_\lambda$ is called the Laplace transform of the semigroup $\{\Phi(t)\}_{t\geq 0}$.

Study of the growth of the Resolvent

Proposition 1.6. *[2]*

$$(R(\lambda, A))^n = \frac{1}{(n-1)!}\int_0^{+\infty} e^{-\lambda t}t^{n-1}\Phi(t)dt.$$

Generalized Yosida approximation

Lemma 1.7. *[2] Let* $A : \mathcal{D}(A) \subset E \to E$ *be a linear operator satisfying the following properties:*

- A *is a closed operator and* $\overline{\mathcal{D}(A)} = E$.

- There exist $\omega \geq 0$ *and* $M \geq 1$ *such that* $A_\omega \subset \rho(A)$ *and for* $\lambda \in A_\omega$ *; we have*

$$\left\|R(\lambda, A)^n\right\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n}; \forall n \in \mathbb{N}^*.$$

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$$\forall \lambda \in A_\omega : \lim_{\operatorname{Re}\lambda \to \infty} \lambda R(\lambda, A)\, y = y; \forall y \in E,$$

thus

$$\lim_{\operatorname{Re}\lambda \to \infty} \lambda AR(\lambda, A)\, y = Ay; \forall y \in \mathcal{D}(A).$$

Remark 1.8. We can say that the bounded operators $\lambda AR(\lambda, A)$ are approximations for the unbounded operator A. This is the reason for introducing the following theorem.

Theorem 1.9. *[2] The family* $\{A_\lambda\}_{\lambda \in A_\omega}$ *; where*

$$A_\lambda = \lambda AR(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda I,$$

is called the generalized Yosida approximation of operator A.

Theorem 1.10. *[13] Let $\{\Phi(t)\}_{t\geq 0}$ be a strongly continuous semigroup in a Banach space E and let $\omega \in \mathbb{R}$ and $M \geq 1$ such that $\|\Phi(t)\| \leq e^{\omega t}$, $\forall t \geq 0$. Then the generator $(A, \mathcal{D}(A))$ of $\{\Phi(t)\}_{t\geq 0}$ has the following equivalent properties :*

1. If $\lambda \in \mathbb{C}$ such that $R(\lambda)y = \int_0^{+\infty} e^{-\lambda\mu}\Phi(\mu)yd\mu$ exist $\forall y \in E$, then $\lambda \in \rho(A)$ and $R(\lambda, A) = R(\lambda)$.

2. If $\operatorname{Re}\lambda > \omega$ then $\lambda \in \rho(A)$ and the resolvent is given as in 1.

3. $\|R(\lambda, A)\| \leq \frac{M}{\operatorname{Re}\lambda - \omega}$, for all $\operatorname{Re}\lambda > \omega$.

Corollary 1.11. *[13] For each $\lambda_0 \in \rho(A)$ we have*

$$d(\lambda_0, \sigma(A)) - \frac{1}{r(R(\lambda_0, A))} \geq \frac{1}{\|R(\lambda_0, A)\|}.$$

Hille-Yosida theorem

Theorem 1.12. *[2] A linear operator $A : \mathcal{D}(A) \subset E \longrightarrow E$ is the infinitesimal generator of a semi group $\Phi(t)_{t\geq 0} \in \mathcal{SG}(M, \omega)$ if and only if*

1. A is a closed operator and $\overline{\mathcal{D}(A)} = E$.

2. It exists $\omega \geq 0$ and $M \geq 1$ such that $A_\omega \subset \rho(A)$ and for $\lambda \in A_\omega$; we have

$$\left\|(\lambda I - A)^{-n}\right\| \leq \frac{M}{(\lambda - \omega)^n}; \forall n \in \mathbb{N}^*.$$

Definition 1.13. For $\lambda > \omega$ we define the approaching yosida of A by $A_\lambda = \lambda^2 R(\lambda, A) - \lambda I = \lambda A R(\lambda, A)$. With

$$\lim_{\lambda\to+\infty} A_\lambda y = Ay; \forall y \in \mathcal{D}(A).$$

1.2 Analytic Semigroups

Sectorial Operator

Definition 1.14. A closed linear operator $(A, \mathcal{D}(A))$, of dense domain in a Banach space E is called sectorial (of angle α) if there exists α, $0 < \alpha \leq \frac{\pi}{2}$ such that the sector

$$\Sigma_{\alpha+\frac{\pi}{2}} = \left\{\lambda \in \mathbb{C} : |\arg\lambda| < \frac{\pi}{2} + \alpha - \{0\}\right\} \subset \rho(A).$$

And if for all $\beta \in (0, \alpha)$, there exists $M_\beta \geq 1$ such that

$$\|R(\lambda, A)\| \leq \frac{M_\beta}{|\lambda|}, \forall 0 \neq \lambda \in \Sigma_{\frac{\pi}{2}+\alpha-\beta}.$$

Definition 1.15. Let $(A, \mathcal{D}(A))$ be a sectorial operator with angle α. We define $\Phi(0) = I$ and the operator $\Phi(y)$ for $y \in \Sigma_\alpha$ by

$$\Phi(y) = \frac{1}{2\pi i}\int_\Gamma e^{\mu y} R(\mu, A)d\mu.$$

Where Γ is a piecewise smooth path (or piecewise smooth curve).

Proposition 1.16. *[13] Let $(A, \mathcal{D}(A))$, a sectorial operator with angle α then for all $y \in \Sigma_\alpha$, $\Phi(y)$ are linear operators bounded on E satisfying the following properties :*

1. $\|\Phi(t)\|$ is uniformly bounded "uniform boundedness" for $y \in \Sigma_{\alpha'}$, if $0 < \alpha' < \alpha$.

2. The application $y \to \Phi(y)$ is analytic in Σ_α.

3. $\Phi(y_1 + y_2) = \Phi(y_1)\Phi(y_2)$ for all $y_1, y_2 \in \Sigma_\alpha$.

4. The application $y \to \Phi(y)$ is strongly continuous in $y \in \Sigma_{\alpha'} \cup \{0\}$, if $0 < \alpha' < \alpha$.

Analytic Semigroup

Definition 1.17. A family of operators $\{\Phi(y)\}_{y\in\Sigma_\alpha\cup\{0\}} \subset \mathcal{L}(E)$ is called analytic semigroup of angle $\alpha \in (0, \frac{\pi}{2}]$ if :

1. $\Phi(0) = I$ and $\Phi(y_1 + y_2) = \Phi(y_1)\Phi(y_2), \forall y_1, y_2 \in \Sigma_\alpha$.
2. The application $y \to \Phi(y)$ is analytic in Σ_α.
3. $\lim\limits_{\Sigma_{\alpha'}\ni y\to 0} \Phi(y)x = x, \forall x \in E$ and $0 < \alpha' < \alpha$.

Definition 1.18. If in addition $\|\Phi(t)\|$ is bounded in $\Sigma_{\alpha'}$ for all $0 < \alpha' < \alpha$, then we say that $\{\Phi(y)\}$ is a bounded analytic semigroup.

It can also be defined in the following equivalent way :

Definition 1.19. Let $0 < \alpha \leq \frac{\pi}{2}$. If the $\mathcal{C}_0$-semigroup $(\Phi(t))_{t\geq 0}$ admits an analytic extension in Σ_α verifying :

$$\lim_{\Sigma_{\alpha'}\ni y\to 0} \Phi(y)x = x, \forall x \in E \ \text{ and } \ \beta \in (0, \alpha).$$

Then $(\Phi(t))$ is said to be an analytic semigroup of angle α its generator is the generator of $(\Phi(t))_{t\geq 0}$. In addition, the analytic semigroup of angle α is said to be bounded if for each $\beta \in (0, \alpha)$, there exists $M_\beta > 0$ such that $\|\Phi(t)\| \leq M_\beta$ of all $t \in \Sigma_\beta$.

It is known that, if A the generator of an analytic semigroup of angle α, then for each $\beta \in (0, \alpha)$, there exists $\omega \in \mathbb{R}$ such that $A - \omega$ is the generator of a bounded analytic semigroup of angle β. The following criterion on the semigroup generators will also be used in the following.

Lemma 1.20. *[13] Let $0 < \alpha \leq \frac{\pi}{2}$. Then the following properties are equivalent :*

1. A is the generator of a bounded analytic semigroup of angle α.

2. For all $\beta \in (0, \alpha)$, there exists $M_\beta > 0$, such that $e^{\pm i\theta} A$ is the generator of a $\mathcal{C}_0$-semigroup $(\Phi_\theta(t))_{t\geq 0}$, satisfying :

$$\|\Phi_\theta(t)\| \leq M_\beta, \forall t \geq 0, \theta \in (0, \alpha).$$

3. The application $]0, +\infty[\ni t \to \Phi(t) \in \mathcal{B}(E)$[1], is differentiable and there is a constant $C > 0$, such that

$$\|\Phi(t)\| \leq \frac{C}{t}, \forall t \geq 0.$$

4. A is the generator of a strongly continuous semigroup $(\Phi(t))_{t\geq 0}$ in E, and there exists a constant $C' > 0$, such that for all $r > 0$, $s \neq 0$, we have

$$\|R(r + is, B)\| \leq \frac{C'}{|s|}.$$

5. A is sectorial.

We now move on to a small introduction to fractional exponents.

Definition 1.21. Let $-A$ be the generator of an analytic semigroup of angle α $(\alpha \in \left(0, \frac{\pi}{2}\right])$ and let $0 \in \rho(A)$ for $b > 0$, the fractional power of A is defined as follows :

$$A^{-b} = -\frac{1}{2\pi i}\int_{\Gamma(\gamma)} \mu^{-b} R(\lambda, A)d\mu, \frac{\pi}{2} - \alpha < \gamma < \pi.$$

Here and in the following λ^b is considered as the main branch. In this definition, the function $\mu \to \mu^{-b}$ is a branch of the fractional power function in $\mathbb{C} - \mathbb{R}^-$, i.e.

$$\mu^{-b} = e^{-b\ \ln \mu}, \ln \mu = \ln |\mu| + i\theta, -\pi < \theta = \arg \mu < \pi.$$

In this work, we need several properties of fractional powers which are grouped in the following lemma :

[1] $\mathcal{B}$ is the Banach algebra.

Lemma 1.22. *[13].*
1. $A^{-b} \in \mathcal{B}(E)$ *is injective for* $b > 0$.
2. A^b *is a closed operator and* $\mathcal{D}(A) \subset \mathcal{D}(A^{b'})$ *for* $b > b' > 0$.
3. $A^b x = A^{b-n} A^n x$, *for* $x \in \mathcal{D}(A^n), n > b, n \in \mathbb{N}$.
4. *If* $B \subset A^b$ *and* $\mathcal{D}(B) \subset \mathcal{D}(A^{b'})$, $b > b' > 0$ *then* B *is closable and* $\overline{B} = A^b$, *where* $\overline{B}$ *is the closure of* B.

Analytic semigroup generated by $-A^b$ and $A - \varepsilon A^b$

In this part we show that the two operators $-A^b$ and $A - \varepsilon A^b$ are generators of analytic semigroups under certain appropriate conditions on the operator A.

Theorem 1.23. *[13] Let* $-A$ *be the generator of a bounded analytic semigroup of angle* α $(0 < \alpha \leq \frac{\pi}{2})$ *and let* $0 \in \rho(A)$. *Then* $-A^b$ *is the generator of a bounded analytic semigroup of angle* $\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right)\right) b$, *where* $b \in \left(1, \frac{\pi}{\pi - 2\alpha}\right)$.

Theorem 1.24. *[13]Suppose that the operator* A *satisfies the conditions of the theorem 1.10. Let* $A_\varepsilon = A - \varepsilon A^b$, *where* $\varepsilon > 0$ *and* $b \in \left(1, \frac{\pi}{\pi - 2\alpha}\right)$. *Then for all* $\beta \in \left(0, \frac{\pi}{2} - (\frac{\pi}{2} - \alpha)b\right)$, A_ε *is the generator of an analytic semigroup* $\{\Phi_\varepsilon(t)\}$ *d 'angle* β, *satisfying :* $\|\Phi_\varepsilon(t)\| \leq M \exp\left(C\varepsilon^{\frac{1}{1-b}} t\right)$ *for* $t \geq 0$, *where* M *and* C *are positive constants independent of* ε.

2 Regularization of a class of ill-posed Cauchy problems associated with generators of analytic semigroups

The ill-posed Cauchy problems are practical problems, have received a lot of attention since the 1960s of the last century. For this reason, the objective of this domain was to study the following abstract Cauchy problem :

$$\begin{cases} y'(t) &= Ay(t); 0 < t \leq T, \\ y(0) &= x, \end{cases} \tag{2.1}$$

where $-A$ is the generator of an analytic semigroup of angle α in the Banach space E, where $0 < \alpha < \frac{\pi}{2}$.

Definition 2.1. The function $y : \mathbb{R}^+ \to E$ is called classical solution of (2.1) if :
1. $y \in \mathcal{C}([0, +\infty], \mathcal{D}(A))$ provides the norm of the graph.
2. $y \in \mathcal{C}^1([0, +\infty], E)$.
3. y checks the equation (2.1) and checks the initial condition $y(0) = x$ with $x \in \mathcal{D}(A)$.

Corollary 2.2. *[1] For a closed operator* $A : \mathcal{D}(A) \subset E \to E$ *associated with the problem (2.1) is well posed if and only if* A *is the generator of a strongly continuous semigroup.*

Lemma 2.3. *Let* $(T(t))_{t \geq 0}$ *be a* $\mathcal{C}_0$*-semigroup of bounded operators, if* $\forall t > 0, T^{-1}(t)$ *exists and it is bounded, then* $\Phi(t) = T^{-1}(t)$ *is a* $\mathcal{C}_0-$*semigroup of bounded operators where its infinitesimal generator is* $-A$. *In addition if*

$$U(t) = \begin{cases} T(t) & \text{for } t \geq 0, \\ T^{-1}(-t) & \text{for } t \leq 0. \end{cases}$$

Then $U(t)$ *is a strongly continuous group of bounded operators.*

Proof. We show that $\Phi(t)$ is a strongly continuous semigroup.
✓ $\Phi(t) = T^{-1}(0) = I$.
✓ $\Phi(t + s) = [T(t + s)]^{-1} = [T(t).T(s)]^{-1} = T^{-1}(s).T^{-1}(t)$, hence

$$\Phi(t + s) = \Phi(s).\Phi(t).$$

✓ Now we show the strong continuity of $\Phi(t)$.
We have $x = T(s)y$ for $s > 0$, then :

$$\begin{aligned}\|\Phi(t)x - x\| &= \left\|T^{-1}(t)x - x\right\| \\ &= \left\|T^{-1}(t)T(t)T(s-t)y - T(s)y\right\| \\ &= \|T(s-t)y - T(s)y\| \to 0, \text{ when } t \to 0.\end{aligned}$$

So $\Phi(t)$ is strongly continuous. Finally, for $x \in \mathcal{D}(A)$ we have

$$\lim_{t\to 0} \frac{\Phi(t)x - x}{t} = \frac{T^{-1}(t)x - x}{t} = -Ax.$$

So $-A$ is the infinitesimal generator of $\Phi(t)$. □

Theorem 2.4. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup of bounded operators. If $0 \in \rho(T(t_0))$ for a certain $t_0 > 0$, then $0 \in \rho(T(t))$ for all $t > 0$ and $T(t)$ can be extended into a $\mathcal{C}_0$-semigroup (strongly continuous group).*

Proof. Since $0 \in \rho(T(t_0))$ then by lemma 2.3 $T(nt_0)$ is bijective, $\forall n \geq 1$.
Let $T(t)x = 0$, choose n such that $nt_0 > t$. We have

$$T(nt_0)x = T(nt_0 - t)T(t)x.$$

So $T(t)x = 0$ implies that $x = 0$. Then $T(t)$ is injective for all $t > 0$.
According to the semi-group properties $\operatorname{Im} T(t) \supset \operatorname{Im} T(t_0)$, for $t \leq t_0$.
For $t > t_0$, let $t = kt_0 + t_1$, with $0 \leq t_1 < t_0$, thus $T(t) = [T(t_0)]^k T(t_1)$, and therefore

$$\operatorname{Im} T(t) \subset \operatorname{Im} T(t_0).$$

So we have : $\operatorname{Im} T(t) = E, \forall t > 0$. This is the proof that the operator is surjective.
$T(t)$ is bijective and $\operatorname{Im} T(t) = E$, for all $t > 0$, and according to the closed graph theorem

$$0 \in \rho(T(t)), \forall t > 0.$$

□

Proposition 2.5. *Let $-A$ be the generator of an analytic semigroup $(\Phi(t))$ and let $0 \in \rho(\Phi(t_0))$ for some $t_0 > 0$. Then $A \in \mathcal{B}(E)$.*

Proof. We can assume without loss of generality that $-A$ is the generator of a bounded analytic semigroup of angle α for a certain $\alpha \in \left(0, \frac{\pi}{2}\right]$, otherwise we consider the analytic semigroup $\{e^{\omega t}\Phi(t)\}$ generator $(\omega - A)$ for a certain $\omega \in \mathbb{R}$.

$$0 \in \rho(\Phi(t_0)) \Longrightarrow 0 \in \rho(\Phi(t)), \forall t \geq 0,$$

thus $0 \in \rho\{e^{\omega t}\Phi(t)\}$ "according to the theorem 2.4".

If $(\omega - A)$ is the generator of an analytic semigroup of angle $\beta \in (0, \alpha)$, then A is a generator of a $\mathcal{C}_0$-semigroup, and according to the Hille-Yosida theorem we have

$$\begin{aligned}&\exists M, \omega' \geq 0 : \{\, \lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega'\} \subset \rho(A) \\ &\text{and } \|R(\lambda, A)\| \leq \tfrac{M}{\operatorname{Re}\lambda - \omega'}, \ \operatorname{Re}\lambda > \omega'.\end{aligned}$$

As A generates a $\mathcal{C}_0$-semigroup $\Phi^{-1}(t)$, and according to the lemma 2.3, $\Phi^{-1}(t)$ bounded, then we can extend it to a analytic semigroup.

And as $-A$ is the generator of an analytic semigroup $(\Phi(t))$ of angle $\alpha \in (0, \frac{\pi}{2})$ then A is the generator of an analytic semigroup of angle $\beta \in \left(\frac{\pi}{2} - \alpha, \frac{\pi}{2}\right) \subset \left(0, \frac{\pi}{2}\right)$.

Then there is a constant $M_\beta > 0$ such that

$$\|R(\lambda, A)\| \leq \frac{M_\beta}{|\lambda|}, \ \operatorname{Re}\lambda > 2\omega', |\arg \ \lambda| \leq \beta.$$

So we have :

$$\begin{cases} 1) & \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \frac{\pi}{2} - \alpha\} \subset \rho(A), \\ 2) & \exists M_\beta > 0 : \|R(\lambda, A)\| \leq \frac{M_\beta}{|\lambda|}, |\arg\ \lambda| \leq \beta, \end{cases}$$

where the constant M_β can be replaced by a larger one if necessary, so

$$\|R(\lambda, A)\| \leq \frac{M_\beta}{|\lambda|}, \ |\lambda| > 2\omega'.$$

And according to the proposition ([8] page 63) we obtain $A \in \mathcal{B}(E)$. □

The problem (2.1) is generally ill-posed. For this, consider the inverse problem corresponding to (2.1)

$$\begin{cases} v'(t) &= -Av(t); 0 < t < T, \\ v(0) &= u. \end{cases} \tag{2.2}$$

Like $-A$ the generator of an analytic semigroup in E, Cauchy's problem (2.2) is well-posed. This means that (2.2) admits a solution for each $u \in E$, and (2.2) is stable. Let us denote by $(\Phi(t))_{t\geq 0}$ the semigroup generated by $-A$. Then $v(t) = \Phi(t)u$, $0 \leq t \leq T$, is the only solution of (2.2). On the other hand if $y(t)$, $(0 \leq t \leq T)$ is the solution of (2.1). Then $y(T-t), (0 \leq t < T)$ is obviously the solution of (2.2) with the initial element $y(T) = u$. By the uniqueness of the solutions of (2.2) we obtain that

$$v(t) = y(T-t);\ 0 \leq t < T.$$

That is

$$\Phi(t)y(T) = y(T-t);\ 0 \leq t < T.$$

For $t = T$, we have

$$\Phi(T)y(T) = y(0) = x.$$

For the change of variable $t = T - t$, the operator A generates the semigroup $\Phi(T - t)$, where $y(T) = u$. Hence

$$y(t) = \Phi(T-t)y(T).$$

Then

$$\Phi(t)y(t) = \Phi(t)\Phi(T-t)y(T); 0 \leq t \leq T.$$

That is

$$\Phi(t)y(t) = \Phi(T)y(T) = x; 0 \leq t \leq T.$$

Since $\Phi(t)$ is inversible for each $t \geq 0$ ([8] page 69), we obtain

$$y(t) = \Phi^{-1}(t)x \ \text{ for } \ 0 \leq t \leq T.$$

According to the proposition 2.5, $\Phi^{-1}(t), t \geq 0$ is not a family of bounded linear operators. So (2.1) is not stable. An important method for dealing with the ill-posed Cauchy problem (2.1) is the quasi-reversibility method. This method leads to the regularization of (2.1). Using the solution of the well-posed Cauchy problem :

$$\begin{cases} y'_\varepsilon(t) &= (A - \varepsilon A^b)y_\varepsilon(t); 0 \leq t \leq T, \\ y_\varepsilon(0) &= x. \end{cases} \tag{2.3}$$

Approach the solution of (2.1) where $\varepsilon > 0$ and A^b, $(b > 1)$ is defined as the fractional power. The main result of this work is : If $-A$ the generator of an analytic semigroup. Then there is the family of regularized for the ill-posed Cauchy problem (2.1). Using the quasi-reversibility method.

2.1 Regularization of 2.1

We start with the definition of regularization families for the operators.

Definition 2.6. A family $\{R_\varepsilon, t, \varepsilon > 0, t \in [0,T]\} \subset \mathcal{B}(E)$ is called a regularization family of operators for (2.1) if for each solution $y(t), (0 \le t \le T)$ of (2.1) with the initial element x and for all $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that :
1. $\varepsilon(\delta) \to 0, (\delta \to 0)$.
2. $\left\| R_{\varepsilon(\delta),t} x_\delta - y(t) \right\| \to 0, (\delta \to 0)$ for each $t \in [0,T]$, when $\| x_\delta - x \| \le \delta$.

We note that the regularization family of operators for (2.1) is not trivial if the problem (2.1) does not have the solution $y(t) \equiv 0$ only.

Indeed, it is known (Voir [8] page 67) that (2.1) has a unique solution for each initial element $x \in D$, where D is a subspace dense in E.

The main result of this work is as follows :

Theorem 2.7. *Suppose that $-A$ is the generator of an analytic semigroup, then there exists a regularization family of operators for the problem (2.1).*

Proof. We first consider the case where $-A$ is the generator of a bounded analytic semigroup of angle α and $0 \in \rho(A)$, where $\left(0 < \alpha < \frac{\pi}{2}\right)$, let $y(t)$ $(0 \le t \le T)$ a solution of (2.1) with the initial element x, checking :

$$\|x_\delta - x\| \le \delta.$$

By using the quasi-reversibility method for (2.3) with the initial element x_δ, the approximate problem admits a unique solution

$$y_{\varepsilon,\delta}(t) = V_\varepsilon(t) x_\delta.$$

Where $V_\varepsilon(t)$ is the semigroup generated by A_ε in the theorem 1.23.

We define

$$R_{\varepsilon,t} = V_\varepsilon(t) \text{ for } \varepsilon > 0 \text{ and } 0 \le t \le T.$$

Then

$$\{R_{\varepsilon,t}, \varepsilon > 0, t \in [0,T]\} \subset \mathcal{B}(E).$$

When $t = 0$ it is clear that

$$\begin{aligned} \|R_{\varepsilon,0} x_\delta - y(0)\| &= \|V_\varepsilon(0) x_\delta - y(0)\| \\ &= \|x_\delta - x\| \to 0 \text{ when } \delta \to 0. \end{aligned}$$

When $t \in [0,T]$, we have

$$\begin{aligned} \|R_{\varepsilon,t} x_\delta - y(t)\| &\le \|R_{\varepsilon,t} x_\delta - R_{\varepsilon,t} x\| + \|R_{\varepsilon,t} x - y(t)\| \\ &= \triangle_1 + \triangle_2, \varepsilon > 0. \end{aligned}$$

For estimate $\triangle_2$, we note that $x = \Phi(t)\ y(t)$. Using the inverse problem (2.2) where $\{\Phi(t)\}$ is the semigroup generated by $-A$.

According to the representation of the analytic semigroup we have

$$\Phi(t) = \frac{1}{2\pi i} \int_{\Gamma(\eta)} e^{\lambda t} R(\lambda, -A) d\lambda.$$

Where $\frac{\pi}{2} < \eta < \pi - \gamma$.

According to the proof of the theorem 1.24 (see [13]), the resolving identity and the Cauchy theorem that

$$\begin{aligned} R_{\varepsilon,t} x = V_\varepsilon(t) x &= -\frac{1}{2\pi i} \int_{\Gamma(\gamma)} e^{(\mu - \varepsilon \mu^b)t} R(\mu, A) x d\mu \\ &= -\frac{1}{2\pi i} \int_{\Gamma(\gamma)} e^{(\mu - \varepsilon \mu^b)t} R(\mu, A) \left\{ \frac{1}{2\pi i} \int_{\Gamma(\eta)} e^{\lambda t} R(\lambda, -A) d\lambda \right\} y(t) d\mu. \end{aligned}$$

Where $\frac{\pi}{2} - \alpha < \gamma < \frac{\pi}{2b}$.
Because $x = \Phi(t)\, y(t)$. We have

$$R_{\varepsilon,t}x = -\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma(\gamma)} e^{(\mu-\varepsilon\mu^b)t} \left\{\int_{\Gamma(\eta)} e^{\lambda t} \frac{R(\mu,A)+R(\lambda,-A)}{\mu+\lambda} y(t)d\lambda\right\} d\mu,$$

since

$$R(\mu,A)\times R(\lambda,-A) = \frac{R(\mu,A)+R(\lambda,-A)}{\mu+\lambda}.$$

Thus

$$\begin{aligned} R_{\varepsilon,t}x &= -\frac{1}{2\pi i}\int_{\Gamma(\gamma)} e^{(\mu-\varepsilon\mu^b)t}R(\mu,A)\left\{\frac{1}{2\pi i}\int_{\Gamma(\eta)} e^{\lambda t}(\mu+\lambda)^{-1}y(t)d\lambda\right\}d\mu \\ &\quad -\frac{1}{2\pi i}\int_{\Gamma(\eta)} e^{\lambda t}R(\lambda,-A)\left\{\frac{1}{2\pi i}\int_{\Gamma(\gamma)}(\mu+\lambda)^{-1}e^{(\mu-\varepsilon\mu^b)t}y(t)d\mu\right\}d\lambda \\ &= -\frac{1}{2\pi i}\int_{\Gamma(\eta)} e^{(\mu-\varepsilon\mu^b)t-\mu t}R(\mu,A)y(t)d\mu, \end{aligned}$$

since

$$\frac{1}{2\pi i}\int_{\Gamma(\eta)}\frac{e^{\lambda t}}{\lambda-(-\mu)}y(t)d\lambda = e^{-\mu t}y(t) \text{ and } \frac{1}{2\pi i}\int_{\Gamma(\eta)}(\mu+\lambda)^{-1}e^{(\mu-\varepsilon\mu^b)t}y(t)d\mu = 0.$$

According to the proof of the theorem 1.23 (see [13]), $R_{\varepsilon,t}x = U(\varepsilon t)y(t)$ where $\{U(t)\}$ is the semigroup generated by $-A^b$ in the theorem 1.10. By the strong continuity of $\{U(t)\}$ we obtain :

$$\triangle_2 = \|R_{\varepsilon,t}x - y(t)\| = \|U(\varepsilon t)u(t) - y(t)\| \to 0, (\delta\to 0). \tag{2.4}$$

When $\varepsilon\to 0$, $(\delta\to 0)$. Concerning $\triangle_1$, it follows from Theorem 1.23 that

$$\begin{aligned} \triangle_1 &= \|R_{\varepsilon,t}x_\delta - R_{\varepsilon,t}x\| \le \|x_\delta - x\| \,.\, \|R_{\varepsilon,t}\| \\ &\le \delta\,\|R_{\varepsilon,t}\| \le \delta\,\|V_\varepsilon(t)\| \\ &\le \delta\, M\ \exp\left(C\varepsilon^{\frac{1}{1-b}}t\right), \end{aligned}$$

where C, $M > 0$ are independent of ε and t. We choose

$$\varepsilon = \left[-TC\left(\ln\sqrt{\delta}\right)^{-1}\right]^{b-1}, 0<\delta<1. \tag{2.5}$$

Then $\varepsilon\to 0$, $(\delta\to 0)$ and $\triangle_1 \le \delta M \exp\left(C\varepsilon^{\frac{1}{1-b}}T\right)$ of (2.5), we find

$$\triangle_1 \le \delta\, M\ \exp\left(TC\left(\left[-TC\left(\ln\sqrt{\delta}\right)^{-1}\right]^{b-1}\right)^{\frac{1}{1-b}}\right),$$

where

$$\begin{aligned} \triangle_1 &\le \delta\, M\ \exp\left(TC\left[-TC\left(\ln\sqrt{\delta}\right)^{-1}\right]^{-1}\right) \\ &\le \delta\, M\ \exp\left(-\ln\sqrt{\delta}\right) = \frac{\delta M}{\sqrt{\delta}} = \sqrt{\delta}M. \end{aligned}$$

Thus

$$\triangle_1 \le \sqrt{\delta}M \to 0, (\delta\to 0). \tag{2.6}$$

By combining (2.4) with (2.6), we get

$$\forall t \in [0,T], \|R_{\varepsilon,t}x_\delta - y(t)\| \to 0; (\delta \to 0).$$

So $\{R_{\varepsilon,t}\}$ is a regularization family of operators of (2.1). Let us deal with the general case when $-A$ is the generator of an analytica semigroup. From the remark appearing after the definition 1.18 there exists a constant $\omega \in \mathbb{R}$ such that $(A-\omega)$ is the generator of a bounded analytic semigroup and $0 \in \rho(A-\omega)$. As above, there is a family of regularizing operators $\{R_{\varepsilon,t}\}$ for the problem :

$$\begin{cases} v'(t) &= (A-\omega)v(t), 0 < t \leq T, \\ v(0) &= x. \end{cases} \tag{2.7}$$

Let $y(t)$, $(0 \leq t \leq T)$ be a solution of (2.1) with the initial element x. So

$$v(t) = e^{-\omega t}y(t), 0 \leq t \leq T,$$

is the solution of (2.7) with the initial element x. So for each $t \in [0,T]$, we have

$$\begin{aligned} \left\|e^{\omega t}R_{\varepsilon(\delta),t}x_\delta - y(t)\right\| &= e^{\omega t}\left\|R_{\varepsilon(\delta),t}x_\delta - e^{-\omega t}y(t)\right\| \\ &\leq e^{\omega t}\left\|R_{\varepsilon(\delta),t}x_\delta - v(t)\right\| \to 0; (\delta \to 0). \end{aligned}$$

That is to say $\{e^{\omega t}R_{\varepsilon,t}\}$ is the regularization family of operators for (2.1). □

Remark 2.8. From the proof of the theorem 2.7 and (Theorem 1.24 (see [13])) we can give the representation of the regularization family of operators for (2.1) as follows :

$$R_{\varepsilon,t} = -\frac{1}{2\pi i}\int_{\Gamma(\gamma)} e^{(w+\mu-\varepsilon\mu^b)t}R(\omega+\mu, A)d\mu; \varepsilon > 0, t > 0.$$

Where $\frac{\pi}{2} - \alpha < \delta < \frac{\pi}{2b}$ and ω is the bound in the exponential estimate of the semigroup $e^{-At}, t \geq 0$.

3 Conclusion

Our contribution in this work is the ill-posed inverse problems. Exactly study the regularization of a class of ill-posed Cauchy problems associated with generators of analytic semigroups.

The method of regularization is consists in replacing an ill-posed problem by a well-posed problem and whose solutions are approximations of the solutions of the initial problem.

For this, the first section consists of a reminder of some mathematical concepts and properties of the theory of analytic semigroups.

The second section is devoted to the example of ill-posed Cauchy problem associated with the generators of analytic semigroups.

As a perspective, we propose to study an ill-posed inverse problem by the Tikhonov regularization method or the Lavrentiv method.

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