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SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH MITTAG–LEFFLER-TYPE POISSON DISTRIBUTION

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Abstract In this paper, we find the necessary and sufficient conditions and inclusion relations for normalized Mittag-Leffler-type Poisson distribution series to be in the class in the class $\mathcal{T}(\lambda,\mu)$ of analytic functions with negative coefficients defined in the open unit disk. Further, we consider an integral operator related to Mittag-Leffler-type Poisson distribution series. Several corollaries and consequences of the main results are also considered.

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad z \in \mathbb{U}. \tag{1.2}$$

A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{T}(\lambda,\mu)$ if it satisfies

$$\Re\{(zf'(z))' + \lambda z(zf'(z))''\} > \mu, \qquad (\lambda \geq 0, 0 \leq \mu < 1, z \in \mathbb{U}). \tag{1.3}$$

In particular, a function $f \in \mathcal{T}$ is said to be in the class $\mathcal{T}(0,\mu) = \mathcal{T}(\mu)$ if it satisfies

$$\Re\{zf''(z) + f'(z)\} > \mu,\ (0 \leq \mu < 1, z \in \mathbb{U}). \tag{1.4}$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A,B)$, $\tau \in \mathbb{C}\backslash\{0\}$, $-1 \leq B < A \leq 1$, if it satisfies the inequality

$$\left|\frac{f'(z) - 1}{(A-B)\tau - B[f'(z) - 1]}\right| < 1, \quad z \in \mathbb{U}.$$

This class was introduced by Dixit and Pal [11] and they proved the following lemma.

Lemma 1.1. *[11] If $f \in \mathcal{R}^{\tau}(A,B)$ is of the form (1.1), then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \qquad n \in \mathbb{N} \setminus \{1\}.$$

The result is sharp for the function

$$f(z) = \int_0^z \left(1 + \frac{(A-B)\,|\tau|\,t^{n-1}}{1 + Bt^{n-1}}\right)dt, \qquad (z \in \mathbb{U}; n \in \mathbb{N} \setminus \{1\}).$$

Now we recall the well Mittag-Leffler function $E_\alpha(z)$ studied by Mittag-Leffler [25] and given by

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad (z\in\mathbb{C},\Re(\alpha)>0\ ).$$

A more general function $E_{\alpha,\beta}$ generalizing $E_\alpha(z)$ was introduced by Wiman [38, 39] and defined by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad (z,\alpha,\beta\in\mathbb{C},\ \Re(\beta)>0,\ \Re(\alpha)>0). \tag{1.5}$$

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g. in [4, 6, 14, 20, 21, 23].
Observe that Mittag-Leffler function $E_{\alpha,\beta}$ does not belong to the family $\mathcal{A}$. Therefore, we consider the following normalization of the Mittag-Leffler function (see,[6, 33])

$$\begin{aligned}\mathbb{E}_{\alpha,\beta}(z) &= \Gamma(\beta)zE_{\alpha,\beta}(z)\\ &= z+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}z^n,\end{aligned} \tag{1.6}$$

where $z,\alpha,\beta\in\mathbb{C};\beta\neq 0,-1,-2,\cdots$ and $\Re(\beta)>0$, $\Re(\alpha)>0$.
Whilst formula (1.6) holds for complex-valued α,β and $z\in\mathbb{C}$, however in this paper, we shall restrict our attention to the case of real-valued α,β and $z\in\mathbb{U}$. Observe that the function $\mathbb{E}_{\alpha,\beta}$ contains many well-known functions as its special case, for example,
$\mathbb{E}_{2,1}(z)=z\cosh\sqrt{z}$, $\mathbb{E}_{2,2}(z)=\sqrt{z}\sinh\sqrt{z}$, $\mathbb{E}_{2,3}(z)=2[\cosh\sqrt{z}-1]$ and $\mathbb{E}_{2,4}(z)=6[\sinh\sqrt{z}-\sqrt{z}]/\sqrt{z}$.
The probability mass function of the Mittag–Leffler-type Poisson distribution is given by Porwal et al. [35] ,

$$P(X=k)=\frac{m^k}{\Gamma(\alpha k+\beta)\mathbb{E}_{\alpha,\beta}(m)},\quad k=0,1,2,3,\cdots, \tag{1.7}$$

where $m>0$, $\alpha>0$ and $\beta>0$ and using the normalized form of Mittag–Leffler function as assumed in (1.6), we define a power series whose coefficients are probabilities of Mittag–Leffler-type Poisson distribution series, as below:

$$\Psi^m_{\alpha,\beta}(z):=z+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)}z^n,\qquad z\in\mathbb{U}.$$

Further, we define the series

$$\Phi^m_{\alpha,\beta}(z):=2z-\Psi^m_{\alpha,\beta}(z)=z-\sum_{n=2}^{\infty}\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)}z^n,\qquad z\in\mathbb{U}. \tag{1.8}$$

Using the concept convolution or Hadamard product of two series, we introduce the convolution operator

$$\mathcal{I}^m_{\alpha,\beta}f(z)=\Psi^m_{\alpha,\beta}(z)*f(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)}a_nz^n,\qquad z\in\mathbb{U},$$

where the function $f(z)$ of the form (1.1) and $*$ denote the convolution or Hadamard product of two series.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, using hypergeometric functions (see for example, [3, 9, 19, 24, 27, 36]), generalized Bessel functions (see for example, [5, 18, 26, 30]), Struve functions (see for example, [10, 22, 37]), Poisson distribution series (see for example, [12, 15, 17, 28, 29, 32, 34]) and Pascal distribution series (see for example, [7, 8, 13, 16, 31]), in this paper we determine the necessary and sufficient conditions for $\Phi^m_{\alpha,\beta}$ to be in the class $\mathcal{T}(\lambda,\mu)$. Furthermore, we give sufficient conditions for $\mathcal{I}^m_{\alpha,\beta}\left(\mathcal{R}^\tau(A,B)\right)\subset\mathcal{T}(\lambda,\mu)$. Finally, we give necessary and sufficient conditions for the integral operator $\mathcal{G}^m_{\alpha,\beta}(z)=\int_0^z\frac{\Phi^m_{\alpha,\beta}(t)}{t}dt$ to be in the class $\mathcal{T}(\lambda,\mu)$.

2 Necessary and sufficient conditions

To establish our main results, we shall require the following lemma.

Lemma 2.1. *A function $f \in \mathcal{T}$ in the form (1.2) belongs to the class $\mathcal{T}(\lambda, \mu)$ if and only if*

$$\sum_{n=2}^{\infty} n^2[1-\lambda+n\lambda]|a_n| \leq 1-\mu. \tag{2.1}$$

Lemma 2.1 can be proved using the same technique as in [1] (see also, [2]).
Unless otherwise mentioned, we shall assume in this paper that $\alpha, m > 0, \lambda \geq 0$ and $0 \leq \mu < 1$.
Firstly, we obtain the necessary and sufficient conditions for $\Phi^m_{\alpha,\beta}$ to be in the class $\mathcal{T}(\lambda, \mu)$.

Theorem 2.2. *If $\beta > 3$, then $\Phi^m_{\alpha,\beta}(z) \in \mathcal{T}(\lambda, \mu)$ if and only if*

$$\begin{aligned}
&\frac{\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\Bigg[\frac{\lambda}{\alpha^3}\left(E_{\alpha,\beta-3}(m)-\frac{1}{\Gamma(\beta-3)}\right)\\
&+\left(\frac{(6-3\beta)\lambda}{\alpha^3}+\frac{(2\lambda+1)}{\alpha^2}\right)\left(E_{\alpha,\beta-2}(m)-\frac{1}{\Gamma(\beta-2)}\right)\\
&+\left(\frac{\lambda(3\beta^2-9\beta+7)}{\alpha^3}+\frac{(2\lambda+1)(3-2\beta)}{\alpha^2}+\frac{(\lambda+2)}{\alpha}\right)\left(E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right)\\
&+\left(\frac{\lambda(1-\beta)^3}{\alpha^3}+\frac{(2\lambda+1)(1-\beta)^2}{\alpha^2}+\frac{(\lambda+2)(1-\beta)}{\alpha}+1\right)\left(E_{\alpha,\beta}(m)-\frac{1}{\Gamma(\beta)}\right)\Bigg]\\
\leq\ & 1-\mu.
\end{aligned} \tag{2.2}$$

Proof. In view of Lemma 2.1 and (2.1) it suffices to show that

$$\Lambda_1=\sum_{n=2}^{\infty} n^2[1-\lambda+n\lambda]\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)} \leq 1-\mu.$$

Now

$$\begin{aligned}
\Lambda_1 &= \sum_{n=1}^{\infty}(n+1)^2[1-\lambda+(n+1)\lambda]\frac{\Gamma(\beta)m^n}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)}\\
&= \sum_{n=1}^{\infty}[\lambda n^3+(2\lambda+1)n^2+(\lambda+2)n+1]\frac{\Gamma(\beta)m^n}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)}\\
&= \Bigg[\sum_{n=1}^{\infty}\frac{\lambda}{\alpha^3}[(\alpha n+\beta-1)(\alpha n+\beta-2)(\alpha n+\beta-3)\\
&\quad+(6-3\beta)(\alpha n+\beta-1)(\alpha n+\beta-2)\\
&\quad+(3\beta^2-9\beta+7)(\alpha n+\beta-1)+(1-\beta)^3]\\
&\quad+\frac{2\lambda+1}{\alpha^2}[(\alpha n+\beta-1)(\alpha n+\beta-2)\\
&\quad+(3-2\beta)(\alpha n+\beta-1)+(1-\beta)^2]\\
&\quad+\frac{\lambda+2}{\alpha}[(\alpha n+\beta-1)+(1-\beta)]+1\Bigg]\frac{\Gamma(\beta)m^n}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)}.
\end{aligned}$$

Now by simple computation,

$$\begin{aligned}
\Lambda_1 &= \frac{\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\left[\frac{\lambda}{\alpha^3}\left(E_{\alpha,\beta-3}(m)-\frac{1}{\Gamma(\beta-3)}\right)\right. \\
&\quad+\left(\frac{(6-3\beta)\lambda}{\alpha^3}+\frac{(2\lambda+1)}{\alpha^2}\right)\left(E_{\alpha,\beta-2}(m)-\frac{1}{\Gamma(\beta-2)}\right) \\
&\quad+\left(\frac{\lambda(3\beta^2-9\beta+7)}{\alpha^3}+\frac{(2\lambda+1)(3-2\beta)}{\alpha^2}+\frac{(\lambda+2)}{\alpha}\right)\left(E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right) \\
&\quad+\left.\left(\frac{\lambda(1-\beta)^3}{\alpha^3}+\frac{(2\lambda+1)(1-\beta)^2}{\alpha^2}+\frac{(\lambda+2)(1-\beta)}{\alpha}+1\right)\left(E_{\alpha,\beta}(m)-\frac{1}{\Gamma(\beta)}\right)\right] \\
&\leq 1-\mu,
\end{aligned}$$

by the given hypothesis (2.2). This completes the proof of Theorem 2.2. □

3 Inclusion Property

Making use of Lemma 1.1 we will study the action of the Mittag–Leffler-type Poisson distribution series on the class $\mathcal{T}(\lambda,\mu)$.

Theorem 3.1. *Let $\beta > 2$. If $f \in \mathcal{R}^\tau(A,B)$ and the inequality*

$$\begin{aligned}
&\frac{(A-B)\,|\tau|\,\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\left[\frac{\lambda}{\alpha^2}\left(E_{\alpha,\beta-2}(m)-\frac{1}{\Gamma(\beta-2)}\right)\right. \\
&+\left(\frac{\lambda(3-2\beta)+\alpha(\lambda+1)}{\alpha^2}\right)\left(E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right) \\
&+\left.\left(\frac{\lambda(1-\beta)^2}{\alpha^2}+\frac{(\lambda+1)(1-\beta)}{\alpha}+1\right)\left(E_{\alpha,\beta}(m)-\frac{1}{\Gamma(\beta)}\right)\right] \\
\leq\; & 1-\mu \qquad (3.1)
\end{aligned}$$

is satisfied then $\mathcal{I}^m_{\alpha,\beta} f \in \mathcal{T}(\lambda,\mu)$.

Proof. In view of Lemma 2.1 and (2.1) it suffices to show that

$$\sum_{n=2}^{\infty} n^2[1-\lambda+n\lambda]\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)}\,|a_n| \leq 1-\mu.$$

Since $f \in \mathcal{R}^\tau(A,B)$, using Lemma 1.1 we have

$$|a_n| \leq \frac{(A-B)\,|\tau|}{n},\; n \in \mathbb{N}\setminus\{1\},$$

therefore, it is enough to show that

$$(A-B)\,|\tau|\left[\sum_{n=2}^{\infty} n[1-\lambda+n\lambda]\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)}\right] \leq 1-\mu.$$

Now,let

$$\begin{aligned}
\Lambda_2 &= (A-B)\,|\tau| \left[\sum_{n=2}^{\infty} n[n\lambda+1-\lambda]\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)}\right] \\
&= (A-B)\,|\tau| \left[\sum_{n=1}^{\infty}(n+1)[1-\lambda+(n+1)\lambda]\frac{\Gamma(\beta)m^{n}}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)}\right] \\
&= (A-B)\,|\tau| \left[\sum_{n=1}^{\infty}[\lambda n^2+(\lambda+1)n+1]\frac{\Gamma(\beta)m^{n}}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)}\right] \\
&= (A-B)\,|\tau|\,\Gamma(\beta) \left[\frac{\lambda}{\alpha^2}\sum_{n=1}^{\infty}[(\alpha n+\beta-1)(\alpha n+\beta-2)\right. \\
&\quad +(3-2\beta)(\alpha n+\beta-1)+(1-\beta)^2]\frac{m^n}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)} \\
&\quad +\frac{\lambda+1}{\alpha}\sum_{n=1}^{\infty}[(\alpha n+\beta-1)+(1-\beta)]\frac{m^n}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)} \\
&\quad \left. +\sum_{n=1}^{\infty}\frac{m^n}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)}\right]
\end{aligned}$$

Now by simple computation,

$$\begin{aligned}
\Lambda_2 &= \frac{(A-B)\,|\tau|\,\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\left[\frac{\lambda}{\alpha^2}\sum_{n=1}^{\infty}\frac{m^n}{\Gamma(\alpha n+\beta-2)}\right. \\
&\quad +\frac{\lambda(3-2\beta)+\alpha(\lambda+1)}{\alpha^2}\sum_{n=1}^{\infty}\frac{m^n}{\Gamma(\alpha n+\beta-1)} \\
&\quad \left.+\left(\frac{\lambda(1-\beta)^2}{\alpha^2}+\frac{(\lambda+1)(1-\beta)}{\alpha}+1\right)\sum_{n=1}^{\infty}\frac{m^n}{\Gamma(\alpha n+\beta)}\right] \\
&= \frac{(A-B)\,|\tau|\,\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\left[\frac{\lambda}{\alpha^2}\left(E_{\alpha,\beta-2}(m)-\frac{1}{\Gamma(\beta-2)}\right)\right. \\
&\quad +\left(\frac{\lambda(3-2\beta)+\alpha(\lambda+1)}{\alpha^2}\right)\left(E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right) \\
&\quad \left.+\left(\frac{\lambda(1-\beta)^2}{\alpha^2}+\frac{(\lambda+1)(1-\beta)}{\alpha}+1\right)\left(E_{\alpha,\beta}(m)-\frac{1}{\Gamma(\beta)}\right)\right] \\
&\leq 1-\mu.
\end{aligned}$$

□

4 An integral operator

Theorem 4.1. *If $\beta > 2$, then the integral operator*

$$\mathcal{G}^m_{\alpha,\beta}(z) := \int_0^z \frac{\Phi^m_{\alpha,\beta}(t)}{t}dt,\ z\in\mathbb{U}, \tag{4.1}$$

is in the class $\mathcal{T}(\lambda, \mu)$ *if and only if*

$$\begin{aligned} &\frac{\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\left[\frac{\lambda}{\alpha^2}\left(E_{\alpha,\beta-2}(m)-\frac{1}{\Gamma(\beta-2)}\right)\right. \\ &+\left(\frac{\lambda(3-2\beta)+\alpha(\lambda+1)}{\alpha^2}\right)\left(E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right) \\ &\left.+\left(\frac{\lambda(1-\beta)^2}{\alpha^2}+\frac{(\lambda+1)(1-\beta)}{\alpha}+1\right)\left(E_{\alpha,\beta}(m)-\frac{1}{\Gamma(\beta)}\right)\right] \\ \leq\ & 1-\mu. \end{aligned} \tag{4.2}$$

Proof. According to (1.8) it follows that

$$\mathcal{G}^m_{\alpha,\beta}(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)} \frac{z^n}{n}.$$

Using Lemma 2.1 and (2.1), the integral operator $\mathcal{G}^m_{\alpha,\beta}(z)$ belongs to $\mathcal{T}(\lambda,\mu)$ if and only if

$$\sum_{n=2}^{\infty} n[1-\lambda+n\lambda]\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)} \leq 1-\mu.$$

By a similar proof like those of Theorem 3.1 we get that $\mathcal{G}^m_{\alpha,\beta} \in \mathcal{T}(\lambda,\mu)$ if and only if (4.2) holds. □

5 Corollaries and consequences

In this section, we apply our main results in order to deduce each of the following corollaries and consequences.

Corollary 5.1. *If* $\beta > 2$, *then* $\Phi^m_{\alpha,\beta}(z) \in \mathcal{T}(\mu)$ *if and only if*

$$\begin{aligned} &\frac{\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\left[\frac{1}{\alpha^2}\left(E_{\alpha,\beta-2}(m)-\frac{1}{\Gamma(\beta-2)}\right)\right. \\ &+\left(\frac{3+2\alpha-2\beta}{\alpha^2}\right)\left(E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right) \\ &\left.+\left(\frac{(1-\beta)^2}{\alpha^2}+\frac{2(1-\beta)}{\alpha}+1\right)\left(E_{\alpha,\beta}(m)-\frac{1}{\Gamma(\beta)}\right)\right] \\ \leq\ & 1-\mu. \end{aligned} \tag{5.1}$$

Corollary 5.2. *Let* $\beta > 1$. *If* $f \in \mathcal{R}^\tau(A,B)$ *and the inequality*

$$\begin{aligned} &\frac{(A-B)\,|\tau|}{\alpha\mathbb{E}_{\alpha,\beta}(m)}\left[E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right. \\ &\left.+(1+\alpha-\beta)\left(E_{\alpha,\beta}(m)-\frac{1}{\Gamma(\beta)}\right)\right] \\ \leq\ & 1-\mu \end{aligned} \tag{5.2}$$

is satisfied then $\mathcal{I}^m_{\alpha,\beta} f \in \mathcal{T}(\mu)$.

Corollary 5.3. *If* $\beta > 1$, *then the integral operator given by (4.1) is in the class* $\mathcal{T}(\mu)$ *if and only if*

$$\begin{aligned} &\frac{\Gamma(\beta)}{\alpha\mathbb{E}_{\alpha,\beta}(m)}\left[E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right. \\ &\left.+(1+\alpha-\beta)\left(E_{\alpha,\beta}(m)-\frac{1}{\Gamma(\beta)}\right)\right] \\ \leq\ & 1-\mu. \end{aligned} \tag{5.3}$$

Concluding remark By fixing $\lambda = 1$ in (1.3) one can define a new class

$$\mathcal{M}(\mu) = \{f \in T : \mathfrak{Re}\{f'(z) + 3zf''^2 f'''(z)\} > \mu,, \ (0 \le \mu < 1, z \in \mathbb{U})\}$$

for which by Lemma 2.1,we state the necessary and sufficient conditions as below

$$\sum_{n=2}^{\infty} n^3 |a_n| \le 1 - \mu. \tag{5.4}$$

Thus by taking $\lambda = 1$ in Theorems 2.2- 4.1, one can easily state analogues results for the function class $\mathcal{M}(\mu)$, we left this as an exercise to interested readers.

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