Palestine Journal of Mathematics

Vol. 11(1)(2022) , 504–523

Entropy solutions for unilateral parabolic problems with L^1-data in Musielak-Orlicz-Sobolev spaces

Nourdine EL AMARTY, Badr EL HAJI and Mostafa EL MOUMNI

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 35K55 ; Secondary 35K85 .

Keywords and phrases: Musielak-Orlicz-Sobolev spaces, Parabolic inequalities, entropy solutions, truncations .

The authors are grateful to Professor **Abdelmoujib Benkirane,** *for his comments and suggestions.*

Abstract We prove the existence of entropy solution for the obstacle parabolic equations : $\frac{\partial u}{\partial t} - \text{div}\Big(a(x,t,u,\nabla u) + \Phi(u)\Big) + g(u)\varphi(x, |\nabla u|) = f$ in Q, where $-\text{div}\Big(a(x,t,u,\nabla u)\Big)$ is a Leray-Lions operator, $\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)$,The function $g(u)\varphi(x, |\nabla u|)$ is a nonlinear lower order term with natural growth with respect to $|\nabla u|$, without satisfying the sign condition and the datum is assumed belongs to $L^1(Q)$.

1 Introduction

Let Q be the cylinder $\Omega\times(0,T)$, $T > 0$, Ω is a bounded domain of $\mathbb{R}^N$ with the segment property, and let φ and ψ two complementary Musielak Orlicz functions. In this work, we consider the following boundary value problem:

$$\begin{cases} u \geq \zeta \text{ a.e. in } Q, \\ \frac{\partial u}{\partial t} - \text{div}\Big(a(x,t,u,\nabla u) + \Phi(u)\Big) + g(u)\varphi(x, \mid \nabla u \mid) = f \text{ in } Q, \\ u = 0 \text{ on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \text{ in } \Omega, \end{cases} \tag{1.1}$$

Let $A : D(A) \subset W_0^{1,x}L_\varphi(Q) \longrightarrow W^{-1,x}L_\psi(Q)$ be a mapping given by

$$A(u) = -\text{div}(a(x,t,u,\nabla u)),$$

where $a : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, measurable with respect to x in Ω for every (t,s,ξ) in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) such that for all ξ and ξ^* in $\mathbb{R}^N$, $\xi \neq \xi^*$,

$$a(x,t,s,\xi)\xi \geq \alpha\varphi(x, \mid \xi \mid), \tag{1.2}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi^*)][\xi - \xi^*] > 0, \tag{1.3}$$

There exist two Musielak Orlicz functions φ and P such that $P \prec\prec \varphi$ such that for a.e. $(x,t) \in Q$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$

$$\mid a(x,t,s,\xi) \mid \leq c(x,t) + k_1\overline{P}^{-1}\varphi(k_2 \mid s \mid) + k_3\psi_x^{-1}\varphi(k_4 \mid \xi \mid), \tag{1.4}$$

where $c(x,t)$ belongs to $E_\psi(Q)$, $c \geq 0$, k_i $(i = 1,2,3,4)$ to $\mathbb{R}^+$ and $\alpha \in \mathbb{R}^+_*$.

We assume that there exists a positive function M such that

$$\lim_{t\longrightarrow\infty} \frac{M(t)}{t} = \infty, \quad M(t) \leq \text{ess}\inf_{x\in\Omega} \varphi(x,t). \tag{1.5}$$

$$\Phi : \mathbb{R} \to \mathbb{R}^N \text{ is a continuous function,} \tag{1.6}$$

$$f \in L^1(Q),\ f \geq 0, \tag{1.7}$$

$$u_0 \in L^1(\Omega),\ u_0 \geq \zeta(x) \text{ and } \zeta \in L^\infty(\Omega) \cap W_0^{1,x} E_\varphi(\Omega), \tag{1.8}$$

and

$$g : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is an integrable function on } \mathbb{R}^+. \tag{1.9}$$

In the classical Sobolev spaces Dall'aglio-Orsina [17] and Porretta [34] proved the existence of solutions for the problem $(\mathcal{P})$, where $b(u) = u$ and g is a nonlinearity with the following "natural" growth condition (of order p):

$$|g(x,t,s,\xi)| \leq b(s) \left(|\xi|^p + c(x,t)\right), \tag{1.10}$$

and which satisfies the classical sign condition,

$$g(x,t,s,\xi)s \geq 0. \tag{1.11}$$

The right hand side f is assumed to belong to $L^1(Q)$. This result generalizes analogous one of Boccardo - Gallouët [14], see also [15, 16] for related topics.

$$|g(x,t,s,\xi)| \geq \beta|\xi|^p \text{ for } |s| \geq \gamma$$

In the framework of Orlicz-Sobolev spaces,in [2] the autors have studied the existence and uniqueness result to the nonlinear parabolic equations whose prototype is

$$\begin{cases} \frac{\partial b(u)}{\partial t} - \Delta_M u - \operatorname{div}\left(\bar{c}(x,t)\bar{M}^{-1}M\left(\frac{\alpha_0}{\lambda}|b(u)|\right)\right) = f \text{ in } Q_T, \\ u(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ b(u)(t=0) = b\left(u_0\right) \qquad \text{in } \Omega. \end{cases} \tag{1.12}$$

where $-\Delta_M u = -\operatorname{div}\left((1+|u|)^2 Du \frac{\log(e+Du)}{|Du|}\right)$, $\bar{c} \in \left(L^\infty\left(Q_T\right)\right)^N$ and $M(t) = t\log(e+t)$ is an N -function. The data f and $b\left(u_0\right)$ in $L^1\left(Q_T\right)$ and $L^1(\Omega)$.
Another approach to define a suitable generalized solution is that of entropy solution which was introduced in [7] in the elliptic case and by Prignet [33] in the parabolic case.

Aharouch and Bennouna [3] have proved the existence and uniqueness of entropy solutions in the framework of Orlicz-Sobolev spaces $W_0^1 L_M(\Omega)$ assuming the Δ_2 condition on the N -function M.

In the generalized-Orlicz spaces, the work [4] is a continuation of [3] where AlHawmi, Benkirane, Hjiaj and Touzani proved the existence and uniqueness of entropy solution for

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) = f & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Phi = 0$ and $\bar{M}$ satisfy the Δ_2 -condition. Antontsev and Shmarev [5] proved theorems of existence and uniqueness of weak solutions of Dirichlet problem for a class of nonlinear parabolic equations with nonstandard anisotropic growth conditions in the variable exponent Lebesgue spaces. Equations of this class generalize the evolution $p(x,t)$ -Laplacian of the type

$$\begin{cases} \frac{\partial u}{\partial t} - \sum_i \frac{\partial}{\partial x_i}\left[a_i(x,t,u)\left|D_i u\right|^{p_i(x,t)-2} D_i u + b_i(x,t,u)\right] = 0 & \text{in } Q_T \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases} . \tag{1.13}$$

In general Musielak-Sobolev spaces, the authors in [1] have proved the existence of solutions of the unilateral problem

$$Au - \operatorname{div}\Phi(x,u) + H(x,u,\nabla u) = \mu$$

where A is a Leray-Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega), \mu \in L(\Omega) + W^{-1}E_{\bar{M}}(\Omega)$, where M and $\bar{M}$ are two complementary Musielak-Orlicz functions and both the first and the second lower terms Φ and H satisfies only the growth condition and $u \geq \zeta$ where ζ is a measurable function, and further works can be found in[8, 6, 9, 10, 11, 12, 13, 27, 28, 29, 30, 31, 32, 36].

This paper is motivated by recent advances in mathematical modeling of non-Newtonian fluids and elastic mechanics, in particular, the electro-rheological fluids (smart fluids). This important class of fluids is characterized by the change of viscosity which is not easy and which depends on the electric field. These fluids, which are known under the name ER fluids, have many applications in elastic mechanics, fluid dynamics etc.

The aim of this work is to solve the obstacle problem associated to (1.1) in the case where $f \in L^1(Q)$ and without assuming any growth restriction on φ, $\Phi(u) \not\equiv 0$, while the function $g(u)\varphi(x,|\nabla u|)$ is not satisfying the sign condition. The existence of solutions is proved via a sequence of penalized problems.

2 Background

Here we give some definitions and properties that concern Musielak-Orlicz spaces (see [37]).

2.1 Musielak-Orlicz functions

Let Ω be an open subset of $\mathbb{R}^n$.

A Musielak-Orlicz function φ is a real-valued function defined in $\Omega \times \mathbb{R}_+$ such that
a) $\varphi(x,t)$ is an N-function i.e. convex, nondecreasing, continuous, $\varphi(x,0) = 0, \varphi(x,t) > 0$ for all $t > 0$ and

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0, \qquad \lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0.$$

b) $\varphi(\cdot,t)$ is a Lebesgue measurable function.

Now, let $\varphi_x(t) = \varphi(x,t)$ and let φ_x^{-1} be the non-negative reciprocal function with respect to t, i.e the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi\left(x, \varphi_x^{-1}(t)\right) = t.$$

The Musielak-orlicz function φ is said to satisfy the Δ_2 -condition if for some $k > 0$, and a non negative function h, integrable in Ω, we have

$$\varphi(x,2t) \leq k\varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \tag{2.1}$$

When 2.1 holds only for $t \geq t_0 > 0$, then φ is said to satisfy the Δ_2 -condition near infinity. Let φ and γ be two Musielak-orlicz functions, we say that φ dominate γ and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x \in \Omega$

$$\gamma(x,t) \leq \varphi(x,ct) \text{ for all } t \geq t_0, \quad (\text{ resp. for all } t \geq 0 \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma \prec\prec \varphi$ if for every positive constant c we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0, \quad \left(\text{resp. } \lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0\right).$$

Remark 2.1. (see [30]) If $\gamma \prec\prec \varphi$ near infinity, then $\forall\varepsilon > 0$ there exists a nonnegative integrable function h, such that

$$\gamma(x,t) \leq \varphi(x,\varepsilon t) + h(x). \text{ for all } t \geq 0 \text{ and for a. e. } x \in \Omega. \tag{2.2}$$

2.2 Musielak-Orlicz-Sobolev spaces

For a Musielak-Orlicz function φ and a measurable function $u:\Omega\longrightarrow\mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u)=\int_{\Omega}\varphi(x,|u(x)|)dx.$$

The set $K_{\varphi}(\Omega)=\{u:\Omega\longrightarrow\mathbb{R}\text{ measurable }/\rho_{\varphi,\Omega}(u)<\infty\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_{\varphi}(\Omega)$ is the vector space generated by $K_{\varphi}(\Omega)$, that is, $L_{\varphi}(\Omega)$ is the smallest linear space containing the set $K_{\varphi}(\Omega)$. Equivalently

$$L_{\varphi}(\Omega)=\left\{u:\Omega\longrightarrow\mathbb{R}\text{ measurable }/\rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right)<\infty,\text{ for some }\lambda>0\right\}.$$

For a Musielak-Orlicz function φ we put: $\psi(x,s)=\sup_{t>0}\{st-\varphi(x,t)\}$, ψ is the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sens of Young with respect to the variable s in the space $L_{\varphi}(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega}=\inf\left\{\lambda>0/\int_{\Omega}\varphi\left(x,\frac{|u(x)|}{\lambda}\right)dx\leq 1\right\},$$

which is called the Luxemburg norm and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega}=\sup_{\|v\|_{\psi}\leq 1}\int_{\Omega}|u(x)v(x)|dx,$$

where ψ is the Musielak Orlicz function complementary to φ. These two norms are equivalent (see [37])

We will also use the space $E_{\varphi}(\Omega)$ defined by

$$E_{\varphi}(\Omega)=\left\{u:\Omega\longrightarrow\mathbb{R}\text{ measurable }/\rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right)<\infty,\text{ for all }\lambda>0\right\}.$$

A Musielak function φ is called locally integrable on Ω if $\rho_{\varphi}(t\chi_D)<\infty$ for all $t>0$ and all measurable $D\subset\Omega$ with meas $(D)<\infty$ Let φ a Musielak function which is locally integrable. Then $E_{\varphi}(\Omega)$ is separable (see [37], Theorem 7.10) .

We say that sequence of functions $u_n\in L_{\varphi}(\Omega)$ is modular convergent to $u\in L_{\varphi}(\Omega)$ if there exists a constant $\lambda>0$ such that

$$\lim_{n\to\infty}\rho_{\varphi,\Omega}\left(\frac{u_n-u}{\lambda}\right)=0.$$

For any fixed nonnegative integer m we define

$$W^mL_{\varphi}(\Omega)=\{u\in L_{\varphi}(\Omega):\forall|\alpha|\leq m, D^{\alpha}u\in L_{\varphi}(\Omega)\},$$

and

$$W^mE_{\varphi}(\Omega)=\{u\in E_{\varphi}(\Omega):\forall|\alpha|\leq m, D^{\alpha}u\in E_{\varphi}(\Omega)\},$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$ with nonnegative integers α_i, $|\alpha|=|\alpha_1|+\ldots+|\alpha_n|$ and $D^{\alpha}u$ denote the distributional derivatives.
The space $W^mL_{\varphi}(\Omega)$ is called the Musielak Orlicz Sobolev space.

Let

$$\bar{\rho}_{\varphi,\Omega}(u)=\sum_{|\alpha|\leq m}\rho_{\varphi,\Omega}(D^{\alpha}u)\text{ and }\|u\|^m_{\varphi,\Omega}=\inf\left\{\lambda>0:\bar{\rho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right)\leq 1\right\}$$

for $u\in W^mL_{\varphi}(\Omega)$.
These functionals are a convex modular and a norm on $W^mL_{\varphi}(\Omega)$, respectively, and the pair $\left(W^mL_{\varphi}(\Omega),\|\|^m_{\varphi,\Omega}\right)$ is a Banach space if φ satisfies the following condition (see[37]):

$$\text{there exist a constant } c_0>0 \text{ such that } \inf_{x\in\Omega}\varphi(x,1)\geq c_0. \tag{2.3}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.

The space $W_0^m L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. and the space $W_0^m E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W_0^m L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$, the following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \right\},$$

and

$$W^{-m} E_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \right\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \bar{\rho}_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

For φ and her complementary function ψ, the following inequality is called the Young inequality (see[37]):

$$ts \leq \varphi(x,t) + \psi(x,s), \quad \forall t, s \geq 0, x \in \Omega, \tag{2.4}$$

this inequality implies that

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1. \tag{2.5}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} > 1, \tag{2.6}$$

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} \leq 1. \tag{2.7}$$

For two complementary Musielak Orlicz functions φ and ψ, let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, then we have the Holder inequality (see[37]):

$$\left| \int_\Omega u(x)v(x)dx \right| \leq \|u\|_{\varphi,\Omega} \| |v| \|_{\psi,\Omega}. \tag{2.8}$$

Lemma 2.2. *[30]*

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:

i)There exists a constant $c > 0$ such that inf $_{x\in\Omega}\varphi(x,1) \geq c$.

ii) There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$ we have

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq |t|^{\left(\frac{A}{\log\left(\frac{1}{\sqrt{x-y}}\right)}\right)}, \quad \forall t \geq 1. \tag{2.9}$$

iii)

$$\textit{If } D \subset \Omega \textit{ is a bounded measurable set, then } \int_D \varphi(x,1)dx < \infty. \tag{2.10}$$

iv) There exists a constant $C > 0$ such that $\psi(x,1) \leq C$ a.e in Ω.

Under this assumptions, $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\bar{\Omega})$ is dense in $W^1 L_\varphi(\Omega)$ the modular convergence.

Consequently, the action of a distribution S in $W^{-1} L_\psi(\Omega)$ on an element u of $W_0^1 L_\varphi(\Omega)$ is well defined. It will be denoted by $< S, u >$.

2.3 Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let Ω a bounded open subset of $\mathbb{R}^N$ and let $Q = \Omega\times]0,T[$ with some given $T > 0$. Let φ and ψ be two complementary Musielak-Orlicz functions. For each $\alpha \in \mathbb{N}^N$ denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows.

$$W^{1,x}L_\varphi(Q) = \{u \in L_\varphi(Q) : \forall|\alpha| \leq 1 D_x^\alpha u \in L_\varphi(Q)\}$$

et

$$W^{1,x}E_\varphi(Q) = \{u \in E_\varphi(Q) : \forall|\alpha| \leq 1 D_x^\alpha u \in E_\varphi(Q)\}.$$

This second space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\leq 1} \|D_x^\alpha u\|_{\varphi,Q}$$

These spaces constitute a complementary system since Ω satisfies the segment property.These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q)$ which has $(N+1)$ copies.

We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$ If $u \in W^{1,x}L_\varphi(Q)$ then the function $t \to u(t) = u(\cdot,t)$ is defined on $[0,T]$ with values in $W^1L_\varphi(\Omega)$. If $u \in W^{1,x}E_\varphi(Q)$, then $u \in W^1E_\varphi(\Omega)$ and it is strongly measurable. Furthermore, the imbedding $W^{1,x}E_\varphi(Q) \subset L^1(0,T,W^1E_\varphi(\Omega))$ holds. The space $W^{1,x}L_\varphi(Q)$ is not in general separable, for $u \in W^{1,x}L_\varphi(Q)$ we cannot conclude that the function $u(t)$ is measurable on $[0,T]$.

However, the scalar function $t \to \|u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_\varphi(Q)$ is defined as the norm closure of $\mathcal{D}(Q)$ in $W^{1,x}E_\varphi(Q)$. We can easily show as in [23] that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q)$ with respect of the weak $*$ topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is a limit in $W^{1,x}L_\varphi(Q)$ of some subsequence $(v_j) \in \mathcal{D}(Q)$ for the modular convergence, i.e. there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$

$$\int_Q \varphi\left(x, \left(\frac{D_x^\alpha v_j - D_x^\alpha u}{\lambda}\right)\right) dxdt \to 0 \text{ as } j \to \infty,$$

this implies that (v_j) converges to u in $W^{1,x}L_\varphi(Q)$ for the weak topology $\sigma(\Pi L_\varphi, \Pi L_\psi)$ Consequently

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi,\Pi E_\psi)} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi,\Pi L_\psi)}$$

The space of functions satisfying such a property will be denoted by $W_0^{1,x}L_\psi(Q)$ Furthermore, $W_0^{1,x}E_\varphi(Q) = W_0^{1,x}L_\varphi(Q) \cap \Pi E_\varphi(Q)$. Thus, both sides of the last inequality are equivalent norms on $W_0^{1,x}L_\varphi(Q)$. We then have the following complementary system:

$$\begin{pmatrix} W_0^{1,x}L_\varphi(Q) & F \\ W_0^{1,x}E_\varphi(Q) & F_0 \end{pmatrix}$$

where F states for the dual space of $W_0^{1,x}E_\varphi(Q)$. and can be defined, except for an isomorphism, as the quotient of ΠL_ψ by the polar set $W_0^{1,x}E_\varphi(Q)^\perp$. It will be denoted by $F = W^{-1,x}L_\psi(Q)$, where

$$W^{-1,x}L_\psi(Q) = \left\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_\psi(Q) \right\}$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\psi,Q}$$

where the infimum is taken over all possible decompositions

$$f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha, \quad f_\alpha \in L_\psi(Q)$$

The space F_0 is then given by

$$F_0 = \left\{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_\psi(Q) \right\}$$

and is denoted by $F_0 = W^{-1,x} E_\psi(Q)$.

3 Truncation Operator

$T_k, k > 0$, denotes the truncation function at level k defined on $\mathbb{R}$ by $T_k(r) = \max(-k, \min(k, r))$. The following abstract lemmas will be applied to the truncation operators.

Lemma 3.1. *([38]) Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak-Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$ Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial u}{\partial x_i} & a.e\ in \{x \in \Omega : u(x) \in D\} \\ 0 & a.e\ in\ \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

Lemma 3.2. *[40] (Poincare inequality). Let φ a Musielak Orlicz function which satisfies the assumptions of lemma 2.2, suppose that $\varphi(x, t)$ decreases with respect of one of coordinate of x Then, there exists a constant $c > 0$ depends only of Ω such that*

$$\int_\Omega \varphi(x, |u(x)|) dx \leq \int_\Omega \varphi(x, c|\nabla u(x)|) dx, \quad \forall u \in W_0^1 L_\varphi(\Omega).$$

Lemma 3.3. *Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then, there exists a sequence $(u_n) \subset \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_\varphi(\Omega)$$

Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.

Lemma 3.4. *[25] Let (f_n) , $f \in L^1(\Omega)$ such that*
i) $f_n \geq 0$ a.e in Ω
ii) $f_n \longrightarrow f$ a.e in Ω
iii) $\int_\Omega f_n(x) dx \longrightarrow \int_\Omega f(x) dx$
then $f_n \longrightarrow f$ strongly in $L^1(\Omega)$.

Lemma 3.5. *(Jensen inequality). [39] Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ a convex function and $g : \Omega \longrightarrow \mathbb{R}$ is function measurable, then*

$$\varphi\left(\int_\Omega g d\mu\right) \leq \int_\Omega \varphi \circ g d\mu.$$

Lemma 3.6. *(The Nemytskii Operator)[30] . Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak-Orlicz functions. Let $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:*

$$|f(x, s)| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|). \tag{3.1}$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$. Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is continuous from

$$\mathcal{P}\left(E_\varphi(\Omega), \frac{1}{k_2}\right) = \left\{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2}\right\}.$$

into $L_\psi(\Omega)$.
Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\mathcal{P}\left(E_\varphi(\Omega), \frac{1}{k_2}\right)$ to $E_\gamma(\Omega)$.

4 Existence results

This section is devoted to study the following existence theorem.

Theorem 4.1. *Assume that (1.2)-(1.9) hold. Then there exists at least one solution of the problem (1.1), in the following sense:*

$$\begin{cases} u \geq \zeta \text{ a.e. in } Q,\ T_k(u) \in W_0^{1,x}L_\varphi(Q),\ S_k(u(.,t)) \in L^1(\Omega). \\ \displaystyle\int_\Omega S_k(u(T)-v(T))\,dx + \Big\langle \frac{\partial v}{\partial t}, T_k(u-v)\Big\rangle + \int_Q a(x,t,u,\nabla u)\nabla T_k(u-v)\,dx\,dt \\ \qquad \displaystyle + \int_Q \Phi(u)\nabla T_k(u-v)\,dx\,dt \leq \int_Q g(u)\varphi(x,|\nabla u|)T_k(u-v)\,dx\,dt \\ \qquad\qquad \displaystyle + \int_Q fT_k(u-v)\,dx\,dt + \int_\Omega S_k(u_0-v(0))\,dx, \end{cases}$$

for every $k>0$, and for all $v \in W_0^{1,x}L_\varphi(Q)\cap L^\infty(Q)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q) + L^1(Q)$ and $v \geq \zeta$. S_k is the truncation defined by $S_k(\tau) = \int_0^\tau T_k(s)\,ds$.

The proof of this Theorem is divided into six steps.

Step 1: Approximate problems and a priori estimate

Let's consider the following approximate problem:

$$\begin{cases} \frac{\partial u_n}{\partial t} - \text{div}\Big(a(x,t,u_n,\nabla u_n) + \Phi_n(u_n)\Big) - nT_n((u_n-\zeta)^-) \\ \qquad = g(u_n)\varphi(x,|\nabla u_n|) + f_n \text{ in } Q, \\ u_n(x,0) = u_{0n}(x) \text{ in } \Omega, \end{cases} \tag{4.1}$$

where Φ_n is a Lipschitz continuous bounded function from $\mathbb{R}$ into $\mathbb{R}^N$, $f_n \subset D(Q)$ such that $f_n \to f$ strongly in $L^1(Q)$ and $(u_{0n}) \subset D(\Omega)$ such that $u_{0n} \to u_0$ strongly in $L^1(\Omega)$. By Lemma 3.1 of [26], there exists at least one weak solution $u_n \in W_0^{1,x}L_\varphi(Q)$ of the problem (4.1). Let $h>0$ and consider the following test function $v = T_h(u_n - T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)$ in (4.1), we have

$$\begin{aligned} &\Big\langle \tfrac{\partial u_n}{\partial t}, T_h(u_n - T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\Big\rangle \\ &\qquad + \int_{\{k<u_n\leq k+h\}} a(.,u_n,\nabla u_n)\nabla u_n \exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt \\ &\qquad + \int_Q a(.,u_n,\nabla u_n)\nabla u_n T_h(u_n - T_k(u_n))g(u_n)\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt \\ &\qquad + \int_Q \Phi_n(u_n)\nabla\Big(T_h(u_n-T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\Big)\,dx\,dt \\ &\qquad - \int_Q nT_n((u_n-\psi)^-)T_h(u_n-T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt \\ &= \int_Q g(u_n)\varphi(x,|\nabla u_n|)T_h(u_n-T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt \\ &\qquad + \int_Q f_n T_h(u_n-T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt \end{aligned}$$

The Liptschitz character of Φ_n, Stokes formula together with the boundary condition $u_n = 0$ on $(0,T)\times\partial\Omega$, make it possible to obtain

$$\int_Q \Phi_n(u_n)\nabla\Big(T_h(u_n-T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\Big)\,dx\,dt = 0. \tag{4.2}$$

Using (4.2) and (1.2), we have then,

$$\begin{aligned}&\Big\langle \tfrac{\partial u_n}{\partial t}, T_h(u_n - T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\Big\rangle\\ &\qquad + \int_{\{k<u_n\le k+h\}} \varphi(x, |\nabla u_n|)\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\\ &\qquad - n\int_Q T_n((u_n-\psi)^-)T_h(u_n - T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt\\ &\le \int_Q f_n T_h(u_n - T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt.\end{aligned}$$

We have

$$\begin{aligned}&\Big\langle \tfrac{\partial u_n}{\partial t}, T_h(u_n - T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\Big\rangle\\ &= \int_\Omega\int_0^{u_n(x,T)} T_h(s - T_k(s))\exp\Big(\int_0^{s} g(s)\,ds\Big)\\ &\qquad - \int_\Omega\int_0^{u_{0n}} T_h(s - T_k(s))\exp\Big(\int_0^{s} g(s)\,ds\Big).\end{aligned}$$

So, we obtain

$$-n\int_Q T_n((u_n-\zeta)^-)T_h(u_n - T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt \le Ch,$$

and also

$$-\int_Q nT_n((u_n-\zeta)^-)\frac{T_h(u_n - T_k(u_n))}{h}\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt \le C.$$

Let us now fix $k > \| \psi \|_\infty$, we deduce the fact that

$$nT_n(u_n-\zeta)(u_n-k)\chi_{\{u_n\le\psi\}}\chi_{\{k<u_n\le k+h\}} \ge 0.$$

Letting h to tend to zero, one has

$$n\int_Q T_n((u_n-\zeta)^-)\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt \le C,$$

and also,

$$n\int_Q T_n((u_n-\zeta)^-) \le C. \tag{4.3}$$

Let us use as test function in (4.1), $v = T_k(u_n)\exp\Big(\int_0^{u_n} g(s)\,ds\Big)$, then as above we obtain

$$\int_Q \varphi(x, |\nabla T_k(u_n)|)\exp\Big(\int_0^{u_n} g(s)\,ds\Big) \le C_1 k. \tag{4.4}$$

By using the Lemma 3.2, we have

$$\int_\Omega \varphi\Big(x, \frac{|T_k(u_n)|}{c}\Big)dx \le \int_\Omega \varphi\Big(x, |\nabla T_k(u_n)|\Big)\exp\Big(\int_0^{u_n} g(s)\,ds\Big) \le C_1 k, \tag{4.5}$$

where c is the constant of Lemma 3.2

Then $(T_k(u_n))_n$ is bounded in $W_0^{1,x}L_\varphi(Q)$, and then there exist some $w_k \in W_0^{1,x}L_\varphi(Q)$ such that

$$T_k(u_n) \rightharpoonup w_k \text{ weakly in } W_0^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi),$$

$$T_k(u_n) \to w_k \text{ strongly in } E_\varphi(Q) \text{ and a.e. in Q.}$$

Let consider the C^2 function defined by

$$\eta_k(s) = \begin{cases} s & |\, s \,|\leq \frac{k}{2}, \\ k \text{ sign}(s) & |\, s \,|\geq k. \end{cases} .$$

Multiplying the approximating equation by $\eta_k^{'}(u_n)$, we get

$$\frac{\partial \eta_k(u_n)}{\partial t} - \text{div}\Big(a(x,t,u_n,\nabla u_n)\eta_k^{'}(u_n)\Big) + a(x,t,u_n,\nabla u_n)\eta_k^{''}(u_n)$$

$$-\text{div}\Big(\Phi_n(u_n)\eta_k^{'}(u_n)\Big) + \Phi_n(u_n)\eta_k^{''}(u_n)$$

$$= g(u_n)\varphi(x,|\,\nabla u_n\,|)\eta_k^{'}(u_n) + f_n\eta_k^{'}(u_n) + nT_n((u_n-\zeta)^-)\eta_k^{'}(u_n)$$

in the distributions sense. we deduce then, $\eta_k(u_n)$ is bounded in $W_0^{1,x}L_\varphi(Q)$ and $\frac{\partial \eta_k(u_n)}{\partial t}$ in $W^{-1,x}L_\psi(Q) + L^1(Q)$. By Corollary 1 of [?], $\eta_k(u_n)$ is compact in $L^1(Q)$.

4.1 Step 2: Convergence in measure of $(u_n)_n$

Let $k > 0$ large enough, by using (4.5) and (1.5), we have

$$\begin{aligned} M(k)meas\{|u_n| > k\} &= \int_{\{|u_n|>k\}} M(|T_k(u_n)|)dx \\ &\leq \int_{\{|u_n|>k\}} \varphi\Big(x, |T_k(u_n)|\Big)dx \\ &\leq \int_Q \varphi\Big(x, |T_k(u_n)|\Big)dxdt \\ &\leq C_1 k. \end{aligned}$$

Where c_3 is a constant not dependent on k ,hence

$$meas\{|u_n| > k\} \leq \frac{C_1 k}{M(k)} \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

For every $\lambda > 0$ we have

$$\begin{aligned} meas\{|u_n - u_m| > \lambda\} &\leq meas\{|u_n| > k\} \\ &+ meas\{|u_m| > k\} \\ &+ meas\{|T_k(u_n) - T_k(u_m)| > \lambda\}. \end{aligned} \tag{4.6}$$

Consequently, by (4.5) we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Q. Let $\varepsilon > 0$, then by (4.6) there exists some $k = k(\varepsilon) > 0$ such that

$$meas\{|u_n - u_m| > \lambda\} < \varepsilon, \quad \text{for all } n, m \geq h_0(k(\varepsilon), \lambda).$$

Which means that $(u_n)_n$ is a Cauchy sequence in measure in Q, thus converge almost every where to some measurable functions u. Then

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W_0^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \\ T_k(u_n) \longrightarrow T_k(u) & \text{strongly in } E_\varphi(Q). \end{cases} \tag{4.7}$$

Now using the estimation (4.3) and Fatou's Lemma, we obtain

$$(u-\zeta)^- = 0,$$

and so,

$$u \geq \zeta.$$

Step 3: Almost everywhere convergence of the gradients

Lemma 4.2. *Let u_n be a solution of the approximate problem (4.1). Then, there exists a subsequence also denoted by u_n such that*

$$\nabla u_n \rightarrow \nabla u \text{ a.e. in } Q.$$

we deduce then that,

$$a(.,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(.,T_k(u),\nabla T_k(u)) \text{ in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi).$$

Proof. of lemma 4.2: For $m > k$, we define the function

$$\rho_m(s) = \begin{cases} 1 & |s| \leq m \\ m+1-|s| & m < |s| < m+1 \\ 0 & |s| \geq m+1 \end{cases}$$

and we set

$$\begin{aligned} T_k^*(s) &= \left(\int_0^{T_k(s)} \exp\left(\int_0^t g(s)\mathrm{d}s\right)\mathrm{d}t\right)\left(\exp\left(-\int_0^\infty g(s)ds\right)\right) \\ R_m(s) &= \int_0^s \rho_m(t)\exp\left(\int_0^t g(s)\mathrm{d}s\right)\mathrm{d}t \\ \omega_{\mu,j}^i &= T_k(v_j)_\mu + \mathrm{e}^{-\mu t}\tau_k(\psi_i) \end{aligned}$$

where $v_j \in D(Q)$ such that $v_j \geq T_k^*(\zeta)$ and $v_j \rightarrow T_k^*(u)$ with the modular convergence in $W_0^{1,x}L_\varphi(Q)$ (for the existence of such a function see [24] since $\zeta \in L^\infty(\Omega) \cap W_0^1E_\varphi(\Omega)$.$\zeta_i$ is a smooth function such that $\zeta_i \rightarrow T_k^*(u_0)$ strongly in $L^1(\Omega)$)

and $\|\zeta_i\|_\infty \leq \|T_k^*(u_0)\|_\infty \cdot \omega_\mu$ is the mollifier function defined in [35]), the function $\omega_{\mu,j}^i$ has the following properties:

$$\begin{cases} \frac{\partial \omega_{\mu,j}^i}{\partial t} = \mu\left(T_k(v_j) - \omega_{\mu,j}^i\right), \quad \omega_{\mu,j}^i(0) = T_k(\zeta_i), \quad \left|\omega_{\mu,j}^i\right| \leq k \\ \omega_{\mu,j}' \rightarrow T_k^*(u)_\mu + \mathrm{e}^{-\mu t}T_k(\zeta_i) \quad \text{in } W_0^1L_\varphi(Q) \text{ for the modular convergence with respect to } j \\ T_k^*(u)_\mu + \mathrm{e}^{-\mu}T_k(\zeta_i) \rightarrow T_k^*(u) \quad \text{in } W_0^1L_\varphi(Q) \text{ for the modular convergence with respect to } \mu \end{cases}$$

Now,by taking $v = \left(T_k^*(u_n) - \omega_{\mu,j}^i\right)\rho_m(u_n)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)$ as a test function, we get

$$\left\langle\left\langle \frac{\partial u_n}{\partial t}, v\right\rangle\right\rangle + \int_Q a(.,u_n,\nabla u_n)\left(\nabla T_k^*(u_n) - \nabla\omega_{\mu,j}^i\right)\rho_m(u_n)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right) \tag{4.8}$$

$$+\int_Q a(.,u_n,\nabla u_n)\nabla u_n\left(T_k^*(u_n) - \omega_{\mu,j}^i\right)\rho_m'(u_n)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right) \tag{4.9}$$

$$+\int_Q a(.,u_n,\nabla u_n)\nabla u_n g(u_n)\left(T_k^*(u_n) - \omega_{\mu,j}^i\right)\rho_m(u_n)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right) \tag{4.10}$$

$$= \int_Q f_n v\mathrm{d}x\mathrm{d}t + n\int_Q T_n\left((u_n-\zeta)^-\right)v\mathrm{d}x\mathrm{d}t + \int_Q g(u_n)\varphi(x,|\nabla u_n|)v\mathrm{d}x\mathrm{d}t$$

$$=: (4) + (5) + (6)$$

Let us recall that for $u_n \in W_0^{1,x}L_\varphi(Q)$, there exists a smooth function $u_{n\sigma}$ (see [21]) such that $u_{n\sigma} \rightarrow u_n$ for the modular convergence in $W_0^{1,x}L_\varphi(Q)$ $\frac{\partial u_{n\sigma}}{\partial t} \rightarrow \frac{\partial u_n}{\partial t}$ for the modular convergence in $W^{-1,x}L_\psi(Q) + L^1(Q)$.

$$\left\langle\left\langle \frac{\partial u_n}{\partial t}, v\right\rangle\right\rangle = \lim_{\sigma\rightarrow 0^+}\int_e (u_{n\sigma})'\left(T_k^*(u_{n\sigma}) - \omega_{\mu,j}^i\right)\rho_m(u_{n\sigma})\exp\left(\int_0^{u_{n\sigma}} g(s)\mathrm{d}s\right)$$

$$= \lim_{\sigma\to 0^+}\left(\int_Q \left(R_m\left(u_{n\sigma}\right)-T_k^*\left(u_{n\sigma}\right)\right)'\left(T_k^*\left(u_{n\sigma}\right)-\omega_{\mu,j}^i\right)\mathrm{d}x\mathrm{d}t+\int_0\left(T_k^*\left(u_{n\sigma}\right)^T T_k^*\left(u_{n\sigma}\right)-\omega_{\mu,j}^i\right)\mathrm{d}x\mathrm{d}t\right)$$

$$= \lim_{\sigma\to 0^+}\left[\int_\Omega \left(R_m\left(u_{n\sigma}\right)-T_k^*\left(u_{n\sigma}\right)\right)\left(T_k^*\left(u_{n\sigma}\right)-\omega_\mu^j\right)\mathrm{d}x\right]_0^T$$

$$-\int_Q \left(R_m\left(u_{n\sigma}\right)-T_k^*\left(u_{n\sigma}\right)\right)\left(T_k^*\left(u_{n\sigma}\right)-\omega_{\mu,j}^i\right)'\mathrm{d}x\mathrm{d}\,t$$

$$+\int_\Omega^{T_k^*}\left(u_{n\sigma}\right)'\left(T_k^*\left(u_{n\sigma}\right)-\omega_{\mu,j}^i\right)\mathrm{d}x\mathrm{d}t =: I_1+l_2+l_3$$

Remark also that,

$$R_m\left(u_{n\sigma}\right)\ge T_k^*\left(u_{n\sigma}\right) \quad \text{if } u_{n\sigma}<k \text{ and } R_m\left(u_{n\sigma}\right)>k=T_k^*\left(u_{n\sigma}\right)\ge\left|\omega_{\mu,j}^i\right| \text{ if } u_{n\sigma}\ge k$$

$$I_1=\int_\Omega \left(R_m\left(u_{n\sigma}\right)(T)-T_k^*\left(u_{n\sigma}\right)(T)\right)\left(T_k^*\left(u_{n\sigma}\right)(T)-\omega_{\mu,j}^i(T)\right)\mathrm{d}x$$

$$-\int_{\Omega_n}\left(R_m\left(u_{n\sigma}\right)(0)-T_k^*\left(u_{n\sigma}\right)(0)\right)\left(T_k^*\left(u_{n\sigma}\right)(0)-\omega_{\mu,j}^i(0)\right)\mathrm{d}x =: I_1^1+I_1^2$$

$$I_1^1\ge\int_{u_{n\sigma}(T)\le k}\left(R_m\left(u_{n\sigma}\right)(T)-T_k^*\left(u_{n\sigma}\right)(T)\right)\left(T_k^*\left(u_{n\sigma}\right)(T)-\omega_{\mu,j}^i(T)\right)\mathrm{d}x$$

and it is easy to see that

$$\limsup_{\sigma\to 0^+} I_1^1\ge\epsilon(n,j,\mu)$$

$$I_1^2=-\int_{u_{n\sigma}(0)\le k}\left(R_m\left(u_{n\sigma}\right)(0)-T_k^*\left(u_{n\sigma}\right)(0)\right)\left(T_k^*\left(u_{n\sigma}\right)(0)-\zeta_i\right)\mathrm{d}x$$

$$-\int_{u_{n\sigma}(0)>k}\left(R_m\left(u_{n\sigma}\right)(0)-T_k^*\left(u_{n\sigma}\right)(0)\right)\left(T_k^*\left(u_{n\sigma}\right)(0)-\zeta_i\right)\mathrm{d}x$$

For the first part, it is the same as I_1^1 and for the second part, we have

$$I_1^2\ge\epsilon(n,j,\mu)-\int_{u_{n\sigma}(0)\ge k}\left(R_m\left(u_{n\sigma}\right)(0)-T_k^*\left(u_{n\sigma}\right)(0)\right)\left(T_k^*\left(u_{n\sigma}\right)(0)-\zeta_i\right)\mathrm{d}x$$

$$\limsup_{\sigma\to 0^+} I_1\ge\epsilon(n,j,\mu)-\int_{u_{0n}\ge k}\left(R_m\left(u_{0n}\right)-T_k^*\left(u_{0n}\right)\right)\left(T_k^*\left(u_{0n}\right)-\zeta_i\right)\mathrm{d}x =: J_1$$

Now by letting $n\to\infty$, we get

$$\lim_{n\to\infty} J_1=-\int_{u_0\ge k}\left(R_m\left(u_0\right)-T_k^*\left(u_0\right)\right)\left(T_k^*\left(u_0\right)-\psi_i\right)\mathrm{d}x\mathrm{d}t$$

and by letting $i\to\infty$, one has

$$\limsup_{\varepsilon\to 0^t} I_1\ge\epsilon(n,j,i,\mu)$$

About I_2, we remark that $T_k^*\left(u_{n\sigma}\right)'=0$ if $u_{n\sigma}>k$, then

$$I_2=-\int_{d_{n_0}\le k}\left(R_m\left(u_{n\sigma}\right)-T_k^*\left(u_{n\sigma}\right)\right)\left(T_k^*\left(u_{n\sigma}\right)-\omega_{\mu,j}^i\right)'\mathrm{d}x\mathrm{d}\,t$$

$$+\int_{u_n\sigma>k}\left(R_m\left(u_{n\sigma}\right)-T_k^*\left(u_{n\sigma}\right)\right)\left(\omega_{\mu,j}^i\right)'\mathrm{d}x\mathrm{d}\,t =: I_2^1+l_2^2$$

As in $I_1, I_2^1\ge\epsilon(n,j,\mu)$, and

$$l_2^2=\int_{d_{n_0}>k}\left(R_m\left(u_{n\sigma}\right)-T_k^*\left(u_{n\sigma}\right)\right)\left(\omega_{\mu,j}^i\right)'\mathrm{d}x\mathrm{d}\,t$$

$$\geq \mu \int_{u_{n\sigma}>k} \left(R_m\left(u_{n\sigma}\right)-T_k^*\left(u_{n\sigma}\right)\right)\left(T_k\left(v_j\right)-T_k^*\left(u_{n\sigma}\right)\right)' \mathrm{dxd}\, t$$

thus by using the fact that

$$\left(R_m\left(u_{n\sigma}\right)-T_k^*\left(u_{n\sigma}\right)\right)\left(T_k^*\left(u_{n\sigma}\right)-\omega_{\mu,j}^i\right) \chi_{u_n\sigma}>_k \geq 0.$$

$$\limsup_{\sigma\to 0^+} \imath_2^2 \geq \mu \int_{u_2>k} \left(R_m\left(u_n\right)-T_k^*\left(u_n\right)\right)\left(T_k\left(v_j\right)-T_k^*\left(u_n\right)\right)' \mathrm{dxd}\, t=\epsilon(n,j)$$

Concerning I_3

$$\begin{aligned} I_3 &= \int_{\mathrm{e}} T_k^*\left(u_{n\sigma}\right)'\left(T_k^*\left(u_{n\sigma}\right)-\omega_{\mu,j}^i\right) \mathrm{d}x\mathrm{d}t \\ &= \int_{\mathrm{Q}}\left(T_k^*\left(u_{n\sigma}\right)-\omega_{\mu,j}^i\right)'\left(T_k^*\left(u_{n\sigma}\right)-\omega_{\mu,j}^i\right) \mathrm{d}x\mathrm{d}t+\int_{\mathrm{Q}}\left(\omega_{\mu,j}^i\right)'\left(T_k^*\left(u_{n\sigma}\right)-\omega_{\mu,j}^i\right) \mathrm{d}x\mathrm{d}t \end{aligned}$$

set $\Upsilon(s)=s^2/2, \Upsilon \geq 0$, then

$$I_3=\left[\int_{\Omega} \Upsilon\left(T_k^*\left(u_{n\sigma}\right)-\omega_{\mu,j}^i\right) \mathrm{d}x\right]_0^T+\mu \int_Q\left(T_k\left(v_j\right)-\omega_{\mu,j}^i\right)\left(T_k^*\left(u_{n\sigma}\right)-\omega_{\mu,j}^i\right) \mathrm{d}x\mathrm{d}t$$

$$\geq \epsilon(n,j,\mu)-\int_{\Omega} \Upsilon\left(T_k^*\left(u_{n\sigma}(0)\right)-\zeta_i\right) \mathrm{d}x+\mu \int_Q\left(T_k\left(v_j\right)-T_k^*\left(u_{n\sigma}\right)\right)\left(T_k^*\left(u_{n\sigma}\right)-\omega_{\mu,j}^i\right) \mathrm{d}x\mathrm{d}t \quad \text{(as in } I_2\text{)}.$$

so,

$$\begin{aligned} \limsup_{\sigma\to 0^+} &\geq \epsilon(n,j,\mu)-\int_{\Omega} \Upsilon\left(T_k^*\left(u_{0n}\right)-\zeta_i\right) \mathrm{d}x+\mu \int_Q\left(T_k\left(v_j\right)-T_k^*\left(u_n\right)\right)\left(T_k^*\left(u_n\right)-\omega_{\mu,j}^i\right) \mathrm{d}x\mathrm{d}t \\ &= -\int_{\Omega} \Upsilon\left(T_k^*\left(u_0\right)-\zeta_i\right) \mathrm{d}x+\mu \int_Q\left(T_k\left(v_j\right)-T_k^*(u)\right)\left(T_k^*(u)-\omega_{\mu,j}^i\right) \mathrm{d}x\mathrm{d}t+\epsilon(n,j,\mu) \end{aligned}$$

and we deduce

$$\limsup_{\sigma\to 0^+} \geq \epsilon(n,j,i,\mu).$$

Then we conclude that

$$\left\langle\left\langle\frac{\partial u_n}{\partial t},\left(T_k^*\left(u_n\right)-\omega_{\mu,j}^i\right) \rho_m\left(u_n\right) \exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\right\rangle\right\rangle \geq \epsilon(n,j,i,\mu).$$

Now for the terms of (4.8),(4.9),(4) and (5).

Let us remark that

$$\text{(i) } \nabla T_k^*(u)=\left(\exp\left(-\int_0^{\infty} g(s)\mathrm{d}s\right)\right) \exp\left(\int_0^{T_k(u)} g(s)\mathrm{d}s\right) \nabla T_k(u)=: \lambda(u)\nabla T_k(u).$$

Concerning (4.8)

$$\begin{aligned} &\int_Q a\left(., u_n, \nabla u_n\right)\left(\nabla T_k^*\left(u_n\right)-\nabla \omega_{\mu,j}^i\right) \rho_m\left(u_n\right) \exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right) \mathrm{d}x\mathrm{d}t \\ &\quad= \int_{u_n\leq k} a\left(., u_n, \nabla u_n\right)\left(\nabla T_k^*\left(u_n\right)-\nabla \omega_{\mu,j}^i\right) \rho_m\left(u_n\right) \exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right) \mathrm{d}x\mathrm{d}t \\ &\quad+\int_{u_n>k} a\left(., u_n, \nabla u_n\right)\left(\nabla T_k^*\left(u_n\right)-\nabla \omega_{\mu,j}^i\right) \rho_m\left(u_n\right) \exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right) \mathrm{d}x\mathrm{d}t \\ &= \int_Q a\left(., T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\left(\nabla T_k^*\left(u_n\right)-\nabla \omega_{\mu,j}^i\right) \exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right) \mathrm{d}x\mathrm{d}t \\ &\quad+\int_{u_n>k} a\left(., u_n, \nabla u_n\right)\left(\nabla T_k^*\left(u_n\right)-\nabla \omega_{\mu,j}^i\right) \rho_m\left(u_n\right) \exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right) \mathrm{d}x\mathrm{d}t \end{aligned}$$

recall that $\rho_m\left(u_n\right)=1$ on $\left\{\left|u_n\right| \leq k\right\}$

Let $s>0, Q_s=\{(x,t) \in Q:|\nabla T_k(u)| \leq s\}, Q_j^s=\left\{(x,t) \in Q:\left|\nabla T_k\left(v_j\right)\right| \leq s\right\}$

$$\int_Q a(., u_n, \nabla u_n)\left(\nabla T_k^*(u_n) - \nabla\omega^i_{\mu,j}\right)\rho_m(u_n)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$

$$= \int_{\mathrm{Q}}\left(a(., T_k(u_n), \nabla T_k(u_n)) - a(., T_k(u_n), \nabla T_k(v_j)\chi_j^s)\right)$$

$$\times\left(\nabla T_k^*(u_n) - \nabla T_k(v_j)\chi_j^s\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$

$$+\int_{\mathrm{Q}} a\left(., T_k(u_n), \nabla T_k(v_j)x_j^s\right)\left(\nabla T_k^*(u_n) - \nabla T_k(v_j)\chi_j^s\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$

$$+\int_{\mathrm{Q}} a(., T_k(u_n), \nabla T_k(u_n))\nabla T_k(v_j)x_j^s\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$

$$-\int_{\mathrm{Q}} a(., u_n, \nabla u_n)\nabla\omega^i_{\mu,j}\rho_m(u_n)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$

$$= J_1 + J_2 + J_3 + J_4$$

thanks to (4.7) we have

$$T_k^*(u_n) \rightharpoonup T_k^*(u) \text{ weakly in } W_0^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \tag{4.11}$$

$$T_k^*(u_n) \to T_k^*(u) \text{ strongly in } E_\varphi(Q) \text{ and a.e in Q} \tag{4.12}$$

By using (1.4), we can deduce the existence of a measurable function h_k such that

$$a(., T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \text{ in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_M, \Pi E_\psi)$$

$$J_2 = \int_Q a\left(., T_k(u), \nabla T_k(v_j)\chi_j^s\right)\left(\nabla T_k^*(u) - \nabla T_k(v_j)\chi_j^s\right)\exp\left(\int_0^u g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t + \epsilon(n)$$

since

$$a\left(., T_k(u_n), \nabla T_k(v_j)x_j^s\right) \to a\left(., T_k(u), \nabla T_k(v_j)\chi_j^s\right) \text{ strongly in } (E_\psi(Q))^N$$

$$a\left(., T_k(u), \nabla T_k(v_j)\chi_j^s\right) \to a(., T_k(u), \nabla T_k(u)\chi^s) \quad \text{strongly in } (E_\varphi(Q))^N$$

and

$$\nabla T_k(v_j)\chi_j^s \to \nabla T_k^*(u)\chi^s \text{ strongly in } (L_\psi(Q))^N$$

Then,

$$J_2 = \epsilon(n, j)$$

Following the same way as in J_2, one has

$$J_3 = \int_Q h_k \nabla T_k^*(u)\exp\left(\int_0^u g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t + \epsilon(n, j, \mu, s)$$

For the terms J_4 :

$$J_4 = -\int_Q a(\cdot, T_{m+1}(u_n), \nabla T_{m+1}(u_n))\nabla\omega^i_{\mu,j}\rho_m(u_n)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$

$$= -\int_{\|_n|\leq k} a(., T_{m+1}(u_n), \nabla T_{m+1}(u_n))\nabla\omega^i_{\mu,j}\rho_m(u_n)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$

$$-\int_{k<|u_n|\leq m+1} a(., T_{m+1}(u_n), \nabla T_{m+1}(u_n))\nabla\omega^i_{\mu,j}\rho_m(u_n)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$

Letting $n \to \infty$, then

$$J_4 = -\int_{k<|u|\leq m+1} h_k \nabla \omega_{\mu j}^i \rho_m(u) \exp\left(\int_0^u g(s)\mathrm{d}s\right) \mathrm{d}x\mathrm{d}t$$

$$-\int_{|u|\leq k} h_{m+1} \nabla \omega_{\mu,j}^i \rho_m(u) \exp\left(\int_0^u g(s)\mathrm{d}s\right) \mathrm{d}x\mathrm{d}t + \epsilon(n)$$

By letting firstly $j \to \infty$ and after that $\mu \to \infty$, we get

$$J_4 = -\int_Q h_k \nabla T_k^*(u) \exp\left(\int_0^u g(s)\mathrm{d}s\right) \mathrm{d}x\mathrm{d}t + \epsilon(n,j,\mu).$$

Then,

$$(1) = \int_{\mathrm{Q}} \left(a\left(.,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right) - a\left(..T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)\left(\nabla T_k^*\left(u_n\right) - \nabla T_k\left(v_j\right)x_j^s\right)$$
$$\times \exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t + \epsilon(n,j,\mu,s)$$

Concerning(4.9)

$$\left|\int_Q a\left(.,u_n,\nabla u_n\right)\nabla u_n\left(T_k^*\left(u_n\right) - \omega_{n,j}^i\right)\rho_m'\left(u_n\right)\exp\left(\int_0^{u_n} g(s)ds\right)\right|$$
$$\leq C(k)\int_{m<|u_1|\leq m+1} a\left(.,u_n,\nabla u_n\right)\nabla u_n \exp\left(\int_0^{u_n} g(s)ds\right)dxdt$$

Let $\Theta_m(s) = T_1\left(s - T_m(s)\right)$ and $\Theta^*(s) = \int_0^s \Theta_m(t)\exp\left(\int_0^t g(s)\mathrm{d}s\right)\mathrm{d}t$.
Using $v = \Theta_m\left(u_n\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)$ as a test function in the approximated problem (4.1),

$$\left\langle u_n', v\right\rangle + \int_{\mathrm{Q}} a\left(.,u_n,\nabla u_n\right)\nabla u_n \Theta_m'\left(u_n\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$
$$\leq \int_{\mathrm{Q}} f_n v\mathrm{d}x\mathrm{d}t + n\int_{\mathrm{Q}} T_n\left(\left(u_n-\zeta\right)^-\right)\Theta_m'\left(u_n\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$

So

$$\int_\Omega \left[\Theta_m^*\left(u_n(t)\right)\right]_0^T + \int_{m<|u_n|\leq m+1} a\left(.,u_n,\nabla u_n\right)\nabla u_n \exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$
$$\leq \int_{\|_n|\geq m} f_n v\mathrm{d}x\mathrm{d}t + n\int_{|u_n|\geq m} T_n\left(\left(u_n-\zeta\right)^-\right)\Theta_m'\left(u_n\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$

and we easily obtain (since $\Theta_m^* \geq 0$):

$$\left|\int_{m<|u_n|\leq m+1} a\left(.,u_n,\nabla u_n\right)\nabla u_n \exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t\right| \leq \epsilon(n,m)$$

so,

$$(4.9) \leq \epsilon(n,m)$$

With the same techniques as above, we can deduce that

$$(4) \leq \epsilon(n,j,\mu)$$

Concerning (5), we have

$$(5) = n\int_{\mathrm{e}} T_n\left(u_n-\zeta\right)^{-}\Big)\left(T_k^*\left(u_n\right)-T_k\left(v_j\right)_\mu\right)\rho_m\left(u_n\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$
$$+\,n\int_{\mathrm{Q}} T_n\left(\left(u_n-\zeta\right)^{-}\right)\left(T_k\left(v_j\right)_\mu-\omega_{\mu,j}^i\right)\rho_m\left(u_n\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$
$$\leq n\int_{\mathrm{Q}} T_n\left(\left(u_n-\zeta\right)^{-}\right)\left(T_k^*(\zeta)-T_k\left(v_j\right)_\mu\right)\rho_m\left(u_n\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$
$$+\,n\int_{\mathrm{Q}} T_n\left(\left(u_n-\zeta\right)^{-}\right)\left(T_k\left(v_j\right)_\mu-\omega_{\mu,j}^i\right)\rho_m\left(u_n\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$

since $T_k\left(v_j\right)_\mu \geq T_k^*(\zeta)$ and $T_k\left(v_j\right)_\mu-\omega_{\mu,j}^i \leq 0$, we deduce that

$$(5) \leq 0$$

Taking now into account the estimation of (4.8),(4.9),(4) and (5), we obtain

$$\int_{\mathrm{Q}}\left(a\left(.,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(.,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)$$
$$\times\left(\nabla T_k^*\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$
$$\leq \epsilon(n,j,\mu,i,s,m).$$

On the other hand,

$$\int_{\mathrm{Q}}\left(a\left(.,T_k\left(u_{\mathrm{n}}\right),\nabla T_k\left(u_n\right)\right)-a\left(..T_k\left(u_n\right),\nabla T_k^*(u)\chi^s\right)\right)\left(\nabla T_k^*\left(u_n\right)-\nabla T_k^*(u)\chi^s\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$
$$-\int_{\mathrm{Q}}\left(a\left(.,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(.,T_k\left(u_n\right),\nabla T_k\left(v_j\right)x_j^s\right)\right)$$
$$\times\left(\nabla T_k^*\left(u_n\right)-\nabla T_k\left(v_j\right)x_j^s\right)\exp\left(\int_0^{v_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$
$$=\int_{\mathrm{Q}} a\left(.,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k^*(u)\chi^s\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$
$$-\int_{\mathrm{Q}} a\left(.,T_k\left(u_n\right),\nabla T_k^*(u)\chi^s\right)\left(\nabla T_k\left(v_j\right)x_j^s-\nabla T_k^*(u)\chi^s\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$
$$+\int_{\mathrm{Q}} a\left(.,T_k\left(u_n\right),\nabla T_k^*\left(v_j\right)\chi_j^s\right)\left(\nabla T_k^*\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t,$$

each term of the last right hand side is of the form $\epsilon(n,j,s)$, which gives

$$\int_Q\left(a\left(.,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(.,T_k\left(u_n\right),\nabla T_k^*(u)\chi^s\right)\right)\left(\nabla T_k^*\left(u_n\right)-\nabla T_k^*(u)\chi^s\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$
$$=\int_Q\left(a\left(.,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(.,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)$$
$$\times\left(\nabla T_k^*\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\exp\left(\int_0^{u_n} g(s)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t$$

$$+\epsilon(n, j, s)$$

Following the same technique used in [20] we have for all $r < s$:

$$\int_Q \left(a\left(., T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(., T_k\left(u_n\right), \nabla T_k^*(u)\right)\right)\left(\nabla T_k^*\left(u_n\right) - \nabla T_k^*(u)\right) \mathrm{d}x\mathrm{d}t \to 0 \tag{4.13}$$

On the other hand, we have

$$\left(\lambda\left(u_n\right) - \lambda(u)\right) \nabla T_k(u) \chi_{\{\nabla T_k(u) \leq r\}} \to 0 \quad \text{strongly in } \left(E_\varphi(Q)\right)^N$$

and

$$a\left(., T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(., T_k\left(u_n\right), \nabla T_k(u)\right) \rightharpoonup h_k - a\left(., T_k(u), \nabla T_k(u)\right) \quad \text{weakly in } \left(L_\psi(Q)\right)^N$$

which gives

$$\int_{0_t} \left(a\left(., T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(., T_k\left(u_n\right), \nabla T_k(u)\right)\right) \nabla T_k(u)\left(\lambda\left(u_n\right) - \lambda(u)\right) \mathrm{d}x\mathrm{d}t \to 0 \tag{4.14}$$

By using the decomposition:

$$\nabla T_k^*\left(u_n\right) - \nabla T_k^*(u) = \lambda\left(u_n\right)\left(\nabla T_k\left(u_n\right) - \nabla T_k(u)\right) + \left(\lambda\left(u_n\right) - \lambda(u)\right) \nabla T_k(u)$$

and taking into account of (4.11), (4.12), (4.13),(4.14) and the monotonicity condition,we get

$$\lim_{n \to \infty} \int_{Q_r} \left(a\left(., T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(., T_k\left(u_n\right), \nabla T_k(u)\right)\right)\left(\nabla T_k\left(u_n\right) - \nabla T_k(u)\right) \mathrm{d}x\mathrm{d}t = 0$$

thus,there exists a subsequence also denoted by u_n such that

$$\nabla u_n \to \nabla u \text{ a.e.in } Q$$

We deduce then that,

$$a\left(., T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \rightharpoonup a\left(., T_k(u), \nabla T_k(u)\right) \quad \text{in } \left(L_\psi(Q)\right)^N \quad \text{for } \sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$$

□

Step 4: Modular convergence of the truncations

We have proved that

$$\int_Q \Big(a(., T_k(u_n), \nabla T_k(u_n)) - a(., T_k(u_n), \nabla T_k(v_j)\chi_j^s)\Big)\Big(\nabla T_k^*(u_n) - \nabla T_k(v_j)\chi_j^s\Big) \times \exp\Big(\int_0^{u_n} g(s)\, ds\Big)\, dx\, dt \leq \varepsilon(n, j, \mu, i, s, m),$$

where

$$T_k^*(s) = \Big(\int_0^{T_k(s)} \exp\Big(\int_0^t g(s)\, ds\Big)\, dt\Big)\Big(\exp\Big(-\int_0^{+\infty} g(s)\, ds\Big)\Big).$$

We can also deduce that

$$\begin{aligned} &\int_Q \Big(a(., T_k(u_n), \nabla T_k(u_n)) - a(., T_k(u_n), \nabla T_k^*(u)\chi^s)\Big)\Big(\nabla T_k^*(u_n) - \nabla T_k^*(u)\chi^s\Big) \times \exp\Big(\int_0^{u_n} g(s)\, ds\Big)\, dx\, dt \\ &= \int_Q \Big(a(., T_k(u_n), \nabla T_k(u_n)) - a(., T_k(u_n), \nabla T_k(v_j)\chi_j^s)\Big)\Big(\nabla T_k^*(u_n) - \nabla T_k(v_j)\chi_j^s\Big) \times \exp\Big(\int_0^{u_n} g(s)\, ds\Big)\, dx\, dt + \varepsilon(n, j, s). \end{aligned}$$

Then,

$$
\begin{aligned}
&\int_Q a(., T_k(u_n), \nabla T_k(u_n))\nabla T_k^*(u_n)\,dx\,dt \\
&\quad \leq \int_Q a(., T_k(u_n), \nabla T_k(u_n))\nabla T_k^*(u)\chi^s\,dx\,dt \\
&\quad\quad + \int_Q a(., T_k(u_n), \nabla T_k^*(u)\chi^s)(\nabla T_k^*(u_n) - T_k(u)\chi^s)\,dx\,dt + \varepsilon(n, j, \mu, i, s, m).
\end{aligned}
$$

We deduce that,

$$
\begin{aligned}
\limsup_n &\int_Q a(., T_k(u_n), \nabla T_k(u_n))\nabla T_k^*(u_n)\,dx\,dt \\
&\leq \int_Q a(., T_k(u), \nabla T_k(u))\nabla T_k^*(u)\chi^s\,dx\,dt + \lim_n \varepsilon(n, j, \mu, i, s, m),
\end{aligned}
$$

then,

$$
\begin{aligned}
\limsup_n \int_Q a(., T_k(u_n), \nabla T_k(u_n))\nabla T_k^*(u_n)\,dx\,dt& \\
&\leq \int_Q a(., T_k(u), \nabla T_k(u))\nabla T_k^*(u)\chi^s\,dx\,dt \\
&\leq \liminf_n \int_Q a(., T_k(u_n), \nabla T_k(u_n))\nabla T_k^*(u_n)\,dx\,dt,
\end{aligned}
$$

as $n \to \infty$, we deduce

$$
a(., T_k(u_n), \nabla T_k(u_n))\nabla T_k^*(u_n) \to a(., T_k(u), \nabla T_k(u))\nabla T_k^*(u) \text{ in } L^1(Q).
$$

Using the same argument as above, we obtain

$$
a(., T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n) \to a(., T_k(u), \nabla T_k(u))\nabla T_k(u) \text{ in } L^1(Q),
$$

by Vitali's theorem and (1.2) we get

$$
\nabla T_k(u_n) \to \nabla T_k(u) \text{ for the modular convergence in } (L_\varphi(Q))^N.
$$

Step 5: Passing to the limit

Using the approximated function of the Lemma 3.2 of [26], the passing to the limit is easy by adapting the same way as in [20, 21, 22].

As a conclusion of Step 1 to Step 5, the proof of our existence result is achieved.

References

[1] Aberqi.A, Bennouna.j and Elmassoud.iM:*Nonlinear alliptic equations with measure data in orlicz spaces* Gulf Journal of Mathematics Vol 6, Issue 4 (2018) 79-100.

[2] Aberqi. A, Bennouna.J, Elmassoudi.M, · Hammoumi. M *Existence and uniqueness of a renormalized solution of parabolic problems in Orlicz spaces* Monatshefte für Mathematik https://doi.org/10.1007/s00605-018-01260-8.

[3] Aharouch.L, Bennouna.J,: *Existence and uniqueness of solutions of unilateral problems in Orlicz spaces.* Nonlinear Anal. $72, 3553 - 3565(2010)$

[4] Al-Hawmi.M, Benkirane.A, Hjiaj, Touzani.H, A.: *Existence and uniqueness of entropy solution for some nonlinear elliptic unilateral problems in Musielak-Orlicz-Sobolev spaces,*Ann. Univ. Craiova Math. Comput. Sci. Ser. $44(1), 1 - 20(2017)$

[5] Antontsev.S, Shmarev.S, : *Anisotropic parabolic equations with variable nonlinearity.* Publ. Math. $53, 355 - 399(2009)$

[6] Ait Kellou .M.A, Benkirane.A, Douiri.S.M, *An inequality of type Poincarke in Musielak spaces and applications to some nonlinear elliptic problems with L^1 -data, Complex Variablos and Elliptic Equations* $60, 1217 - 1242, (2015)$

[7] Benilan.P, Boccardo.L, Gallouet.T, Gariepy.R, Pierre.M, Vazquez.J,: *An L^1 -theory of existence and uniqueness of solutions of nonlinear elliptic equations,* Ann. Scuola Norm. Super. Pisa 22, 241 − 273 (1995)

[8] Ait Khellou.M, Benkirane.A,: *Elliptic inequalities with L^1 data in Musielak-Orlicz spaces,* Monatsh Math, 183, 1–33, (2017).

[9] Benkirane.A, El Hadfi.Y and El Moumni.M, Existence results for nonlinear parabolic equations with two lower order terms and L^1-data. *Ukraïn. Mat. Zh.* 71 , no. 5, (2019) 610–630.

[10] Benkirane.A, El Hadfi.Y and El Moumni.M, Renormalized solutions for nonlinear parabolic problems with L^1- data in orlicz-sobolev spaces. *Bulletin of Parana's Mathematical Society* (3s.) Volume 35, Number 1, (2017), pp: 57–84 .

[11] Benkirane.A, El Haji.B, and El Moumni.M; *On the existence of solution for degenerate parabolic equations with singular terms,* Pure and Applied Mathematics Quarterly Volume 14, Number 3-4, 591-606(2018).

[12] Benkirane.A, El Moumni.M, and Fri.A, An approximation of Hedberg's type in Sobolev spaces with variable exponent and application. *Chinese Journal of Mathematics*, Volume 2014 , Article ID 549051, (2014)7 pages .

[13] Benkirane.A, El Moumni.M, and Fri.A, Renormalized solution for strongly nonlinear elliptic problems with lower order terms and L^1-data. *Izvestiya RAN: Ser. Mat.* (2017) 81:3 3–20 .

[14] Boccardo.L, Gallouët.T,: *Nonlinear elliptic and parabolic equations involving measure data.* J. Funct. Anal. 87, 149-169 (1989)

[15] Boccardo.L, Murat.F,: *Strongly nonlinear Cauchy problems with gradient dependent lower order nonlinearity.* Pitman Res. Notes Math. 208, 247-254 (1989)

[16] Boccardo.L, Murat.F,: *Almost everywhere convergence of the gradients,* Nonlinear Anal. 19 (6) 581-597, (1992).

[17] DallAglio.A, Orsina.L,: Nonlinear parabolic equations with natural growth conditions and L^1 data. Nonlinear Anal. 27, 5973 (1996)

[18] Donaldson.T, *Inhomogeneous Orlicz-Sobolev spaces and nonlinear parabolic initial-boundary value problems*, J. Dif. Equations, 16 (1974), 201–256.

[19] Elmahi.A, *Strongly nonlinear parabolic initial-boundary value problems in Orlicz spaces*, Electron. J. Dif. Eqns., Conf. 09 (2002), 203–220.

[20] Elmahi.A, Meskine.D, *Strongly nonlinear parabolic equations having natural growth terms in Orlicz spaces,* Nonlinear Analysis, 60 (2005) 1-35.

[21] Elmahi.A, Meskine.D, *Parabolic equations in Orlicz space*, J. London Math. Soc. 2 (72) (2005) 410–428.

[22] Elmahi.A, Meskine.D, *Strongly nonlinear parabolic equations with natural growth terms and L^1 data in Orlicz spaces*, Port. Math. 62 (2) (2005).

[23] Gossez. J.-P,: *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients,* Trans. Amer. Math. Soc. 190, 163-205, (1974).

[24] Gossez.J.-P, Mustonen.V, *Variational inequality in Orlicz-Sobolev spaces*, Nonlinear Anal. Theory Appl. 11 (1987) 379–392.

[25] Hewitt.E, Stromberg.K,: *Real and Abstract Analysis. Springer, Berlin (1965)*

[26] Kbiri Alaoui.M, Meskine.D, Souissi.A, *On some class of nonlinear parabolic inequalities in Orlicz spaces*, Nonlinear Analysis, 74 (2011) 5863–5875.

[27] El Haji.B, El Moumni.M, Talha.A, *Entropy solutions for nonlinear parabolic equations in Musielak Orlicz spaces without Δ_2-conditions* - Gulf Journal of Mathematics,9 (1) 2020

[28] El Haji.B, El Moumni.M, Kouhaila.K, *On a nonlinear elliptic problems having large monotonocity with L^1-data in weighted Orlicz-Sobolev spaces*, Moroccan J. of Pure and Appl. Anal. (MJPAA) Vol 5(1), 104-116(2019).

[29] El Moumni.M, *Entropy solution for strongly nonlinear elliptic problems with lower order terms and L^1-data*, Annals of the University of Craiova - Mathematics and Computer Science Series. Volume 40, Number 2 , pp: 211–225 (2013).

[30] El Moumni.M, *Nonlinear elliptic equations without sign condition and L^1-data in Musielak-Orlicz-Sobolev spaces*, Acta. Appl. Math. 159:95–117 (2019).

[31] El Moumni.M, *Renormalized solutions for strongly nonlinear elliptic problems with lower order terms and measure data in Orlicz-Sobolev spaces*, Iran. J. Math. Sci. Inform. Vol 14, No 1, pp 95–119 (2019).

[32] El Amarti.N, El Haji.B and El Moumni.M,*Existence of renomalized solution for nonlinear Elliptic boundary value problem without Δ_2 -condition* SeMA 77, 389–414 (2020). https://doi.org/10.1007/s40324-020-00224-z.

[33] Prignet. A,: *Existence and uniqueness of entropy solutions of parabolic problems with L^1 data.* Nonlinear Anal. 28, $1943 - 1954(1997)$

[34] Porretta.A,: Existence results for strongly nonlinear parabolic equations via strong convergence of truncations. Ann. Math. Pura Appl. (IV) 177, 143-172 (1999)

[35] Landes.R, *On the existence of weak solutions for quasilinear parabolic initial-boundary value problems*, Proc. Roy. Soc. Edinburgh sect. A. 89 (1981) 137–217.

[36] Musielak.J,: *Modular spaces and Orlicz spaces,* Lecture Notes in Math. 10-34 (1983).

[37] Musielak.J, *Modular spaces and Orlicz spaces,* Lecture Notes in Math. 1034 (1983).

[38] Porretta.A,: *Existence results for strongly nonlinear parabolic equations via strong conver- gence of truncations,* Ann. Mat. Pura Appl. (IV), 177, 143-172, (1999).

[39] Rudin.W , *Real and Complex Analysis, 3rd ed., McGraw-Hill, New York, 1974.*

[40] Talha. A., Benkirane. A,: *Strongly nonlinear elliptic boundary value problems in Musielak- Orlicz spaces,* Monatsh Math, 184, 1-32, (2017).

Author information

Nourdine EL AMARTY, Department of Mathematics, Faculty of Sciences El Jadida, University Chouaib Doukkali, P. O. Box 20, 24000 El Jadida, Morocco.
E-mail: elamartynourdine@gmail.com

Badr EL HAJI, Department of Mathematics, Laboratory LaR2A, Department of Mathematics, Faculty of Sciences Tetouan, Abdelmalek Essaadi University, BP 2121, Tétouan, Morocco.
E-mail: badr.elhaji@gmail.com

Mostafa EL MOUMNI, Department of Mathematics, Faculty of Sciences El Jadida, University Chouaib Doukkali, P. O. Box 20, 24000 El Jadida, Morocco.
E-mail: mostafaelmoumni@gmail.com

Received: July 17, 2020.
Accepted: November 11, 2020.