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STABILITY OF E-PROXIMINALITY

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Abstract. The notion of E-proximinality was recently introduced. In this paper, we prove that E-proximinality is stable under c_0-direct sum of Banach spaces. We present an example of a proximinal hyperplane which is not E-proximinal. We also provide an alternate definition of E-proximinality and prove its equivalence.

1 Introduction

Let X be a real Banach space. The closed unit ball of X is denoted by B_X and the unit sphere of X is denoted by S_X. Also, X^* denotes the dual of X. For x in X and $r > 0$, we set

$$B[x, r] = \{y \in X : \|x - y\| \leq r\}, \quad B(x, r) = \{y \in X : \|x - y\| < r\}.$$

We consider only closed subspaces in this paper. For any f in X^*, $ker f$ denotes the kernel of f. The set of all norm attaining functionals on X is denoted by $NA(X)$ and $NA_1(X) = NA(X) \cap S_{X^*}$. For any x in X and a subset C of X, the distance of x from C is denoted by $d(x, C)$. Let

$$P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}.$$

The subset C is said to be proximinal in X, if for each $x \in X$, the set $P_C(x)$ is non-empty . For any $\delta > 0$ we set

$$P_C(x, \delta) = \{z \in C : \|x - z\| < d(x, C) + \delta\}.$$

We say a proximinal set C of a normed linear space X is *strongly proximinal* if for each x in X and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sup\{d(z, P_C(x)) : z \in P_C(x, \delta)\} < \epsilon.$$

For any f in X^* we set,

$$J_X(f) = \{x \in S_X : f(x) = \|f\|\}.$$

If $f \in X^*$, then we have

$$f \in NA(X) \Leftrightarrow J_X(f) \neq \emptyset \Leftrightarrow ker\ f \text{ is proximinal in } X.$$

Also, if H is proximinal, for any x in X, we have

$$P_H(x) = x - \frac{f(x)}{\|f\|} J_X(f).$$

The notion of ball proximinality of a closed subspace was introduced in [1], motivated by an example of Saidi, given in [10].

Definition 1.1. A subspace Y of a normed linear space X is *ball proximinal* in X if B_Y is proximinal in X.

It is easily verified (see [1] and [10]) that if Y is ball proximinal in X, then Y is proximinal in X. That the converse is not true, was shown by Saidi's counterexample [10]. Ball proximinality has been studied in [1], [4] and [5]. In [4], subspaces with strong $1\frac{1}{2}$- ball property were shown to be ball proximinal. In [3], an example of a Chebyshev strongly proximinal hyperplane $H = ker f$ that is not ball proximinal, is given. The characterizations of ball proximinal hyperplanes and strongly ball proximinal hyperplanes are given in [5].

Strong proximinality was introduced in [2] and results related to this notion has been studied in many research articles (see [1]-[5] and [7], [8]). In [9], it is shown that if X_i is a Banach space which has the property that any M-ideal of finite codimension in X_i is an intersection of M-ideals of codimension one, for all $i \geq 1$, then $X = (\oplus_{c_0} X_i)_{i\in\mathbb{N}}$ also has the same property. In this paper, we prove that E-proximinality is stable under c_0-direct sum of Banach spaces. We present an example of a proximinal hyperplane which is not E-proximinal. We also provide an alternate definition of E-proximinality and prove its equivalence.

2 Alternate Definition of E-proximinality

We start this section by proving an equivalent definition for E-proximinality. The notion of E-proximinality was introduced in [5].

Definition 2.1. [5] A proximinal subset Y of a Banach space X is called **E-proximinal** if for any x in X and $\epsilon > 0$ there exists an element y in $P_Y(x)$ such that $\|y\| < \alpha(x) + \epsilon$, where

$$\alpha(x) = \inf\{r > 0 : \exists (y_n) \subset Y \ s.t \ \|y_n\| \leq r \ and \ \lim_{n\to\infty} \|x - y_n\| = d(x, Y)\}.$$

The characterizations of E-proximinal hyperplanes and E-proximinal subspaces are given in [6]. It is shown in [6] that every proximinal hyperplane is also E-proximinal in $C(Q)$, where Q is a compact Hausdorff space. It is known from [6] that if X is Banach, $f \in NA_1(X)$ and the pre-duality map is norm-to-weak upper semi continuous at f, then $ker\ f$ is E-proximinal.

Let X be a Banach space and Y be a proximinal subspace of X. For x in X, consider the two conditions:

$$a)\ \ d(x, Y) = d(x, B_Y) \quad and \quad b)\ \ \inf\{\|y\| : y \in P_Y(x)\} \leq 1.$$

Clearly b) implies a). It is easy to check that if Y is strongly proximinal, or Y is ball proximinal then also we have a) implies b) and in these instances, a) and b) are equivalent.

It is clear that strong proximinality implies E-proximinality and ball proximinality implies E-proximinality. In [5], an example of an E-proximinal hyperplane which is not strongly proximinal is given. Also, it is easy to see that the example given in [3] is E-proximinal but it is not ball proximinal.

An alternate and natural way to view the equivalence of a) and b) arises from an observation from [3], which points out the equivalence of the two conditions below, for a subspace Y of a normed linear space X.

(I) Every $x \in X$ such that $d(x, Y) = d(x, B_Y)$ has a nearest point in B_Y,
or equivalently, $d(x, Y) = d(x, B_Y) \Rightarrow P_{B_Y}(x) = P_Y(x) \cap B_Y \neq \emptyset$.

(II) If for $x \in X$,
$\alpha(x) = \inf\{r > 0 : \exists (y_n) \subset Y \ s.t \ \|y_n\| \leq r \ and \ \lim_{n\to\infty} \|x - y_n\| = d(x, Y)\}$
then there is an element y in $P_Y(x)$ with $\|y\| = \alpha(x)$. That is, a nearest point which achieves the smallest norm of a minimizing sequence can be found.

For completeness, we prove the equivalence of the two conditions (I) and (II) given above. We begin by showing that the infimum $\alpha(x)$ is attained. That is, there exists a minimizing sequence (y_n) of x satisfying $\|y_n\| \leq \alpha(x)$ for all $n \geq 1$.

To see this, let $d = d(x, Y)$. Note that for each $k \in \mathbb{N}$, there exists a minimizing sequence $(y_{nk})_{n=1}^{\infty}$ such that

$$\sup_n \|y_{nk}\| < \alpha(x) + \frac{1}{k} \ \ and \ \ \|x - y_{kk}\| < d + \frac{1}{k}.$$

For all $n \in \mathbb{N}$, let $\gamma_n = \frac{\alpha(x)}{\alpha(x)+1/n}$ and $z_n = \gamma_n y_{nn}$. Then we have $\|z_n\| \leq \alpha(x)$ for all $n \geq 1$. Now $\lim_{n\to\infty} \gamma_n = 1$ and

$$\begin{aligned} \|x - z_n\| &\leq \|x - y_{nn}\| + \|y_{nn} - z_n\| \\ &< d + \frac{1}{n} + |1-\gamma_n| \, \|y_{nn}\| . \end{aligned}$$

Hence $\lim_{n\to\infty} \|x - z_n\| \leq d$. But $d \leq \|x - z_n\|$, $\forall\, n \in \mathbb{N}$, since $z_n \in Y$. So $d \leq \lim_{n\to\infty} \|x - z_n\|$. This with the above inquality implies $\lim_{n\to\infty} \|x - z_n\| = d$. Thus $(z_n)_{n=1}^{\infty}$ is the required minimizing sequence.

We now proceed with the proof for equivalence of (I) and (II) which also shows that (I) implies Y is proximinal. We only prove (I) implies (II), the other implication being obvious.

Pick any $x \in X\backslash Y$. Let $(y_n)_{n=1}^{\infty}$ be a minimizing sequence for x. Then $(y_n)_{n=1}^{\infty}$ is a bounded sequence and there exists $r > 0$ such that $\|y_n\| \leq r$ for all $n \geq 1$. Let $z = \frac{x}{r}$. Then $(\frac{y_n}{r})_{n=1}^{\infty}$ is a minimizing sequence for z and clearly $\frac{y_n}{r} \in B_Y$ for all $n \geq 1$. Hence $d(z, Y) = d(z, B_Y)$. By (I), there exists $w \in B_Y \cap P_Y(z)$. Then $rw \in P_Y(x)$ and $\|rw\| = r\,\|w\| \leq r$. This implies $P_Y(x) \neq \phi$ and since $x \in X\backslash Y$ was chosen arbitrarily, Y is proximinal in X. Further, since the choice of the minimizing sequence (y_n) was arbitrary, (II) holds.

We now go on to describe the alternate definition for E-proximinality. For this purpose, we first prove the equivalence of the two conditions given below.
(i) The subspace Y is proximinal and

$$d(x, Y) = d(x, B_Y) \Rightarrow \inf\{\|y\| : y \in P_Y(x)\} \leq 1.$$

(ii) If $(y_n)_{n=1}^{\infty}$ is a minimizing sequence for x with $\|y_n\| \leq r$ for all $n \geq 1$, then given $\epsilon > 0$ there exists $y_0 \in P_Y(x)$ such that $\|y_0\| < r + \epsilon$.
We prove (i) implies (ii), since the other implication is trivial.

Assume (i) and let $(y_n)_{n=1}^{\infty}$ be a minimizing sequence for x with $\|y_n\| \leq r$ for all $n \geq 1$. Then clearly $d(\frac{x}{r}, Y) = d(\frac{x}{r}, B_Y)$. So $\inf\{\left\|\frac{y}{r}\right\| : y \in P_Y(x)\} \leq 1$. This implies $\inf\{\|y\| : y \in P_Y(x)\} \leq r$. So there exists $y_0 \in P_Y(x)$ such that $\|y_0\| < r + \epsilon$.

Hence (i) and (ii) are equivalent and (i) provides an alternate definition of E-proximinality which is summed up in the following theorem.

Theorem 2.2. *Let Y be a proximinal subspace of a Banach space X. If for every x in X satisfying $d(x, Y) = d(x, B_Y)$, we have* $\inf\{\|y\| : y \in P_Y(x)\} \leq 1$*, then Y is E-proximinal.*

3 E-proximinality in Direct Sum Spaces

In this section, we consider the infinite c_0-direct sum of Banach spaces. We first prove a distance formula which is needed in the sequel.

Proposition 3.1. *Let Y_i be a subspace of the Banach space X_i for each $i \in \mathbb{N}$. Consider the c_0-direct sums $X = (\oplus_{c_0} X_i)_{i\in\mathbb{N}}$ and $Y = (\oplus_{c_0} Y_i)_{i\in\mathbb{N}}$. Let $x \in X \setminus B_Y$ and $x = \sum_{i\in\mathbb{N}} x_i$. Then* $d(x, B_Y) = \max\{d(x_i, B_{Y_i}) : 1 \leq i < \infty\}$.

Proof. Since $x \in X\backslash B_Y$, we have $d(x, B_Y) > 0$ and $\lim_{i\to\infty} \|x_i\| = 0$. This implies $\lim_{i\to\infty} d(x_i, B_{Y_i}) = 0$. Hence there exists $n_0 \in \mathbb{N}$ such that $d(x_i, B_{Y_i}) < d(x, B_Y)$ for all $i > n_0$. We know that $d(x, B_Y) = \inf_{y\in B_Y} \|x - y\|$. So given $\epsilon > 0$, there exists $z = (z_i)_{i=1}^{\infty} \in B_Y$ such that

$$\|x - z\| < d(x, B_Y) + \epsilon.$$

Note that $z_i \in B_{Y_i}$ for all $i \in \mathbb{N}$. It is clear that

$$\|x - z\| = \max\{\|x_i - z_i\| : 1 \leq i < \infty\}.$$

Hence

$$\|x_i - z_i\| < d(x, B_Y) + \epsilon, \ \forall\, 1 \leq i < \infty.$$

This implies

$$d(x_i, B_{Y_i}) \leq \|x_i - z_i\| < d(x, B_Y) + \epsilon, \ \forall\, 1 \leq i < \infty.$$

Since this is true for every $\epsilon > 0$, we get,

$$d(x_i, B_{Y_i}) \leq d(x, B_Y), \ \forall\, i \geq 1.$$

Hence

$$\max_{1 \leq i < \infty} d(x_i, B_{Y_i}) \quad \leq \quad d(x, B_Y). \tag{3.1}$$

To prove the reverse inequality, let $d = \max_{1 \leq i < \infty} d(x_i, B_{Y_i})$. Let $\epsilon > 0$ be given. Then there exists $z_i \in B_{Y_i}$ such that

$$\|x_i - z_i\| < d(x_i, B_{Y_i}) + \epsilon, \ \forall\, i \geq 1.$$

Let $z = \sum_{i \in \mathbb{N}} z_i$. Then clearly $z \in B_Y$ and $\max_{1 \leq i < \infty} \|x_i - z_i\| < d + \epsilon$. That is, $\|x - z\| < d + \epsilon$. Since this is true for every $\epsilon > 0$, we get,

$$d(x, B_Y) \quad \leq \quad d \tag{3.2}$$

From (3.1) and (3.2), we get $d(x, B_Y) = \max\{d(x_i, B_{Y_i}) : 1 \leq i < \infty\}$. □

We now show that E-proximinality is stable under infinite c_0-direct sum of Banach spaces.

Theorem 3.2. *Let $\{X_i : i \in \mathbb{N}\}$ be a family of Banach spaces and Y_i be an E-proximinal subspace of X_i for each $i \in \mathbb{N}$. Consider the following direct sums $X = (\oplus_{c_0} X_i)_{i \in \mathbb{N}}$ and $Y = (\oplus_{c_0} Y_i)_{i \in \mathbb{N}}$. Then Y is E-proximinal in X.*

Proof. Pick $x = (x_i) \in X$ such that $d(x, Y) = d(x, B_Y)$. We have to show that $\inf\{\|y\| : y \in P_Y(x)\} \leq 1$. We have $d = d(x, Y) = \sup_{1 \leq i < \infty} d(x_i, Y_i)$. Now $x \in X$ implies that $d(x_i, Y_i) \to 0$ as $i \to \infty$. Choose $i_0 \in \mathbb{N}$ such that

$$d(x_i, Y_i) \leq \|x_i\| \leq \frac{d}{2} < d, \ \ \forall\, i > i_0. \tag{3.3}$$

Since $d(x, Y) = \max_{1 \leq i \leq i_0} d(x_i, Y_i)$, there exists $m \in \mathbb{N}, 1 \leq m \leq i_0$ such that $d(x, Y) = d(x_m, Y_m)$. In this case

$$d(x, Y) = d(x_m, Y_m) \leq d(x_m, B_{Y_m}) \leq d(x, B_Y) = d(x, Y).$$

Hence

$$d(x, Y) = d(x_m, Y_m) = d(x_m, B_{Y_m}) = d(x, B_Y).$$

We now proceed to construct a sequence $(y_n)_{n=1}^{\infty} \subseteq Y$ such that $y_n \in P_Y(x)$ for all $n \geq 1$ and $\sup_n \|y_n\| = 1$. For this, for each $i \in \mathbb{N}$, we construct a sequence $(z_{ni})_{n=1}^{\infty}$ in Y_i such that $\lim_{n \to \infty} \|z_{ni}\| = 1$ and $\|x_i - z_{ni}\| \leq d(x, Y)$ for all $n \geq 1$ and set $y_n = \sum_{i \in \mathbb{N}} z_{ni}$, for all $n \geq 1$.

By Proposition 3.1, we have $d(x, B_Y) = \sup_{1 \leq i < \infty} d(x_i, B_{Y_i})$. For any $n \in \mathbb{N}$, let $z_{ni} = 0$ for all $i > i_0$, where i_0 is given by (3.3). Fix $i \in \{1, 2, ..., i_0\}$. We discuss three cases for the construction of the sequence $(z_{ni})_{n=1}^{\infty} \subseteq Y_i$ as follows.

Case 1. $d(x_i, Y_i) = d(x, Y)$.

Note that

$$d(x, Y) = d(x_i, Y_i) \leq d(x_i, B_{Y_i}) \leq d(x, B_Y) = d(x, Y).$$

Hence $d(x_i, Y_i) = d(x_i, B_{Y_i})$ in this case. Since Y_i is E-proximinal in X_i, we can get a sequence $(z_{ni})_{n=1}^{\infty} \subseteq P_{Y_i}(x_i)$ such that $\lim_{n \to \infty} \|z_{ni}\| = 1$ and $\|x_i - z_{ni}\| = d(x, Y)$.

Case 2. $d(x_i, Y_i) < d(x, Y)$ and $d(x_i, B_{Y_i}) < d(x, B_Y)$.
Since $d(x, B_Y) > d(x_i, B_{Y_i})$, there exists $\eta_i > 0$ such that

$$d(x, B_Y) \quad > \quad d(x_i, B_{Y_i}) + \eta_i.$$

Now get a sequence $(z_{ni})_{n=1}^{\infty} \subseteq B_{Y_i}$ such that

$$\|x_i - z_{ni}\| < d(x_i, B_{Y_i}) + \eta_i, \quad \forall\, n \geq 1.$$

Then note that $\lim_{n\to\infty} \|x_i - z_{ni}\| < d(x, B_Y) = d(x, Y)$.
Case 3. $d(x_i, Y_i) < d(x, Y)$ and $d(x_i, B_{Y_i}) = d(x, B_Y)$.
Note that in this case

$$d(x_i, Y_i) < d(x_i, B_{Y_i}) = d(x, B_Y) = d(x, Y).$$

We now proceed as follows. Let $\epsilon_{ni} > 0$ be such that $\lim_{n\to\infty} \epsilon_{ni} = 0$. Pick any sequence $(v_{ni})_{n=1}^{\infty} \subseteq B_{Y_i}$ such that

$$\|x_i - v_{ni}\| < d(x, Y) + \epsilon_{ni}, \quad \forall\, n \geq 1.$$

Choose $\delta_i > 0$ be such that $d(x_i, Y_i) = d(x, Y) - \delta_i$. Pick any $u_i \in P_{Y_i}(x_i)$ such that $\|u_i\| \leq 1 + \rho_i$, for some $\rho_i > 0$. For any $n \geq 1$, set $\lambda_{ni} = \frac{\delta_i}{\delta_i + \epsilon_{ni}}$. Then note that

$$\lambda_{ni}\epsilon_{ni} - \delta_i(1 - \lambda_{ni}) \quad = \quad 0. \tag{3.4}$$

Set $z_{ni} = \lambda_{ni} v_{ni} + (1 - \lambda_{ni}) u_i$ for all $n \geq 1$. Then

$$\begin{aligned} \|x_i - z_{ni}\| &\leq \lambda_{ni} \|x_i - v_{ni}\| + (1 - \lambda_{ni}) \|x_i - u_i\| \\ &< \lambda_{ni}(d(x, Y) + \epsilon_{ni}) + (1 - \lambda_{ni})(d(x, Y) - \delta_i) \\ &= d(x, Y) + \lambda_{ni}\epsilon_{ni} - \delta_i(1 - \lambda_{ni}) \\ &= d(x, Y) \ \text{ by (3.4).} \end{aligned}$$

Also

$$\begin{aligned} \|z_{ni}\| &\leq \|\lambda_{ni} v_{ni}\| + \|(1 - \lambda_{ni}) u_i\| \\ &\leq \lambda_{ni} + (1 - \lambda_{ni})(1 + \rho_i) \\ &= 1 + \frac{\rho_i \epsilon_{ni}}{\delta_i + \epsilon_{ni}}. \end{aligned}$$

Since $\lim_{n\to\infty} \epsilon_{ni} = 0$, we get $\lim_{n\to\infty} \|z_{ni}\| = 1$.

For any $n \geq 1$, define $z_{ni} \in Y_i$ for all $1 \leq i \leq i_0$ as in the above cases. Recall that we have set $z_{ni} = 0$ for $i > i_0$ and for $n \geq 1$. Then we have $\lim_{i\to\infty} z_{ni} = 0$ for all $n \geq 1$. Now define $y_n \in Y = (\oplus_{c_0} Y_i)_{i\in\mathbb{N}}$ as $y_n = (\oplus_{c_0} z_{ni})_{i\in\mathbb{N}}$ for each $n \geq 1$. Then $(y_n)_{n=1}^{\infty}$ is a sequence in $(\oplus_{c_0} Y_i)_{i\in\mathbb{N}}$.

Let $\epsilon > 0$ be given. Then for any $n \geq 1$, we have $\|y_n\| = \max_{1\leq i\leq i_0} \|z_{ni}\|$. Since $\lim_{n\to\infty} \|z_{ni}\| \leq 1$ for all $1 \leq i \leq i_0$, there exists $N_i \in \mathbb{N}$ such that $|\,\|z_{ni}\| - 1| < \epsilon$ for all $n \geq N_i$. Let $N = \max\{N_i : 1 \leq i \leq i_0\}$. Then $\|y_n\| < 1 + \epsilon$ for all $n \geq N$ and hence $\limsup_n \|y_n\| = 1$.

Having defined the sequence $(z_{ni})_{i=1}^{\infty}$ for all the possible three cases we note that in all the cases $\lim_{n\to\infty} \|z_{ni}\| \leq 1$ and $\|x_i - z_{ni}\| \leq d(x, Y)$. Hence for $y_n = (\oplus_{c_0} z_{ni})_{i\in\mathbb{N}}$, we have $\limsup_n \|y_n\| = 1$ and

$$d(x, Y) \leq \|x - y_n\| = \sup_{1\leq i<\infty} \|x_i - z_{ni}\| \leq d(x, Y).$$

Hence $y_n \in P_Y(x)$ for all $n \geq 1$ and $\limsup_n \|y_n\| = 1$. This implies $\inf\{\|y\| : y \in P_Y(x)\} \leq 1$ if $d(x, Y) = d(x, B_Y)$. Since $x \in X$ was chosen arbitrarily, this implies Y is E-proximinal in X. □

Remark 3.3. The finite ℓ_∞-direct sum can be considered as an infinite c_0-direct sum by adding infinitely many zero subspaces. Hence from the above theorem, it now follows that E-proximinality is stable under finite ℓ_∞-direct sums.

4 Example of a proximinal hyperplane which is not E-proximinal

Now we present an example of a proximinal hyperplane which is not E-proximinal. The construction of the example is based on a renorming in the space ℓ_1.

Example 4.1. Let $(e_n)_{n\geq 1}$ be the canonical basis of $\ell_1(\mathbb{N})$. For $n \geq 3$, we let $v_n = \frac{n-1}{n}(e_1 + e_2 - e_n)$. Set

$$B = \overline{conv}^{\|\cdot\|}\{(\pm e_1) \cup (\pm\frac{3}{4}e_2) \cup (\pm\frac{n-1}{n}e_n)_{n\geq 3} \cup \pm(v_n)_{n\geq 3}\}.$$

The set B is the closed unit ball of an equivalent norm denoted by $\|.\|_B$ on ℓ_1. Note that $d_B(u, H)$ denotes the distance of u from H with respect to $\|.\|_B$ norm. Let $(e_n^*)_{n\geq 1}$ denote the coordinate functionals. We have $\|e_1^*\|_{B^*} = 1$ and $e_1^{*^{-1}}(1) \cap B = \{e_1\}$. Hence $H = ker(e_1^*)$ is proximinal in $(\ell_1, \|.\|_B)$. If $u = e_1 + e_2$, then $P_H(u) = e_2$ and $d_B(u, H) = 1$. Moreover

$$\|u - v_n\|_B = \left\|\frac{n-1}{n}e_n + \frac{1}{n}(e_1 + e_2)\right\|_B \to 1.$$

Hence $d_B(u, B_H) = 1$. Note that $(v_k)_{k\geq 3} \subseteq B$ and $e_2^*(v_k) = \frac{k-1}{k} \to 1$ as $k \to \infty$. Further

$$\begin{aligned}
\|e_2 - v_k\|_1 &= \left\|e_2 - \frac{k-1}{k}(e_1 + e_2 - e_k)\right\|_1 \\
&= \left\|\frac{e_2}{k} + \frac{k-1}{k}(e_k - e_1)\right\|_1 \\
&\leq \frac{1}{k} + \frac{k-1}{k}\|e_k - e_1\|_1 \\
&= \frac{1}{k} + 2\left(1 - \frac{1}{k}\right) \\
&\to 2 \text{ as } k \to \infty.
\end{aligned}$$

Let $(w_k)_{k\geq 1} \subseteq B$ and $w_k = \lim_{n\to\infty} z_{k_n}$, where

$$z_{k_n} = \lambda_{1n}e_1 + \lambda_{2n}\left(\frac{3}{4}e_2\right) + \lambda_{3n}\left(\frac{k-1}{k}e_k\right) + \lambda_{4n}v_k$$

and $\sum_{i=1}^{4}\lambda_{in} = 1, \lambda_{in} > 0$ for all $1 \leq i \leq 4, 1 \leq n < \infty$. Then

$$e_2^*(w_k) = \lim_{n\to\infty} e_2^*(z_{k_n}) = \lim_{n\to\infty}\left[\lambda_{2n}\frac{3}{4} + \lambda_{4n}\frac{k-1}{k}\right].$$

Now $\lim_{k\to\infty} e_2^*(w_k) = 1$ implies $\lim_{n\to\infty} \lambda_{4n} = 1$ and $w_k = v_k$. Hence if $(w_k)_{k\geq 1} \subseteq B$ and $e_2^*(w_k) \to 1$, then $\|e_2 - w_k\|_1 \to 2$. This implies that $\|e_2\|_B > 1$. Note that $d_B(u, H) = d_B(u, B_H) = 1$, $P_H(u) = e_2$ and $\|e_2\|_B > 1$. Hence $\inf\{\|y\|_B : y \in P_H(u)\} > 1$. It now follows that H is proximinal but not E-proximinal.

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