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FEKETE-SZEGÖ INEQUALITIES FOR q-STARLIKE AND q-CONVEX FUNCTIONS INVOLVING q-ANALOGUE OF RUSCHEWEYH-TYPE DIFFERENTIAL OPERATOR

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Abstract In the present paper, the new generalized Ma-Minda type classes of q-starlike and q-convex functions are introduced by using Ruscheweyh-type q-differential operator. Denote by $\mathcal{SR}_q^\lambda(\phi)$ the class of q-starlike functions and $\mathcal{CR}_q^\lambda(\phi)$ the class of q-convex functions associated with Ruscheweyh-type q-differential operator, where ϕ is the function with positive real part. By making use of these classes, we obtain initial coefficient estimates and Fekete-Szegö inequalities for the classes $\mathcal{SR}_q^\lambda(\phi)$ and $\mathcal{CR}_q^\lambda(\phi)$, respectively.

1 Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of all analytic functions in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Let $\mathcal{A}$ be the class of analytic functions $f \in \mathcal{H}(\mathbb{U})$ which are normalized by $f(0) = f'(0) - 1 = 0$ and have the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.1}$$

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ containing all univalent functions in $\mathbb{U}$. For given two functions $f, g \in \mathcal{H}(\mathbb{U})$, we say that f is subordinate to g in $\mathbb{U}$, denoted by $f(z) \prec g(z)$, if there exists a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0, |\omega(z)| < 1$ such that $f(z) = g(\omega(z))$ in the unit disk $\mathbb{U}$ (see [6]).

Let $\mathcal{P}$ be the class of analytic functions p in $\mathbb{U}$ with $p(0) = 1$ and $\Re(p(z)) > 0$ such that $p \in \mathcal{P}$ if and only if $p(z) \prec (1+z)/(1-z)$. It is well known that a function $f \in \mathcal{S}$ is called starlike ($f \in \mathcal{S}^*$) or convex ($f \in \mathcal{C}$) if there exists a function $p \in \mathcal{P}$ such that p must be expressed, respectively, by the relations $zf'(z)/f(z) = p(z)$ and $1 + zf''(z)/f'(z) = p(z)$ for all $z \in \mathbb{U}$. For definitions and properties of these classes see [1] and [6].

Ma and Minda [13] unified various subclasses of starlike and convex functions for which either one of the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a more general superordinate function. The classes $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ of Ma-Minda starlike and Ma-Minda convex functions are, respectively, characterized by $zf'(z)/f(z) \prec \phi(z)$ and $1 + zf''(z)/f'(z) \prec \phi(z)$. Here we assume that $\phi \in \mathcal{P}$ satisfying $\phi(0) = 1, \phi'(0) > 0$ and $\phi(\mathbb{U})$ is symmetric with respect to the real axis. Also, ϕ has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...,$$

where all coefficients are real and $B_1 > 0$.

The coefficient $|a_3 - \mu a_2^2|$ on the normalized analytic functions f in $\mathbb{U}$ plays an important role in geometric functions theory. In 1933, Fekete and Szegö [4] obtained a sharp bound of the functional $\mu a_2^2 - a_3$ with real $0 \le \mu \le 1$ for a univalent function f . Since then, the problem of finding the sharp bounds for this functional of any compact family of functions $f \in \mathcal{A}$ with any complex μ is known as the classical Fekete-Szegö problem or inequality. Many authors have considered the Fekete-Szegö problem for various subclasses of $\mathcal{A}$, the upper bound for $|a_3 - \mu a_2^2|$ was investigated by many different authors (see [11], [12], [16]).

Quantum calculus or $q-$calculus is the calculus where we do not need limits. The great interest is due to its applications in various branches of mathematics and physics, as for example, in the areas of ordinary fractional calculus, orthogonal polynomials, basic hypergeometric functions, combinatorics and so on.

Let $q \in (0,1)$. The $q-$derivative (or $q-$difference) operator, introduced by Jackson [8], is defined as

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \ (z \neq 0).$$

We note that $\lim_{q\to 1^-}(D_q f)(z) = f'(z)$ if f is differentiable at z. For a function f of the form (1.1), we observe that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where $[n]_q = \frac{1-q^n}{1-q}, q \in (0,1)$. Clearly, for $q \to 1^-$, $[n]_q \to n$.

The $q-$Gamma function is given by

$$\Gamma_q(n+1) = [n]_q \Gamma_q(n),$$

and the $q-$factorial is given by

$$[n]_q! = [1]_q[2]_q...[n-1]_q[n]_q, (n > 0) \quad \text{and} \quad [0]_q! = 1.$$

Also, the $q-$shifted factorial is given by

$$([n]_q)_m = [n]_q[n+1]_q...[n+m-1]_q, (m \geq 1) \quad \text{and} \quad ([n]_q)_0 = 1.$$

For definitions and properties of $q-$calculus, one may refer to [5], [8], [9].

In [10], Kanas and Raducanu introduced the Ruscheweyh-type q-differential operator. Let $f \in \mathcal{A}$ be given by (1.1). Then the Ruscheweyh-type q-differential operator $\mathcal{R}^\lambda : \mathcal{A} \to \mathcal{A}$ is defined by

$$\mathcal{R}_q^\lambda f(z) = f(z) * F_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{([\lambda+1]_q)_{n-1}}{[n-1]_q!} a_n z^n \quad (z \in \mathbb{U}, \ \lambda > -1), \tag{1.2}$$

where

$$F_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{([\lambda+1]_q)_{n-1}}{[n-1]_q!} z^n.$$

The Ruscheweyh-type q-differential operator reduces to the differential operator defined by Ruscheweyh [15] in the case when $q \to 1^-$.

The class $\mathcal{S}_q^*$ of q-starlike functions was introduced and studied by Ismail *et al.* in [7]. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_q^*$ if and only if

$$\mathcal{S}_q^* = \left\{ f \in \mathcal{A} \ : \ \Re\left(\frac{zD_q f(z)}{f(z)} \right) > 0, \ q \in (0,1), \ z \in \mathbb{U} \right\}.$$

In the limiting case $q \to 1^-$, the class $\mathcal{S}_q^*$ reduces to the class $\mathcal{S}^*$.

The class $\mathcal{C}_q$ of q-convex functions was introduced by Ahuja *et al.* in [2]. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}_q$ if and only if

$$\mathcal{C}_q = \left\{ f \in \mathcal{A} \ : \ \Re\left(\frac{D_q(zD_q f(z))}{D_q f(z)} \right) > 0, \ q \in (0,1), \ z \in \mathbb{U} \right\}.$$

In the limiting case $q \to 1^-$, the class $\mathcal{C}_q$ reduces to the class $\mathcal{C}$.

By using principle of subordination, we define two new Ma-Minda type classes of q-starlike and q-convex functions associated with the Ruscheweyh-type q-differential operator as below:

$$\mathcal{SR}_q^{\lambda}(\phi) = \left\{ f \in \mathcal{A} \,:\, \frac{zD_q(\mathcal{R}_q^{\lambda} f(z))}{\mathcal{R}_q^{\lambda} f(z)} \prec \phi(z),\ \phi \in \mathcal{P}, \lambda > -1, z \in \mathbb{U} \right\},$$

$$\mathcal{CR}_q^{\lambda}(\phi) = \left\{ f \in \mathcal{A} \,:\, \frac{D_q(zD_q(\mathcal{R}_q^{\lambda} f(z)))}{D_q(\mathcal{R}_q^{\lambda} f(z))} \prec \phi(z),\ \phi \in \mathcal{P}, \lambda > -1, z \in \mathbb{U} \right\}.$$

Remark 1.1. For $\lambda = 0$, we get the following known classes defined in [3]:

$$\mathcal{SR}_q^{0}(\phi) \equiv \mathcal{S}_q^{*}(\phi) \ \text{ and } \ \mathcal{CR}_q^{0}(\phi) \equiv \mathcal{C}_q(\phi).$$

In order to prove our results, we need the following lemma:

Lemma 1.2. [14] Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \dots$, then $|c_n| \le 2$ for $n \ge 1$. If $|c_1| = 2$, then $p(z) \equiv p_1(z) = \frac{1+\gamma_1 z}{1-\gamma_2 z}$ with $\gamma_1 = \frac{c_1}{2}$. Conversely, if $p(z) \equiv p_1(z)$ for some $|\gamma_1| = 1$, then $c_1 = 2\gamma_1$ and $|c_1| = 2$. Furthermore, we have

$$\left| c_2 - \frac{c_1^2}{2} \right| \le 2 - \frac{|c_1|^2}{2}.$$

If $|c_1| < 2$ and $|c_2 - \frac{c_1^2}{2}| \le 2 - \frac{|c_1^2|}{2}$, then $p(z) \equiv p_2(z)$, where

$$p_2(z) = \frac{1 + z\frac{\gamma_2 z + \gamma_1}{1+\overline{\gamma_1}\gamma_2 z}}{1 - z\frac{\gamma_2 z + \gamma_1}{1+\overline{\gamma_1}\gamma_2 z}}$$

and $\gamma_1 = \frac{c_1}{2}$, $\gamma_2 = \frac{2c_2 - c_1^2}{4-|c_1|^2}$. Conversely, if $p(z) \equiv p_2(z)$ for some $|\gamma_1| = 1$ and $|\gamma_2| = 1$, then $\gamma_1 = \frac{c_1}{2}$, $\gamma_2 = \frac{2c_2 - c_1^2}{4-|c_1|^2}$ and $|c_2 - \frac{c_1^2}{2}| \le 2 - \frac{|c_1|^2}{2}$.

The aim of this paper is to investigate the initial coefficient estimates and Fekete-Szegö inequalities for the classes $\mathcal{SR}_q^{\lambda}(\phi)$ of q-starlike functions and $\mathcal{CR}_q^{\lambda}(\phi)$ of q-convex functions defined by Ruscheweyh-type q-differential operator. Several special cases of the main results are also obtained.

2 Main Results

We first give initial coefficient estimates $|a_2|$ and $|a_3|$ for the functions in the class $\mathcal{SR}_q^{\lambda}(\phi)$.

Theorem 2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$, where the coefficients B_n are real and $B_1 > 0$. If f given by (1.1) belongs to the class $\mathcal{SR}_q^{\lambda}(\phi)$, then for $\lambda > -1$ we have*

$$|a_2| \le \frac{B_1}{([2]_q - 1)[\lambda+1]_q}, \tag{2.1}$$

$$|a_3| \le \frac{(2)_q! B_1}{([3]_q - 1)([\lambda+1]_q)_2} \max\left\{ 1, \left| \frac{B_1}{[2]_q - 1} + \frac{B_2}{B_1} \right| \right\}, \tag{2.2}$$

$$\left| a_3 - \frac{([2]_q - 1)^2([\lambda+1]_q)^2 (2)_q! (\frac{B_1}{[2]_q - 1} + \frac{B_2}{B_1} - 1)}{B_1([3]_q - 1)([\lambda+1]_q)_2} a_2^2 \right| \le \frac{(2)_q! B_1}{([3_q] - 1)([\lambda+1]_q)_2}. \tag{2.3}$$

These results are sharp.

Proof. If $f \in \mathcal{SR}_q^\lambda(\phi)$, then there exists a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$\frac{zD_q(\mathcal{R}_q^\lambda f(z))}{\mathcal{R}_q^\lambda f(z)} = \phi(\omega(z)).$$

Define the function p by

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + \ldots \quad .$$

We can note that $p(0) = 1$ and p is a function with positive real part. Therefore

$$\phi(\omega(z)) = \phi\left(\frac{p(z)-1}{p(z)+1}\right) = \phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) z^3 + \ldots\right]\right)$$
$$= 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \ldots \quad . \tag{2.4}$$

Also, computations show that

$$\frac{zD_q(\mathcal{R}_q^\lambda f(z))}{\mathcal{R}_q^\lambda f(z)} = 1 + \frac{([2]_q - 1)[\lambda+1]_q}{(1)_q!} a_2 z$$
$$+ \left[\frac{([3]_q - 1)\,([\lambda+1]_q)_2}{(2)_q!} a_3 - ([2]_q - 1)\left(\frac{[\lambda+1]_q}{(1)_q!}\right)^2 a_2^2\right] z^2 + \ldots \quad . \tag{2.5}$$

From equations in (2.4) and (2.5), we obtain

$$\frac{([2]_q - 1)[\lambda+1]_q}{(1)_q!} a_2 = \frac{B_1 c_1}{2} \tag{2.6}$$

and

$$\frac{([3]_q - 1)\,([\lambda+1]_q)_2}{(2)_q!} a_3 - ([2]_q - 1)\left(\frac{[\lambda+1]_q}{(1)_q!}\right)^2 a_2^2 = \frac{B_1 c_2}{2} - \frac{B_1 c_1^2}{4} + \frac{B_2 c_1^2}{4}. \tag{2.7}$$

Taking into account Lemma 1.2, we obtain

$$|a_2| = \left|\frac{B_1 c_1}{2([2]_q - 1)[\lambda+1]_q}\right| \le \frac{B_1}{([2]_q - 1)[\lambda+1]_q}$$

and

$$|a_3| = \left|\frac{(2)_q! B_1}{2([3]_q - 1)([\lambda+1]_q)_2}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(\frac{B_1}{([2]_q - 1)} + \frac{B_2}{B_1}\right)\right]\right|$$
$$\le \frac{(2)_q! B_1}{2([3]_q - 1)([\lambda+1]_q)_2}\left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(\frac{B_1}{([2]_q - 1)} + \frac{B_2}{B_1}\right)\right]$$
$$\le \frac{(2)_q! B_1}{([3]_q - 1)([\lambda+1]_q)_2} \max\left\{1, \left|\frac{B_1}{([2]_q - 1)} + \frac{B_2}{B_1}\right|\right\}.$$

Now using (2.6) and (2.7), we get

$$\left|a_3 - \frac{(2)_q!([2]_q - 1)^2([\lambda+1]_q)^2}{B_1([3]_q - 1)([\lambda+1]_q)_2}\left(\frac{B_1}{[2]_q - 1} + \frac{B_2}{B_1} - 1\right) a_2^2\right|$$

$$= \frac{(2)_q! |B_1 c_2|}{2([3]_q - 1)([\lambda+1]_q)_2} \le \frac{(2)_q! B_1}{([3]_q - 1)([\lambda+1]_q)_2}.$$

An examination of the proof shows that equality in (2.1) is attained if $c_1 = 2$. Equivalently, we have $p(z) = (1+z)/(1-z)$. Therefore, the extremal function in class $\mathcal{SR}_q^\lambda(\phi)$ is given by

$$\frac{zD_q(\mathcal{R}_q^\lambda f(z))}{\mathcal{R}_q^\lambda f(z)} = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \tag{2.8}$$

In equality (2.2), for the first case, equality holds if $c_1 = 0$, $c_2 = 2$. Equivalently, we have $p(z) = p_2(z) = (1+z^2)/(1-z^2)$. Therefore, the extremal function in $\mathcal{SR}_q^\lambda(\phi)$ is given by

$$\frac{zD_q(\mathcal{R}_q^\lambda f(z))}{\mathcal{R}_q^\lambda f(z)} = \phi\left(\frac{p_2(z)-1}{p_2(z)+1}\right). \tag{2.9}$$

In (2.2), for the second case, the equality holds if $c_1 = 2$, $c_2 = 2$. Therefore, the extremal function in $\mathcal{SR}_q^\lambda(\phi)$ is given by (2.8). Obtained extremal function for (2.1) is also valid for (2.3).

In fact, Theorem 2.1 gives a special case of Fekete-Szego problem for real

$$\mu = \frac{(2)_q!([2]_q-1)^2[\lambda+1]_q^2}{B_1([3]_q-1)([\lambda+1]_q)_2}\left(\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}-1\right),$$

which obtain the naturally and simple estimate. Thus the proof is completed.

Setting $\lambda = 0$, we get the following known result given in [3].

Remark 2.2. Let $\phi(z) = 1 + B_1z + B_2z^2 + ...$, where the coefficients B_n are real and $B_1 > 0$. If f given by (1.1) belongs to the class $\mathcal{S}_q^*(\phi)$, then

$$|a_2| \le \frac{B_1}{[2]_q-1}, \quad |a_3| \le \frac{B_1}{[3]_q-1}\max\left\{1, \left|\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}\right|\right\},$$

$$\left|a_3 - \frac{([2]_q-1)^2(\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}-1)}{B_1([3]_q-1)}a_2^2\right| \le \frac{B_1}{[3]_q-1}.$$

We consider the Fekete-Szegö problem with complex μ in the following theorem. □

Theorem 2.3. *Let μ be a non-zero complex number and let f given by (1.1) belongs to the class $\mathcal{SR}_q^\lambda(\phi)$, then for $\lambda > -1$ we have*

$$|a_3 - \mu a_2^2| \le \frac{(2)_q!B_1}{([3]_q-1)([\lambda+1]_q)_2}\max\left[1, \left|\frac{B_2}{B_1}+\frac{B_1}{([2]_q-1)}\left(1-\frac{\mu([3]_q-1)([\lambda+1]_q)_2}{(2)_q!([2]_q-1)[\lambda+1]_q^2}\right)\right|\right]. \tag{2.10}$$

The result is sharp.

Proof. Applying (2.6) and (2.7), we get

$$a_3 - \mu a_2^2 = \frac{(2)_q!B_1}{2([3]_q-1)([\lambda+1]_q)_2}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left\{\frac{B_2}{B_1}+\frac{B_1}{([2]_q-1)}\left(1-\frac{\mu([3]_q-1)([\lambda+1]_q)_2}{(2)_q!([2]_q-1)[\lambda+1]_q^2}\right)\right\}\right]$$

In view of Lemma 1.2, we arrive at

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \frac{(2)_q!B_1}{2([3]_q-1)([\lambda+1]_q)_2}\left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left\{\left|\frac{B_2}{B_1}+\frac{B_1}{([2]_q-1)}\left(1-\frac{\mu([3]_q-1)([\lambda+1]_q)_2}{(2)_q!([2]_q-1)[\lambda+1]_q^2}\right)\right|\right\}\right] \\ &= \frac{(2)_q!B_1}{2([3]_q-1)([\lambda+1]_q)_2}\left[2 + \frac{|c_1|^2}{2}\left\{\left|\frac{B_2}{B_1}+\frac{B_1}{([2]_q-1)}\left(1-\frac{\mu([3]_q-1)([\lambda+1]_q)_2}{(2)_q!([2]_q-1)[\lambda+1]_q^2}\right)\right| - 1\right\}\right] \\ &\le \frac{(2)_q!B_1}{([3]_q-1)([\lambda+1]_q)_2}\max\left[1, \left|\frac{B_2}{B_1}+\frac{B_1}{([2]_q-1)}\left(1-\frac{\mu([3]_q-1)([\lambda+1]_q)_2}{(2)_q!([2]_q-1)[\lambda+1]_q^2}\right)\right|\right]. \end{aligned}$$

Equality is attained for the first case on choosing $c_1 = 0$, $c_2 = 2$ in (2.9) and for the second case on choosing $c_1 = 2$, $c_2 = 2$ in (2.8) . Thus the proof is completed. □

Setting $\lambda = 0$, we get the following known result given in [3].

Remark 2.4. Let μ be a nonzero complex number and let f given by (1.1) belongs to the class $\mathcal{S}_q^*(\phi)$, then

$$|a_3-\mu a_2^2|\leq\frac{B_1}{[3]_q-1}\max\left\{1,\left|\frac{B_2}{B_1}+\frac{B_1}{[2]_q-1}\left(1-\frac{([3]_q-1)}{([2]_q-1)}\mu\right)\right|\right\}.$$

In the next theorem, we investigate initial coefficient estimates for the class $\mathcal{CR}_q^\lambda(\phi)$.

Theorem 2.5. *Let $\phi(z)=1+B_1z+B_2z^2+...$, where the coefficients B_n are real and $B_1>0$. If f given by (1.1) belongs to the class $\mathcal{CR}_q^\lambda(\phi)$, then for $\lambda>-1$ we have*

$$|a_2|\leq\frac{B_1}{[2]_q([2]_q-1)[\lambda+1]_q},\tag{2.11}$$

$$|a_3|\leq\frac{(2)_q!B_1}{[3]_q([3]_q-1)([\lambda+1]_q)_2}\max\left\{1,\left|\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}\right|\right\},\tag{2.12}$$

$$\left|a_3-\frac{(2)_q![2]_q^2([2]_q-1)^2[\lambda+1]_q^2(\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}-1)}{B_1[3]_q([3]_q-1)([\lambda+1]_q)_2}a_2^2\right|\leq\frac{(2)_q!B_1}{[3]_q([3_q]-1)([\lambda+1]_q)_2}.\tag{2.13}$$

These results are sharp.

Proof. If $f\in\mathcal{CR}_q^\lambda(\phi)$, then there exists a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ such that

$$\frac{D_q(zD_q(\mathcal{R}_q^\lambda f(z)))}{\mathcal{D}_q(\mathcal{R}_q^\lambda f(z))}=\phi(\omega(z)).$$

Computations show that

$$\frac{D_q(zD_q(\mathcal{R}_q^\lambda f(z)))}{\mathcal{D}_q(\mathcal{R}_q^\lambda f(z))}=1+\frac{[2]_q([2]_q-1)[\lambda+1]_q}{(1)_q!}a_2z$$
$$+\left[\frac{[3]_q\,([3]_q-1)\,([\lambda+1]_q)_2}{(2)_q!}a_3-[2]_q^2([2]_q-1)\left(\frac{[\lambda+1]_q}{(1)_q!}\right)^2a_2^2\right]z^2+...\;.\tag{2.14}$$

From equations given in (2.4) and (2.14), we obtain

$$\frac{[2]_q([2]_q-1)[\lambda+1]_q}{(1)_q!}a_2=\frac{B_1c_1}{2}\tag{2.15}$$

and

$$\frac{[3]_q\,([3]_q-1)\,([\lambda+1]_q)_2}{(2)_q!}a_3-[2]_q^2([2]_q-1)\left(\frac{[\lambda+1]_q}{(1)_q!}\right)^2a_2^2=\frac{B_1c_2}{2}-\frac{B_1c_1^2}{4}+\frac{B_2c_1^2}{4}.\tag{2.16}$$

In view of Lemma 1.2, using (2.15) and (2.16) we obtain

$$|a_2|=\left|\frac{B_1c_1}{2[2]_q([2]_q-1)[\lambda+1]_q}\right|\leq\frac{B_1}{[2]_q([2]_q-1)[\lambda+1]_q}$$

and

$$|a_3|\leq\frac{(2)_q!B_1}{[3]_q([3]_q-1)([\lambda+1]_q)_2}\max\left\{1,\left|\frac{B_1}{([2]_q-1)}+\frac{B_2}{B_1}\right|\right\}.$$

Using (2.15) and (2.16), we also get

$$\left|a_3-\frac{[2]_q^2(2)_q!([2]_q-1)^2[\lambda+1]_q^2}{B_1[3]_q([3]_q-1)([\lambda+1]_q)_2}\left(\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}-1\right)a_2^2\right|$$

$$= \frac{(2)_q!|B_1c_2|}{2[3]_q([3]_q-1)([\lambda+1]_q)_2} \le \frac{(2)_q!B_1}{[3]_q([3]_q-1)([\lambda+1]_q)_2}.$$

Equality in (2.11) holds if

$$\frac{D_q(zD_q(\mathcal{R}_q^\lambda f(z)))}{\mathcal{D}_q(R_q^\lambda f(z))} = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \tag{2.17}$$

and in (2.12) holds if

$$\frac{D_q(zD_q(\mathcal{R}_q^\lambda f(z)))}{\mathcal{D}_q(R_q^\lambda f(z))} = \phi\left(\frac{p_2(z)-1}{p_2(z)+1}\right), \tag{2.18}$$

where p_1 and p_2 are given by Lemma 1.2.

Theorem 2.5 gives a special case of Fekete-Szegö problem for real

$$\mu = \frac{(2)_q![2]_q^2([2]_q-1)^2[\lambda+1]_q^2}{B_1[3]_q([3]_q-1)([\lambda+1]_q)_2}\left(\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}-1\right),$$

which obtain the naturally and simple estimate. Thus the proof is completed. □

Putting $\lambda = 0$, we get the following known result given in [3].

Remark 2.6. Let $\phi(z) = 1 + B_1z + B_2z^2 + ...$, where the coefficients B_n are real and $B_1 > 0$. If f given by (1.1) belongs to the class $\mathcal{C}_q(\phi)$, then

$$|a_2| \le \frac{B_1}{[2]_q([2]_q-1)}, \quad |a_3| \le \frac{B_1}{[3]_q([3]_q-1)} \max\left\{1, \left|\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}\right|\right\},$$

$$\left|a_3 - \frac{[2]_q^2([2]_q-1)^2(\frac{B_1}{[2]_q-1}+\frac{B_2}{B_1}-1)}{B_1[3]_q([3]_q-1)}a_2^2\right| \le \frac{B_1}{[3]_q([3]_q-1)}.$$

We now consider Fekete-Szegö inequality for complex μ.

Theorem 2.7. *Let μ be a non-zero complex number and let f given by (1.1) belongs to the class $\mathcal{CR}_q^\lambda(\phi)$, then for $\lambda > -1$ we have*

$$|a_3 - \mu a_2^2| \le \frac{(2)_q!B_1}{[3]_q([3]_q-1)([\lambda+1]_q)_2} \max\left[1, \left|\frac{B_2}{B_1}+\frac{B_1}{([2]_q-1)}\left\{1-\frac{[3]_q([3]_q-1)([\lambda+1]_q)_2}{[2]_q![2]_q^2([2]_q-1)[\lambda+1]_q^2}\mu\right\}\right|\right]. \tag{2.19}$$

The result is sharp.

Proof. Applying (2.15) and (2.16), we obtain

$$a_3-\mu a_2^2 = \frac{(2)_q!B_1}{2[3]_q([3]_q-1)([\lambda+1]_q)_2}\left[c_2-\frac{c_1^2}{2}+\frac{c_1^2}{2}\left\{\frac{B_2}{B_1}+\frac{B_1}{([2]_q-1)}\left(1-\frac{\mu[3]_q([3]_q-1)([\lambda+1]_q)_2}{(2)_q![2]_q^2([2]_q-1)[\lambda+1]_q^2}\right)\right\}\right].$$

In view of Lemma 1.2, we get

$$\begin{aligned}|a_3-\mu a_2^2| &\le \frac{(2)_q!B_1}{2[3]_q([3]_q-1)([\lambda+1]_q)_2}\left[2-\frac{|c_1|^2}{2}+\frac{|c_1|^2}{2}\left\{\left|\frac{B_2}{B_1}+\frac{B_1}{([2]_q-1)}\left(1-\frac{\mu[3]_q([3]_q-1)([\lambda+1]_q)_2}{(2)_q![2]_q^2([2]_q-1)[\lambda+1]_q^2}\right)\right|\right\}\right] \\ &\le \frac{(2)_q!B_1}{[3]_q([3]_q-1)([\lambda+1]_q)_2}\max\left[1, \left|\frac{B_2}{B_1}+\frac{B_1}{([2]_q-1)}\left(1-\frac{\mu[3]_q([3]_q-1)([\lambda+1]_q)_2}{(2)_q![2]_q^2([2]_q-1)[\lambda+1]_q^2}\right)\right|\right].\end{aligned}$$

This result is sharp for the functions given in (2.17) and (2.18). This completes the proof. □

Setting $\lambda = 0$, we get the following known result given in [3].

Remark 2.8. Let μ be a nonzero complex number and let f given by (1.1) belongs to the class $\mathcal{C}_q(\phi)$, then

$$|a_3-\mu a_2^2| \le \frac{B_1}{[3]_q([3]_q-1)}\max\left\{1, \left|\frac{B_2}{B_1}+\frac{B_1}{[2]_q-1}\left(1-\frac{[3]_q([3]_q-1)}{[2]_q^2([2]_q-1)}\mu\right)\right|\right\}.$$

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