Palestine Journal of Mathematics

Vol. 11(1)(2022) , 549–559

DENUMERABLY MANY POSITIVE RADIAL SOLUTIONS FOR THE ITERATIVE SYSTEM OF ELLIPTIC EQUATIONS IN AN ANNULUS

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Communicated by Martin Bohner

MSC 2010 Classifications: Primary 35J66, 35J60; Secondary 34B18, 47H10.

Keywords and phrases: Nonlinear elliptic system, annulus, positive radial solution, Krasnoselskii's fixed point theorem.

Abstract Sufficient conditions are derived for the existence of denumerably many positive radial solutions to the iterative system of elliptic equations

$$\Delta \mathtt{u}_{\mathtt{j}} + \mathtt{P}(|\mathtt{x}|)\mathtt{g}_{\mathtt{j}}(\mathtt{u}_{\mathtt{j}+1}) = 0,\ \mathtt{R}_1 < |\mathtt{x}| < \mathtt{R}_2,$$
$$\mathtt{u}_{\ell+1} = \mathtt{u}_1,\ \mathtt{j} = 1, 2 \cdots, \ell,$$

$\mathtt{x} \in \mathbb{R}^N$, $N > 2$, subject to a linear mixed boundary conditions at $\mathtt{R}_1$ and $\mathtt{R}_2$, by an application of Krasnoselskii's fixed point theorem.

1 Introduction

The system of nonlinear elliptic equations of the form

$$\left.\begin{aligned} \Delta \mathtt{u}_{\mathtt{j}} + \mathtt{g}_{\mathtt{j}}(\mathtt{u}_{\mathtt{j}+1}) = 0 \ \text{ in } \ \Omega, \\ \mathtt{u}_{\mathtt{j}} = 0 \ \text{ on } \ \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\mathtt{j} \in \{1, 2, 3, \cdots, \ell\}$, $\mathtt{u}_{\ell+1} = \mathtt{u}_1$, and Ω is a bounded domain in $\mathbb{R}^N$, has an important applications in population dynamics, combustion theory and chemical reactor theory. The recent literature for the existence, multiplicity and uniqueness of positive solutions for (1.1), see [5, 3, 6, 9, 10, 11] and references therein.

In [7], Dong and Wei established the existence of radial solutions for the following nonlinear elliptic equations with gradient terms in annular domains,

$$-\Delta \mathtt{u} = \mathtt{g}\Big(|\mathtt{x}|, \mathtt{u}, \frac{\mathtt{x}}{|\mathtt{x}|} \cdot \nabla \mathtt{u}\Big) \ \text{ in } \ \Omega_a^b,$$
$$\mathtt{u} = 0 \ \text{ on } \ \partial\Omega_a^b,$$

by using Schauder's fixed point theorem and contraction mapping theorem. In [15], Padhi, Graef and Kanaujiya considered the following elliptic boundary value problem in an annulus,

$$\Delta \mathtt{u} + \lambda \mathtt{h}(|\mathtt{x}|, \mathtt{u}) = 0 \ \text{ in } \ \Omega,$$
$$\mathtt{u} = 0 \ \text{ on } \ \partial\Omega,$$

and established the existence of positive radial solutions by the revised version of Gustaf-son and Schmitt fixed point theorems. In [12], R. Kajikiya and E. Ko established the existence of positive radial solutions for a semipositone elliptic equation of the form,

$$-\Delta \mathtt{u} = \lambda \mathtt{g}(\mathtt{u}) \ \text{ in } \ \Omega,$$
$$\mathtt{u} = 0 \ \text{ on } \ \partial\Omega,$$

where Ω is a ball or an annulus in $\mathbb{R}^N$. Recently, Son and Wang [16] studied positive radial solutions for nonlinear elliptic systems of the form,

$$\Delta u_j + \lambda K_j(|x|)g_j(u_{j+1}) = 0 \text{ in } \Omega,$$
$$u_j = 0 \text{ on } |x| = r_0,$$
$$u_j \to 0 \text{ as } |x| \to +\infty,$$

where $j \in \{1,2,3,\cdots,\ell\}$, $u_{\ell+1} = u_1$, $\lambda > 0$, $N > 2$, $r_0 > 0$, and Ω is an exterior of a ball and established existence, multiplicity and uniqueness results for various nonlinearities in g_j. Motivated by the aforementioned works, in this paper we establish the existence of denumerably many positive radial solutions of the iterative system of nonlinear elliptic equation in an annulus,

$$\Delta u_j + P(|x|)g_j(u_{j+1}) = 0,\ R_1 < |x| < R_2, \tag{1.2}$$

with one of the following sets of boundary conditions:

$$\left.\begin{aligned} &u_j = 0 \text{ on } |x| = R_1 \text{ and } |x| = R_2,\\ &u_j = 0 \text{ on } |x| = R_1 \text{ and } \frac{\partial u_j}{\partial r} = 0 \text{ on } |x| = R_2,\\ &\frac{\partial u_j}{\partial r} = 0 \text{ on } |x| = R_1 \text{ and } u_j = 0 \text{ on } |x| = R_2, \end{aligned}\right\} \tag{1.3}$$

where $j \in \{1,2,3,\cdots,\ell\}$, $u_{\ell+1} = u_1$, $\Delta u = \mathrm{div}(\nabla u)$, $x \in \mathbb{R}^N$, $N > 2$, $P = \prod_{i=1}^n P_i$, each $P_i : (R_1, R_2) \to (0, +\infty)$ is continuous, $r^{2(N-1)}P$ is integrable, may have singularities, by an application of Krasnoselskii's cone fixed point theorem on a Banach space.

The study of positive radial solutions, writing $r = |x|$, the iterative system (1.2) reduces to the study of positive solutions to the following iterative system of ordinary differential equations,

$$u_j''(r) + \frac{N-1}{r}u_j' + P(r)g_j\big(u_{j+1}(r)\big) = 0,\ R_1 \le r \le R_2. \tag{1.4}$$

By the change of variables $v_j(y) = u_j\big(r(y)\big)$ and the transformation $y = -\int_r^{R_2} \tau^{1-N} d\tau$ turns the system (1.4) into

$$v_j''(y) + r^{2(N-1)}(y)P(r(y))g_j\big(v_{j+1}(y)\big) = 0,\ a < y < 0, \tag{1.5}$$

where $v_1 = v_{\ell+1}$ and $a = -\int_{R_1}^{R_2} \tau^{1-N} d\tau$. Further, it can still transform system (1.5) into

$$\varpi_j''(\tau) + Q(\tau)g_j\big(\varpi_{j+1}(\tau)\big) = 0,\ 0 < \tau < 1, \tag{1.6}$$

where $Q(\tau) = a^2 r^{2(N-1)}\big(a(1-\tau)\big)\prod_{i=1}^n Q_i(\tau)$, $Q_i(\tau) = P_i\big(r(a(1-\tau))\big)$, by the change of variables $\varpi_j(\tau) = v_j(y)$ and $\tau = (a-y)/a$. The detailed explanation of the transformation from the equation (1.4) to (1.6) see [4, 13, 14]. By suitable choices of nonnegative real numbers α, β, γ and δ with $d = \alpha\gamma + \alpha\delta + \beta\gamma > 0$, the set of boundary conditions (1.3) reduces to

$$\left.\begin{aligned} \alpha\varpi_j(0) - \beta\varpi_j'(0) = 0,\\ \gamma\varpi_j(1) + \delta\varpi_j'(1) = 0, \end{aligned}\right\} \tag{1.7}$$

where $j \in \{1,2,3,\cdots,\ell\}$ and $\varpi_1 = \varpi_{\ell+1}$. We note that Q_i may have singularities on $[0,1]$. Thus for each $i \in \{1,2,3,\cdots,n\}$, we assume that the following conditions hold throughout the paper:

($\mathcal{H}_1$) $g_j : [0,+\infty) \to [0,+\infty)$ is continuous.

($\mathcal{H}_2$) $Q_i \in L^{p_i}[0,1]$, $(p_i \ge 1)$ and may have denumerably many singularities on $(0, 1/2)$.

($\mathcal{H}_3$) There exists a sequence $\{\tau_k\}_{k=1}^\infty$ such that $0 < \tau_{k+1} < \tau_k < \frac{1}{2}$, $k \in \mathbb{N}$,

$$\lim_{k\to\infty} \tau_k = \tau^* < \frac{1}{2},\ \lim_{\tau\to\tau_k} Q_i(\tau) = +\infty,\ k \in \mathbb{N},\ i = 1,2,3,\cdots,n$$

and each $Q_i(\tau)$ does not vanish identically on any subinterval of $[0,1]$. Moreover, there exists $Q_i^* > 0$ such that

$$Q_i^* < Q_i(\tau) < \infty \text{ a.e. on } [0,1].$$

The rest of the paper is organized in the following fashion. In Section 2, we convert the boundary value problem (1.6)–(1.7) into equivalent integral equation which involves the kernel. Also, we estimate bounds for the kernel which are useful in our main results. In Section 3, we establish a criteria for the existence of denumerably many positive radial solutions for (1.2) by applying Krasnoselskii's cone fixed point theorem in a Banach space. Finally, as an application, an example to demonstrate our results is given.

2 Kernel and Its Bounds

In this section, we constructed kernel to the homogeneous boundary value problem corresponding to (1.6)–(1.7) and established certain lemmas for the bounds of the kernel.

Lemma 2.1. *Let* $\mathtt{V} \in \mathcal{C}[0,1]$. *Then the boundary value problem*

$$\varpi_1''(\tau) + \mathtt{V}(\tau) = 0,\ 0 < \tau < 1, \tag{2.1}$$

$$\left.\begin{aligned} \alpha\varpi_1(0) - \beta\varpi_1'(0) = 0,\\ \gamma\varpi_1(1) + \delta\varpi_1'(1) = 0, \end{aligned}\right\} \tag{2.2}$$

has a unique solution

$$\varpi_1(\tau) = \int_0^1 \aleph(\tau,s)\mathtt{V}(s)ds, \tag{2.3}$$

where

$$\aleph(\tau,s) = \frac{1}{d}\begin{cases} (\beta+\alpha\tau)(\gamma+\delta-\gamma s), & 0 \le \tau \le s \le 1,\\ (\beta+\alpha s)(\gamma+\delta-\gamma\tau), & 0 \le s \le \tau \le 1. \end{cases}$$

Lemma 2.2. *For* $\mathtt{c} \in (0,1/2)$, *let* $\pi(\mathtt{c}) = \min\left\{\frac{\beta+\alpha\mathtt{c}}{\beta+\alpha}, \frac{\delta+\gamma\mathtt{c}}{\delta+\gamma}\right\}$. *The kernel* $\aleph(\tau,s)$ *has the following properties:*

(i) $\aleph(\tau,s)$ *is nonnegative and continuous on* $[0,1]\times[0,1]$,

(ii) $\aleph(\tau,s) \le \aleph(s,s)$ *for* $t,\tau \in [0,1]$,

(iii) there exists $\mathtt{c} \in (0,1/2)$ *such that* $\pi(\mathtt{c})\aleph(s,s) \le \aleph(\tau,s)$ *for* $\tau \in [\mathtt{c}, 1-\mathtt{c}]$, $s \in [0,1]$.

Proof. From the definition of kernel $\aleph(\tau,s)$, it is clear that (i) and (ii) hold. To prove (iii), let $\tau \in [\mathtt{c}, 1-\mathtt{c}]$ and $s \le \tau$, then

$$\frac{\aleph(\tau,s)}{\aleph(s,s)} = \frac{\gamma+\delta-\gamma\tau}{\gamma+\delta-\gamma s} \ge \frac{\delta+\gamma\mathtt{c}}{\delta+\gamma} \ge \pi(\mathtt{c}),$$

and for $\tau \le s$, we have

$$\frac{\aleph(\tau,s)}{\aleph(s,s)} = \frac{\beta+\alpha\tau}{\beta+\alpha s} \ge \frac{\beta+\alpha\mathtt{c}}{\beta+\alpha} \ge \pi(\mathtt{c}).$$

This completes the proof. □

From Lemma 2.1, we note that an ℓ-tuple $(\varpi_1, \varpi_2, \cdots, \varpi_\ell)$ is solution of the boundary value problem (1.6)–(1.7) if and only, if

$$\varpi_1(\tau) = \int_0^1 \aleph(\tau,s_1)\mathtt{Q}(s_1)\mathtt{g}_1\left[\int_0^1 \aleph(s_1,s_2)\mathtt{Q}(s_2)\mathtt{g}_2\left[\int_0^1 \aleph(s_2,s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots \right.\right.$$
$$\left.\left.\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph(s_{\ell-1},s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\varpi_1(s_\ell)\big)ds_\ell\right]\cdots\right]ds_3\right]ds_2\right]ds_1.$$

In general,

$$\varpi_{\mathtt{j}}(\tau) = \int_0^1 \aleph(\tau, s)\mathtt{Q}(s)\mathtt{g}_{\mathtt{j}}\big(\varpi_{\mathtt{j}+1}(s)\big)ds, \ \mathtt{j} = 1, 2, 3, \cdots, \ell,$$
$$\varpi_1(\tau) = \varpi_{\ell+1}(\tau), \ 0 < \tau < 1.$$

Denote the Banach space $\mathcal{C}([0,1],\mathbb{R})$ by $\mathscr{B}$ with the norm $\|\varpi\| = \max\limits_{\tau\in[0,1]} |\varpi(\tau)|$. For $\mathtt{c} \in (0, 1/2)$, the cone $\mathcal{P}_{\mathtt{c}} \subset \mathscr{B}$ is defined by

$$\mathcal{P}_{\mathtt{c}} = \Big\{\varpi \in \mathscr{B} : \varpi(\tau) \geq 0, \min_{\varpi\in[\mathtt{c},\,1-\mathtt{c}]} \varpi(\tau) \geq \pi(\mathtt{c})\|\varpi\|\Big\}.$$

For any $\varpi_1 \in \mathcal{P}_{\mathtt{c}}$, define an operator $\mathcal{T} : \mathcal{P}_{\mathtt{c}} \to \mathscr{B}$ by

$$(\mathcal{T}\varpi_1)(\tau) = \int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\Bigg[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\Bigg[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4\cdots$$
$$\mathtt{g}_{\ell-1}\Bigg[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\varpi_1(s_\ell)\big)ds_\ell\Bigg]\cdots\Bigg]ds_3\Bigg]ds_2\Bigg]ds_1.$$

Lemma 2.3. *For each* $\mathtt{c} \in (0, 1/2)$, $\mathcal{T}(\mathcal{P}_{\mathtt{c}}) \subset \mathcal{P}_{\mathtt{c}}$ *and* $\mathcal{T} : \mathcal{P}_{\mathtt{c}} \to \mathcal{P}_{\mathtt{c}}$ *is completely continuous.*

Proof. Let $\mathtt{c} \in (0, 1/2)$. Since $\mathtt{g}_{\mathtt{j}}(\varpi_{\mathtt{j}+1}(\tau))$ is nonnegative for $\tau \in [0,1]$, $\varpi_1 \in \mathcal{P}_{\mathtt{c}}$. Since $\aleph(\tau, s)$, is nonnegative for all $\tau, s \in [0,1]$, it follows that $\mathcal{T}(\varpi_1(\tau)) \geq 0$ for all $\tau \in [0,1]$, $\varpi_1 \in \mathcal{P}_{\mathtt{c}}$ Now, by Lemma 2.1 and 2.2, we have

$$\begin{aligned}
&\min_{\tau\in[\mathtt{c},1-\mathtt{c}]}(\mathcal{T}\varpi_1)(\tau)\\
&= \min_{\tau\in[\mathtt{c},1-\mathtt{c}]}\Bigg\{\int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\Bigg[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\Bigg[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4\cdots\\
&\qquad \mathtt{g}_{\ell-1}\Bigg[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\varpi_1(s_\ell)\big)ds_\ell\Bigg]\cdots\Bigg]ds_3\Bigg]ds_2\Bigg]ds_1\Bigg\}\\
&\geq \pi(\mathtt{c})\int_0^1 \aleph(s_1, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\Bigg[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\Bigg[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4\cdots\\
&\qquad \mathtt{g}_{\ell-1}\Bigg[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\varpi_1(s_\ell)\big)ds_\ell\Bigg]\cdots\Bigg]ds_3\Bigg]ds_2\Bigg]ds_1\\
&\geq \pi(\mathtt{c})\Bigg\{\int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\Bigg[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\Bigg[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4\cdots\\
&\qquad \mathtt{g}_{\ell-1}\Bigg[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\varpi_1(s_\ell)\big)ds_\ell\Bigg]\cdots\Bigg]ds_3\Bigg]ds_2\Bigg]ds_1\Bigg\}\\
&\geq \pi(\mathtt{c})\max_{\tau\in[0,1]}|\mathcal{T}\varpi_1(\tau)|.
\end{aligned}$$

Thus $\mathcal{T}(\mathcal{P}_{\mathtt{c}}) \subset \mathcal{P}_{\mathtt{c}}$. Therefore, the operator $\mathcal{T}$ is completely continuous by standard methods and by the Arzela-Ascoli theorem. □

3 Denumerably Many Positive Radial Solutions

In this section, for the existence of denumerably many positive radial solutions of (1.2), we apply the following theorems.

Theorem 3.1. *[8] Let* $\mathcal{E}$ *be a cone in a Banach space* $\mathcal{X}$ *and* Λ_1, Λ_2 *are open sets with* $0 \in \Lambda_1, \overline{\Lambda}_1 \subset \Lambda_2$. *Let* $\mathcal{T} : \mathcal{E} \cap (\overline{\Lambda}_2 \backslash \Lambda_1) \to \mathcal{E}$ *be a completely continuous operator such that* *(a)* $\|\mathcal{T}\mathtt{u}\| \le \|\mathtt{u}\|$, $\mathtt{u} \in \mathcal{E} \cap \partial\Lambda_1$, *and* $\|\mathcal{T}\mathtt{u}\| \ge \|\mathtt{u}\|$, $\mathtt{u} \in \mathcal{E} \cap \partial\Lambda_2$, *or*

(b) $\|\mathcal{T}\mathtt{u}\| \ge \|\mathtt{u}\|$, $\mathtt{u} \in \mathcal{E} \cap \partial\Lambda_1$, *and* $\|\mathcal{T}\mathtt{u}\| \le \|\mathtt{u}\|$, $\mathtt{u} \in \mathcal{E} \cap \partial\Lambda_2$.

Then $\mathcal{T}$ *has a fixed point in* $\mathcal{E} \cap (\overline{\Lambda}_2 \backslash \Lambda_1)$.

Theorem 3.2. *(Hölder's) Let* $f \in L^{\mathtt{p}_i}[0,1]$ *with* $\mathtt{p}_i > 1$, *for* $i = 1, 2, \cdots, n$ *and* $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} = 1$. *Then* $\prod_{i=1}^{n} f_i \in L^1[0,1]$ *and* $\|\prod_{i=1}^{n} f_i\|_1 \le \prod_{i=1}^{n} \|f_i\|_{\mathtt{p}_i}$. *Further, if* $f \in L^1[0,1]$ *and* $g \in L^\infty[0,1]$. *Then* $fg \in L^1[0,1]$ *and* $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$.

Consider the following three possible cases for $\mathtt{P}_\mathtt{j} \in L^{\mathtt{p}_i}[0,1]$:

$$\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} < 1, \ \sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} = 1, \ \sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} > 1.$$

Firstly, we seek denumerably many positive radial solutions for the case $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} < 1$.

Theorem 3.3. *Suppose* $(\mathcal{H}_1) - (\mathcal{H}_3)$ *hold, let* $\{\mathtt{c}_k\}_{k=1}^{\infty}$ *be a sequence with* $\tau_{k+1} < \mathtt{c}_k < \tau_k$. *Let* $\{\mathtt{A}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{B}_k\}_{k=1}^{\infty}$ *be such that*

$$\mathtt{A}_{k+1} < \pi(\mathtt{c}_k)\mathtt{B}_k < \mathtt{S}_k < \eta\mathtt{B}_k < \mathtt{A}_k, \ k \in \mathbb{N},$$

where

$$\eta = \max\left\{ \left[\pi(\mathtt{c}_1)a^2 \prod_{i=1}^{n} \mathtt{Q}_i^* \int_{\mathtt{c}_1}^{1-\mathtt{c}_1} \aleph(s,s)\mathtt{r}^{2(N-1)}\big(a(1-s)\big)ds\right]^{-1}, \ 1\right\}.$$

Further, assume that $\mathtt{g}_\mathtt{j}$ *satisfies*

$(\mathcal{J}_1)$ $\mathtt{g}_\mathtt{j}(\varpi(\tau)) \le \mathtt{N}_1\mathtt{A}_k$ *for all* $\tau \in [0,1]$, $0 \le \varpi \le \mathtt{A}_k$,
where

$$\mathtt{N}_1 < \left[a^2\|\aleph\|_q \prod_{i=1}^{n} \|\mathtt{Q}_i\|_{p_i}\right]^{-1}, \ \ \aleph(s) = \aleph(s,s)\mathtt{r}^{2(N-1)}\big(a(1-s_\ell)\big),$$

$(\mathcal{J}_2)$ $\mathtt{g}_\mathtt{j}(\varpi(\tau)) \ge \eta\mathtt{B}_k$ *for all* $\tau \in [\mathtt{c}_k, 1-\mathtt{c}_k]$, $\pi(\mathtt{c}_k)\mathtt{B}_k \le \varpi \le \mathtt{B}_k$.

The iterative system (1.2) *has denumerably many positive radial solutions* $\{(\varpi_1^{[k]}, \varpi_2^{[k]}, \cdots, \varpi_\ell^{[k]})\}_{k=1}^{\infty}$ *such that* $\varpi_\mathtt{j}^{[k]}(\tau) \ge 0$ *on* $(0,1)$, $\mathtt{j} = 1, 2, \cdots, \ell$ *and* $k \in \mathbb{N}$.

Proof. Consider the sequences $\{\Lambda_{1,k}\}_{k=1}^{\infty}$ and $\{\Lambda_{2,k}\}_{k=1}^{\infty}$ of open subsets of $\mathscr{B}$ defined by

$$\Lambda_{1,k} = \{\varpi \in \mathscr{B} : \|\varpi\| < \mathtt{A}_k\}, \ \Lambda_{2,k} = \{\varpi \in \mathscr{B} : \|\varpi\| < \mathtt{B}_k\}.$$

Let $\{\mathtt{c}_k\}_{k=1}^{\infty}$ be as in the hypothesis and note that $\tau^* < \tau_{k+1} < \mathtt{c}_k < \tau_k < \frac{1}{2}$, for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, define the cone $\mathcal{P}_{\mathtt{c}_k}$ by

$$\mathcal{P}_{\mathtt{c}_k} = \left\{\varpi \in \mathscr{B} : \varpi(\tau) \ge 0 \ \text{ and } \min_{\tau \in [\mathtt{c}_k, \, 1-\mathtt{c}_k]} \varpi(t) \ge \pi(\mathtt{c}_k)\|\varpi\|\right\}.$$

Let $\varpi_1 \in \mathcal{P}_{\mathtt{c}_k} \cap \partial\Lambda_{1,k}$. Then, $\varpi_1(s) \le \mathtt{A}_k = \|\varpi_1\|$ for all $s \in [0,1]$. By $(\mathcal{J}_1)$ and $0 < s_{\ell-1} < 1$, we have

$$\begin{aligned}
\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\varpi_1(s_\ell)\big)ds_\ell &\le \int_0^1 \aleph(s_\ell, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\varpi_1(s_\ell)\big)ds_\ell \\
&\le \mathtt{N}_1\mathtt{A}_k \int_0^1 \aleph(s_\ell, s_\ell)\mathtt{Q}(s_\ell)ds_\ell \\
&\le \mathtt{N}_1\mathtt{A}_k \int_0^1 \aleph(s_\ell, s_\ell)a^2\mathtt{r}^{2(n-1)}\big(a(1-s_\ell)\big)\prod_{i=1}^{n}\mathtt{Q}_i(s_\ell)ds_\ell.
\end{aligned}$$

There exists a $\mathtt{q} > 1$ such that $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} + \frac{1}{\mathtt{q}} = 1$. By the first part of Theorem 3.2, we have

$$\begin{aligned}\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\varpi_1(s_\ell)\big)ds_\ell &\le \mathtt{N}_1\mathtt{A}_k a^2\|\aleph\|_q \prod_{i=1}^{n}\|\mathtt{Q}_i\|_{p_i}\\ &\le \mathtt{A}_k.\end{aligned}$$

It follows in similar manner for $0 < s_{\ell-2} < 1$,

$$\begin{aligned}\int_0^1 \aleph(s_{\ell-2}, s_{\ell-1})\mathtt{Q}(s_{\ell-1})\mathtt{g}_{\ell-1}&\left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\varpi_1(s_\ell)\big)ds_\ell\right]ds_{\ell-1}\\ &\le \int_0^1 \aleph(s_{\ell-1}, s_{\ell-1})\mathtt{Q}(s_{\ell-1})\mathtt{g}_{\ell-1}(\mathtt{R}_k)ds_{\ell-1}\\ &\le \mathtt{M}_1\mathtt{R}_k\int_0^1 \aleph(s_{\ell-1}, s_{\ell-1})\mathtt{Q}(s_{\ell-1})ds_{\ell-1}\\ &\le \mathtt{N}_1\mathtt{A}_k a^2\|\aleph\|_q \prod_{i=1}^{n}\|\mathtt{Q}_i\|_{p_i}\\ &\le \mathtt{A}_k.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}(\Omega\varpi_1)(\tau) = \int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1&\left[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\left[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4\cdots\right.\right.\\ &\left.\left.\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\varpi_1(s_\ell)\big)ds_\ell\right]\cdots\right]ds_3\right]ds_2\Bigg]ds_1\\ \le \mathtt{A}_k.&\end{aligned}$$

Since $\mathtt{A}_k = \|\varpi_1\|$ for $\varpi_1 \in \mathcal{P}_{\beta_k} \cap \partial\Lambda_{1,k}$, we get

$$\|\Omega\varpi_1\| \le \|\varpi_1\|. \tag{3.1}$$

Let $\tau \in [\mathtt{c}_k, 1-\mathtt{c}_k]$. Then, $\mathtt{B}_k = \|\varpi_1\| \ge \varpi_1(t) \ge \min_{\tau\in[\mathtt{c}_k,1-\mathtt{c}_k]} \varpi_1(t) \ge \pi(\mathtt{c}_k)\|\varpi_1\| \ge \mathtt{c}_k\mathtt{B}_k$. By $(\mathcal{J}_2)$ and for $s_{\ell-1} \in [\mathtt{c}_k, 1-\mathtt{c}_k]$, we have

$$\begin{aligned}\int_0^1 &\aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\\ &\ge \int_{\mathtt{c}_k}^{1-\mathtt{c}_k} \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\\ &\ge \eta\mathtt{B}_k\int_{\mathtt{c}_k}^{1-\mathtt{c}_k} \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)ds_\ell\\ &\ge \eta\mathtt{B}_k\pi(\mathtt{c}_1)\int_{\mathtt{c}_1}^{1-\mathtt{c}_1} \aleph(s_\ell, s_\ell)\mathtt{Q}(s_\ell)ds_\ell\\ &\ge \eta\mathtt{B}_k\pi(\mathtt{c}_1)a^2\int_{\mathtt{c}_1}^{1-\mathtt{c}_1} \aleph(s_\ell, s_\ell)\mathtt{r}^{2(n-1)}\big(a(1-s_\ell)\big)\prod_{i=1}^{n}\mathtt{Q}_i(s_\ell)ds_\ell\\ &\ge \eta\mathtt{B}_k\pi(\mathtt{c}_1)a^2\prod_{i=1}^{n}\mathtt{Q}_i^*\int_{\mathtt{c}_1}^{1-\mathtt{c}_1} \aleph(s_\ell, s_\ell)\mathtt{r}^{2(n-1)}\big(a(1-s_\ell)\big)ds_\ell\\ &\ge \mathtt{B}_k.\end{aligned}$$

Continuing with bootstrapping argument, we get

$$(\Omega\varpi_1)(\tau) = \int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\left[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\left[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots \right.\right.$$
$$\left.\left.\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\varpi_1(s_\ell)\big)ds_\ell\right]\cdots\right]ds_3\right]ds_2\Bigg]ds_1$$
$$\geq \mathtt{B}_k.$$

Thus, if $\varpi_1 \in \mathcal{P}_{\mathtt{c}_k} \cap \partial\Lambda_{2,k}$, then

$$\|\Omega\varpi_1\| \geq \|\varpi_1\|. \tag{3.2}$$

It is evident that $0 \in \Lambda_{2,k} \subset \overline{\Lambda}_{2,k} \subset \Lambda_{1,k}$. From (3.1),(3.2), it follows from Theorem 3.1 that the operator Ω has a fixed point $\varpi_1^{[k]} \in \mathcal{P}_{\mathtt{c}_k} \cap \big(\overline{\Lambda}_{1,k} \backslash \Lambda_{2,k}\big)$ such that $\varpi_1^{[k]}(\tau) \geq 0$ on $(0, 1)$, and $k \in \mathbb{N}$. Next setting $\varpi_{\ell+1} = \varpi_1$, we obtain denumerably many positive radius solutions $\{(\varpi_1^{[k]}, \varpi_2^{[k]}, \cdots, \varpi_\ell^{[k]})\}_{k=1}^\infty$ of (1.3) given iteratively by

$$\varpi_{\mathtt{j}}(\tau) = \int_0^1 \aleph(\tau, s)\mathtt{Q}(s)\mathtt{g}_{\mathtt{j}}(\varpi_{\mathtt{j}+1}(s))ds, \ \mathtt{j} = 1, 2, \cdots, \ell - 1, \ell,$$
$$\varpi_{\ell+1}(\tau) = \varpi_1(\tau).$$

The proof is completed. □

For $\sum_{i=1}^n \mathtt{p}_i = 1$, we have the following theorem.

Theorem 3.4. *Suppose* $(\mathcal{H}_1) - (\mathcal{H}_3)$ *hold, let* $\{\mathtt{c}_k\}_{k=1}^\infty$ *be a sequence with* $\tau_{k+1} < \mathtt{c}_k < \tau_k$. *Let* $\{\mathtt{A}_k\}_{k=1}^\infty$ *and* $\{\mathtt{B}_k\}_{k=1}^\infty$ *be such that*

$$\mathtt{A}_{k+1} < \pi(\mathtt{c}_k)\mathtt{B}_k < \mathtt{S}_k < \eta\mathtt{B}_k < \mathtt{A}_k, \ k \in \mathbb{N},$$

Further, assume that $\mathtt{g}_{\mathtt{j}}$ *satisfies* $(\mathcal{J}_2)$ *and*
$(\mathcal{J}_3)$ $\mathtt{g}_\iota(\varpi(\tau)) \leq \mathtt{N}_2\mathtt{A}_k$ *for all* $0 \leq \varpi(\tau) \leq \mathtt{A}_k$, $\tau \in [0, 1]$, *where*

$$\mathtt{N}_2 < \min\left\{\left[a^2\|\aleph\|_\infty \prod_{i=1}^n \|\mathtt{Q}_i\|_{\mathtt{p}_i}\right]^{-1}, \eta\right\}.$$

The iterative system (1.2) *has denumerably many positive radial solutions* $\{(\varpi_1^{[k]}, \varpi_2^{[k]}, \cdots, \varpi_\ell^{[k]})\}_{k=1}^\infty$ *such that* $\varpi_{\mathtt{j}}^{[k]}(\tau) \geq 0$ *on* $(0, 1)$, $\mathtt{j} = 1, 2, \cdots, \ell$ *and* $k \in \mathbb{N}$.

Proof. Let $\Lambda_{1,k}$ be as in the proof of Theorem 3.3 and let $\varpi_1 \in \mathcal{P}_{\mathtt{c}_k} \cap \partial\Lambda_{2,k}$. Again $\varpi_1(\tau) \leq \mathtt{A}_k = \|\varpi_1\|$, for all $\tau_1 \in [0, 1]$. By $(\mathcal{J}_3)$ and $0 < \tau_{\ell-1} < 1$, we have

$$\begin{aligned}\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell & \\ &\leq \int_0^1 \aleph(s_\ell, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell \\ &\leq \mathtt{N}_2\mathtt{A}_k \int_0^1 \aleph(s_\ell, s_\ell)\mathtt{Q}(s_\ell)ds_\ell \\ &\leq \mathtt{N}_2\mathtt{A}_k a^2 \int_0^1 \aleph(s_\ell, s_\ell)\mathtt{r}^{2(n-1)}\big(a(1 - s_\ell)\big)\prod_{i=1}^n \mathtt{Q}_i(s_\ell)ds_\ell \\ &\leq \mathtt{N}_2\mathtt{A}_k a^2\|\aleph\|_\infty \prod_{i=1}^n \|\mathtt{Q}_i\|_{\mathtt{p}_i} \\ &\leq \mathtt{A}_k.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$
\begin{aligned}
(\Omega\varpi_1)(\tau) = \int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\Bigg[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\Bigg[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4\cdots \\
\mathtt{g}_{\ell-1}\Bigg[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\varpi_1(s_\ell)\big)ds_\ell\Bigg]\cdots\Bigg]ds_3\Bigg]ds_2\Bigg]ds_1 \\
\leq \mathtt{A}_k.
\end{aligned}
$$

Thus, $\|\Omega\varpi_1\| \leq \|\varpi_1\|$, for $\varpi_1 \in \mathcal{P}_{\mathtt{c}_k} \cap \partial\Lambda_{1,k}$. Rest of the proof is similar to the proof of Theorem 3.3. Hence, the theorem. □

Finally, we deal with the case $\sum_{i=1}^{n} \mathtt{p}_i > 1$.

Theorem 3.5. *Suppose* $(\mathcal{H}_1) - (\mathcal{H}_3)$ *hold, let* $\{\mathtt{c}_k\}_{k=1}^{\infty}$ *be a sequence with* $\tau_{k+1} < \mathtt{c}_k < \tau_k$. *Let* $\{\mathtt{A}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{B}_k\}_{k=1}^{\infty}$ *be such that*

$$\mathtt{A}_{k+1} < \pi(\mathtt{c}_k)\mathtt{B}_k < \mathtt{S}_k < \eta\mathtt{B}_k < \mathtt{A}_k,\ \ k \in \mathbb{N},$$

Further, assume that $\mathtt{g}_\mathtt{j}$ *satisfies* $(\mathcal{J}_2)$ *and*
$(\mathcal{J}_4)$ $\mathtt{g}_\mathtt{j}(\varpi(\tau)) \leq \mathtt{N}_3\mathtt{A}_k$ *for all* $0 \leq \mathtt{u}(\tau) \leq \mathtt{A}_k$, $\tau \in [0, 1]$, *where*

$$\mathtt{N}_3 < \min\left\{\left[a^2\|\aleph\|_\infty \prod_{i=1}^{n}\|\mathtt{Q}_i\|_1\right]^{-1}, \eta\right\}.$$

The iterative system (1.2) *has denumerably many positive radial solutions* $\{(\varpi_1^{[k]}, \varpi_2^{[k]}, \cdots, \varpi_\ell^{[k]})\}_{k=1}^{\infty}$ *such that* $\varpi_\mathtt{j}^{[k]}(\tau) \geq 0$ *on* $(0, 1)$, $\mathtt{j} = 1, 2, \cdots, \ell$ *and* $k \in \mathbb{N}$.

Proof. The proof of the present theorem is similar to the proofs of Theorem 3.3 and Theorem 3.4. So, we omit details here. □

4 Application

In this section, we provide an example to illustrate the applicability of main results.

Example 4.1. Consider the following fractional order boundary value problem,

$$
\left.
\begin{aligned}
\Delta\mathtt{u}_\mathtt{j} + \mathtt{P}(|\mathtt{x}|)\mathtt{g}_\mathtt{j}(\mathtt{u}_{\mathtt{j}+1}) = 0,\ 1 < |\mathtt{x}| < 2, \\
\mathtt{u}_\mathtt{j} = 0\ \text{ on }\ |\mathtt{x}| = 1 \text{ and } |\mathtt{x}| = 2, \\
\mathtt{u}_\mathtt{j} = 0\ \text{ on }\ |\mathtt{x}| = 1 \text{ and } \frac{\partial\mathtt{u}_\mathtt{j}}{\partial\mathtt{r}} = 0 \text{ on } |\mathtt{x}| = 2, \\
\frac{\partial\mathtt{u}_\mathtt{j}}{\partial\mathtt{r}} = 0\ \text{ on }\ |\mathtt{x}| = 1 \text{ and } \mathtt{u}_\mathtt{j} = 0 \text{ on } |\mathtt{x}| = 2,
\end{aligned}
\right\}
\tag{4.1}
$$

where $\mathtt{j} \in \{1, 2\}$, $\mathtt{u}_3 = \mathtt{u}_1$. Let $N = 3$ and $\alpha = \beta = \gamma = \delta = 1$. Then $d = 3$. Now by simple calculations, we get $a = -\dfrac{1}{2}$ and $r(\tau) = \dfrac{2}{1 - 2\tau}$,

$$\mathtt{Q}(\tau) = \frac{1}{4}\left[\frac{2}{2-\tau}\right]^4 \prod_{i=1}^{2}\mathtt{Q}_i(\tau),\ \ \mathtt{Q}_i(\tau) = \mathtt{P}_i\left(\frac{2}{2-\tau}\right),$$

in which

$$\mathtt{P}_1(t) = \frac{1}{|t-1|} \qquad \text{and} \qquad \mathtt{P}_2(t) = \frac{1}{|t-\frac{1}{2}|},$$

$$\mathtt{g_j}(\varpi) = \begin{cases} 0.1\times 10^{-4}, & \varpi \in (10^{-4}, +\infty), \\ \frac{28\times 10^{-(4k+2)}-0.1\times 10^{-4k-10}}{10^{-(4k+2)}-10^{-4k}}(\mathtt{u}-10^{-4k}) + 0.1\times 10^{-4k-10}, & \varpi \in \left[10^{-(4k+2)}, 10^{-4k}\right], \\ 28\times 10^{-(4k+2)}, & \varpi \in \left(\frac{1}{5}\times 10^{-(4k+2)}, 10^{-(4k+2)}\right), \\ \frac{28\times 10^{-(4k+2)}-0.1\times 10^{-(4k+4)}}{\frac{1}{5}\times 10^{-(4k+2)}-10^{-(4k+4)}}(\varpi - 10^{-(4k+4)}) + 0.1\times 10^{-(4k+4)}, & \varpi \in \left(10^{-(4k+4)}, \frac{1}{5}\times 10^{-(4k+2)}\right], \\ 0, & \varpi = 0, \end{cases}$$

$\mathtt{j} = 1, 2$. Let

$$\tau_k = \frac{31}{64} - \sum_{r=1}^{k} \frac{1}{4(r+1)^4},\ \mathtt{c}_k = \frac{1}{2}(\tau_k + \tau_{k+1}),\ k = 1, 2, 3, \cdots,$$

then

$$\mathtt{c}_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32}$$

and

$$\tau_{k+1} < \mathtt{c}_k < \tau_k,\ \pi(\mathtt{c}_k) = \frac{1+\mathtt{c}_k}{2} > \frac{1}{5}.$$

It is easy to see

$$\tau_1 = \frac{15}{32} < \frac{1}{2},\ \tau_k - \tau_{k+1} = \frac{1}{4(k+2)^4},\ k = 1, 2, 3, \cdots.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, it follows that

$$\tau^* = \lim_{k\to\infty} \tau_k = \frac{31}{64} - \sum_{i=1}^{\infty} \frac{1}{4(i+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} > \frac{1}{5},$$

Also, $\mathtt{P}_1, \mathtt{P}_2 \in L^{\mathtt{p}}[0,1]$, $\prod_{i=1}^{2} \mathtt{Q}_i^* = 2$, and

$$\pi(\mathtt{c}_1)a^2 \prod_{i=1}^{n} \mathtt{Q}_i^* \int_{\mathtt{c}_1}^{1-\mathtt{c}_1} \aleph(s,s)\mathtt{r}^{2(N-1)}\big(a(1-s)\big)ds \approx 0.03636905790.$$

$$\eta = \max\left\{ \left[\pi(\mathtt{c}_1)a^2 \prod_{i=1}^{n} \mathtt{Q}_i^* \int_{\mathtt{c}_1}^{1-\mathtt{c}_1} \aleph(s,s)\mathtt{r}^{2(N-1)}\big(a(1-s)\big)ds\right]^{-1},\ 1\right\} \approx 27.49590057.$$

and let $\mathtt{q} = 2, \mathtt{p}_1 = \mathtt{p}_2 = 1/4$, then $\|\aleph\|_q = 4.230401435$, $\|\mathtt{Q}_1\|_{p_1} = 4.895788358$, $\|\mathtt{Q}_2\|_{p_2} = 1.199795099$, and

$$\mathtt{N}_1 < \left[a^2\|\aleph\|_q \prod_{i=1}^{n} \|\mathtt{Q}_i\|_{p_i}\right]^{-1} \approx 0.1609713891.$$

So, let $\mathtt{N}_1 = 0.15$. In addition if we take

$$\mathtt{A}_k = 10^{-4k},\ \mathtt{B}_k = 10^{-(4k+2)},$$

then

$$\mathtt{A}_{k+1} = 10^{-(4k+4)} < \frac{1}{5} \times 10^{-(4k+2)} < \mathtt{c}_k \mathtt{B}_k$$
$$< \mathtt{B}_k = 10^{-(4k+2)} < \mathtt{A}_k = 10^{-4k},$$

and g_1, g_2 satisfies the following growth conditions:

$$\mathtt{g_j}(\varpi) \leq \mathtt{N}_1 \mathtt{A}_k = 0.15 \times 10^{-4k}, \quad \varpi \in \left[0, 10^{-4k}\right],$$

$$\mathtt{g_j}(\varpi) \geq \eta \mathtt{B}_k = 27.49590057 \times 10^{-(4k+2)}, \quad \varpi \in \left[\frac{1}{5} \times 10^{-(4k+2)}, 10^{-(4k+2)}\right].$$

Then all the conditions of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, the boundary value problem (4.1) has denumerably many positive radial solutions $\{(\varpi_1^{[k]}, \varpi_2^{[k]})\}_{k=1}^{\infty}$ such that $10^{-(4k+2)} \leq \|\varpi_{\mathtt{j}}^{[k]}\| \leq 10^{-4k}$ for each $k = 1, 2, 3, \cdots$, and $\mathtt{j} = 1, 2$.

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Received: August 23, 2020.
Accepted: November 8, 2020.