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# DIAGONALIZATION METHOD AND SEMILINEAR DIFFERENTIAL EQUATIONS ON THE HALF LINE

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**Abstract** This paper deals with some existence of bounded solutions for two classes of semlinear differential equations. We consider the cases when the linear part of the equation generates a $C_0$ semigroup as well as an integrated semigroup. An application is made of a Darbo fixed point theorem associated with the diagonalization method and the concept of measure of noncompactness. The present results initiate the application of such method to semilinar differential equations on the half line.

## 1 Introduction

There has been a significant development in semilinear functional evolution equations in recent years; see the monographs [1, 6, 21, 25, 27], the papers [2, 3, 4, 8, 9], and the references therein. Some global existence results for functional evolution equations and inclusions in the space of continuous and bounded functions are presented in [11, 12]. In [2], an iterative method is used for the existence of mild solutions of evolution equations and inclusions. In the previous papers some restrictions are supposed like the compactness of the semigroup, the Lipschitz conditions on the nonlinear term or the boundedness of the obtained mild solutions.

Many techniques have been developed for studying the existence and uniqueness of solutions of initial and boundary value problem for fractional differential equations. Several authors tried to develop a technique that depends on the Darbo or the Mönch fixed point theorems with the Hausdorff or Kuratowski measure of noncompactness. The notion of the measure of noncompactness was defined in many ways. In 1930, Kuratowski [26] defined the measure of noncompactness, $\alpha(A)$, of a bounded subset $A$ of a metric space $(X, d)$, and in 1955, Darbo [17] introduced a new type of fixed point theorem for set contractions.

In [5], the authors used a generalization of the classical Darbo fixed point theorem combined with the concept of measure of noncompactness in Fréchet spaces to prove some existence of mild solutions for the following evolution problem

$$\begin{cases} u''(t) - A(t)u(t) = f(t, u_t); \;\; \text{if } t \in \mathbb{R}_+ := [0, \infty), \\ u_0 = \Phi \in \mathcal{B}, \;\; u'(0) = \bar{u} \in E, \end{cases} \tag{1.1}$$

where $\mathcal{B}$ is an abstract phase space, $(E, \|\cdot\|)$ is a (real or complex) Banach space, $\{A(t)\}_{t>0}$ is a family of linear closed operators from $E$ into $E$ that generate an evolution system of bounded linear operators $\{U(t, s)\}_{(t,s)\in\Lambda}$; with $\Lambda := \{(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 \le s \le t < +\infty\}$.

In [14], the authors used the diagonalization method to prove some existence of bounded solutions to an initial value problem for fractional differential equations on the half line. In [13], by using the Schauder fixed point theorem combined with the diagonalization process, the authors provide sufficient conditions for the existence of bounded solutions for the following class of Caputo fractional differential equations

$$\begin{cases} {}^C D^\alpha y(t) = f(t, y(t), {}^C D^{\alpha-1} y(t)); & \text{if } t \in \mathbb{R}_+, \ 1 < \alpha \le 2, \\ y(0) = y_0, \ \ y \text{ is bounded on } \mathbb{R}_+. \end{cases} \tag{1.2}$$

In this paper, we discuss the existence of bounded mild solutions for the evolution equation

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)); & \text{if } t \in \mathbb{R}_+, \\ u(0) = u_0 \in E, \ \ u \text{ is bounded on } \mathbb{R}_+, \end{cases} \tag{1.3}$$

where $f : \mathbb{R}_+ \times E \to E$ is a given function, $(E, \|\cdot\|)$ is a (real or complex) Banach space, and $A : D(A) \subset E \to E$ is the infinitesimal generator of a $C_0$-semigroup $T(t), \ t \ge 0$.

Next, we shall be concerned with existence of integral mild solutions for problem (1.3), in the case where $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator on the Banach space $E$.

This paper initiates the application of the diagonalization method to first order semilinear differential equations in case that the linear part of the equation generates a $C_0$-semigroup as well as an integrated semigroup.

## 2 Preliminaries

Let $I := [0, T]; \ T > 0$. A measurable function $u : I \to E$ is Bochner integrable if and only if $\|u\|$ is Lebesgue integrable. For properties of the Bochner integral, see for instance, Yosida [28]. By $B(E)$ we denote the Banach space of all bounded linear operators from $E$ into $E$, with the norm

$$\|N\|_{B(E)} = \sup_{\|u\|=1} \|N(u)\|.$$

As usual, $L^1(I, E)$ denotes the Banach space of measurable functions $u : I \to E$ which are Bochner integrable and normed by

$$\|u\|_{L^1} = \int_0^T \|u(t)\| dt.$$

As usual, by $\mathcal{C} := C(I)$ we denote the Banach space of all continuous functions from $I$ into $E$ with the norm $\|\cdot\|_\infty$ defined by

$$\|u\|_\infty = \sup_{t \in I} \|u(t)\|.$$

Now, we define the Kuratowski and Hausdorf measures of noncompactness and give their basic properties.

**Definition 2.1.** [10] Let $E$ be a Banach space and $\Omega_E$ the bounded subsets of $E$. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty)$ defined by

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and } \operatorname{diam}(B_i) \le \epsilon\}; \text{ here } B \in \Omega_E,$$

where

$$\operatorname{diam}(B_i) = sup\{\|u - v\| : u, v \in B_i\}.$$

The Kuratowski measure of noncompactness satisfies the following properties:

**Lemma 2.2.** *[10, 23] Let $A$ and $B$ be bounded sets.*

*(a)* $\alpha(B) = 0 \Leftrightarrow \overline{B}$ *is compact ($B$ is relatively compact), where $\overline{B}$ denotes the closure of $B$.*

*(b)* $\alpha(B) = \alpha(\overline{B}) = \alpha(convB)$, *where $convB$ is the convex hull of $B$.*

*(c) monotonicity:* $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

*(d) algebraic semi-additivity :* $\alpha(A+B) \leq \alpha(A) + \alpha(B)$, *where*

$$A + B = \{x + y : x \in A, \ \ y \in B\}.$$

*(e) semi-homogencity :* $\alpha(\lambda B) = |\lambda|\alpha(B)$; $\lambda \in \mathbb{R}$. *where* $\lambda(B) = \{\lambda x : x \in B\}$.

*(f) invariance under translations:* $\alpha(B + x_0) = \alpha(B)$ *for any* $x_0 \in E$.

**Lemma 2.3.** *[20] Let* $V \subset C(I, E)$ *be a bounded and equicontinuous set, then*

*(i) the function* $t \to \alpha(V(t))$ *is continuous on* $I$, *and*

$$\alpha_c(V) = \sup_{t \in I} \alpha(V(t)).$$

*(ii)* $\alpha\left(\int_0^T u(s)ds : u \in V\right) \leq \int_0^T \alpha(V(s))ds,$

*where*

$$V(t) = \{u(t) : u \in V\}; \ t \in I.$$

**Lemma 2.4.** *[15] If* $Y$ *is a bounded subset of a Banach space* $X$, *then for each* $\epsilon > 0$, *there is a sequence* $\{y_k\}_{k=1}^{\infty} \subset Y$ *such that*

$$\mu(Y) \leq 2\mu(\{y_k\}_{k=1}^{\infty}) + \epsilon.$$

**Lemma 2.5.** *[24] If* $\{u_k\}_{k=1}^{\infty} \subset L^1(I)$ *is uniformly integrable, then* $\mu(\{u_k\}_{k=1}^{\infty})$ *is measurable and*

$$\mu\left(\left\{\int_0^t u_k(s)ds\right\}_{k=1}^{\infty}\right) \leq 2\int_0^t \mu(\{u_k(s)\}_{k=1}^{\infty})ds.$$

For our purpose we will need the following fixed point theorem.

**Theorem 2.6.** *(Darbo's Fixed Point Theorem) [18, 19] Let* $X$ *be a Banach space and* $C$ *be a bounded, closed, convex and nonempty subset of* $X$. *Suppose a continuous mapping* $N : C \to C$ *is such that for all closed subsets* $D$ *of* $C$,

$$\alpha(T(D)) \leq k\alpha(D), \tag{2.1}$$

*where* $0 \leq k < 1$, *and* $\alpha$ *is the Kuratowski measure of noncompactness. Then* $T$ *has a fixed point in* $C$.

**Remark 2.7.** Mappings satisfying the Darbo-condition (2.1) have subsequently been called k-*set contractions*.

## 3 Bounded Mild Solutions

Let us start by defining what we mean by a bounded mild solution of the problem (1.3).

**Definition 3.1.** By a bounded mild solution of the problem (1.3) we mean a continuous and bounded function satisfying the integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s)\, f(s, u(s))ds.$$

The following hypotheses will be used in the sequel.

$(H_1)$ $A : D(A) \subset E \to E$ is the infinitesimal generator of a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$,

$(H_2)$ The function $t \mapsto f(t,u)$ is measurable for each $u \in E$, and the function $u \mapsto f(t,u)$ is continuous for a.e. $t \in \mathbb{R}_+$,

$(H_3)$ There exists a locally integrable function $p_n : \mathbb{R}_+ \to \mathbb{R}_+$; $n \in \mathbb{N}$ such that

$$\|f(t,u)\| \le p_n(t)(1 + \|u\|), \ for\ a.e.\ t \in I_n := [0,n],\ and\ each\ u \in E,$$

$(H_4)$ For each bounded set $B \subset E$, we have

$$\alpha(f(t,B)) \le p_n(t)\alpha(B),\ a.e.\ t \in I_n.$$

Set

$$M = \sup_{t\in\mathbb{R}_+} \|T(t)\|_{B(E)},\ p_n^* = \int_0^n p_n(t)dt,\ \widetilde{p}_n = \int_0^n e^{-\omega t} p_n(t)dt,\ n \in \mathbb{N}.$$

Now, we shall prove the following theorem concerning the existence of bounded mild solutions of problem (1.3).

**Theorem 3.2.** *Assume that the hypotheses $(H_1) - (H_4)$ hold. If $4Mp_n^* < 1$, then the problem (1.3) has at least one bounded mild solution defined on $\mathbb{R}_+$.*

*Proof.* The proof will be given in two parts. Fix $n \in \mathbb{N}$ and consider the problem

$$\begin{cases} u'(t) = Au(t) + f(t,u(t));\ t \in I_n, \\ u(0) = u_0 \in E. \end{cases} \tag{3.1}$$

**Part1 .** We begin by showing that (3.1) has a solution $u_n \in C(I_n)$ with $\|u_n\|_\infty \le R_n$ where

$$R_n \ge \frac{M\|u_0\| + Mp_n^*}{1 - Mp_n^*}.$$

Consider the operator $N : C(I_n) \to C(I_n)$ defined by:

$$(Nu)(t) = T(t)u_0 + \int_0^t T(t-s)\, f(s,u(s))ds. \tag{3.2}$$

Clearly, the fixed points of the operator $N$ are mild solution of the problem (3.1).
For any $u \in C(I_n)$, and each $t \in I_n$ we have

$$\begin{aligned} \|(Nu)(t)\| &\le M\|u_0\| + M\int_0^t \|f(s,u(s))\|ds \\ &\le M\|u_0\| + M\int_0^t p_n(s)(1 + \|u(s)\|)ds \\ &\le M\|u_0\| + Mp_n^*(1 + R_n). \end{aligned}$$

Thus

$$\|N(u)\|_\infty \le M\|u_0\| + Mp_n^*(1 + R_n) \le R_n. \tag{3.3}$$

This proves that $N$ transforms the ball $B_{R_n} := B(0,R_n) = \{w \in C(I_n) : \|w\|_\infty \le R_n\}$ into itself. We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 2.6. The proof will be given in two steps.

**Step 1.** *$N : B_{R_n} \to B_{R_n}$ is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in $B_{R_n}$. Then, for each $t \in I_n$, we have

$$\|(Nu_n)(t) - (Nu)(t)\| \le \int_0^t T(t-s)\|f(s,u_n(s)) - f(s,u(s))\|ds. \tag{3.4}$$

Since $u_n \to u$ as $n \to \infty$ and $f$ is Carathéodory, then by the Lebesgue dominated convergence theorem, equation (3.4) implies

$$\|N(u_n) - N(u)\|_\infty \to 0 \quad \text{as } n \to \infty.$$

**Step 2.** *For each bounded and equicontinuous subset $D$ of $C(I_n)$, $\mu(N(D)) \le \ell\mu(D)$.* From Lemmas 2.4 and 2.5, for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^\infty \subset D$, such that for all $t \in [0,n]$, we have

$$\begin{aligned} \mu((ND)(t)) &= \mu\left(\left\{T(t)u_0 + \int_0^t T(t-s)\, f(s,u(s))ds;\ u \in D\right\}\right) \\ &\le 2\mu\left(\left\{\int_0^t T(t-s)f(s,u_k(s))ds\right\}_{k=1}^\infty\right) + \epsilon \\ &\le 4\int_0^t \mu\left(\|T(t-s)\|_{B(E)}\{f(s,u_k(s))\}_{k=1}^\infty\right) ds + \epsilon \\ &\le 4M\int_0^t \mu\left(\{f(s,u_k(s))\}_{k=1}^\infty\right) ds + \epsilon \\ &\le 4M\int_0^t p_n(s)\mu\left(\{u_k(s)\}_{k=1}^\infty\right) ds + \epsilon \\ &\le 4Mp_n^*\mu_c(D) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu_c(ND) \le 4Mp_n^*\mu_c(D).$$

As a consequence of these two steps together with Theorem 2.6, we can conclude that $N$ has a fixed point in $u_n \in B_{R_n}$ which is a mild solution of problem (3.1).

**Part 2.** The diagonalization process.
Now, we use the following diagonalization process. For $k \in \mathbb{N}$ let

$$\begin{cases} w_k(t) = u_{n_k}(t);\ t \in [0,n_k], \\ w_k(t) = u_{n_k}(n_k);\ t \in [n_k,\infty). \end{cases}$$

Here $\{n_k\}_{k\in\mathbb{N}^*}$ is a sequence of numbers satisfying

$$0 < n_1 < n_2 < \ldots n_k < \ldots \uparrow \infty.$$

Let $S = \{w_k\}_{k=1}^\infty$. Notice that

$$\|w_{n_k}(t)\|_E \le R_n :\ for\ t \in [0,n_1],\ k \in \mathbb{N}.$$

Also, if $k \in \mathbb{N}$ and $t \in [0,n_1]$, we have

$$w_{n_k}(t) = T(t)u_0 + \int_0^t T(t-s)\, f(s,w_{n_k}(s))ds.$$

Thus, for $k \in \mathbb{N}$ and $t,x \in [0,n_1]$, we have

$$\|w_{n_k}(t) - w_{n_k}(x)\| \le \int_0^{n_1} \|T(t-s) - T(x-s))\|_{B(E)}\|f(s,w_{n_k}(s))\|ds.$$

Hence

$$\|w_{n_k}(t) - w_{n_k}(x)\| \le p_1^* \int_0^{n_1} \|T(t-s) - T(x-s))\|_{B(E)}ds.$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence $\mathbb{N}_1^*$ of $\mathbb{N}$ and a function $z_1 \in C([0,n_1])$ with $u_{n_k} \to z_1$ as $k \to \infty$ in $C([0,n_1])$ through $\mathbb{N}_1^*$. Let $\mathbb{N}_1 = \mathbb{N}_1^* - \{1\}$.
Notice that

$$|w_{n_k}(t)| \le R_n :\ for\ t \in [0,n_2],\ k \in \mathbb{N}.$$

Also, if $k \in \mathbb{N}$ and $t, x \in [0, n_2]$, we have

$$\|w_{n_k}(t) - w_{n_k}(x)\| \leq p_2^* \int_0^{n_2} \|T(t-s) - T(x-s))\|_{B(E)} ds.$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence $\mathbb{N}_2^*$ of $\mathbb{N}_1$ and a function $z_2 \in C([0, n_2])$ with $u_{n_k} \to z_2$ as $k \to \infty$ in $C([0, n_2])$ through $\mathbb{N}_2^*$. Note that $z_1 = z_2$ on $[0, n_1]$ since $\mathbb{N}_2^* \subset \mathbb{N}_1$. Let $\mathbb{N}_2 = \mathbb{N}_2^* - \{2\}$. Proceed inductively to obtain for $m = 3, 4, \ldots$ a subsequence $\mathbb{N}_m^*$ of $\mathbb{N}_{m-1}$ and a function $z_m \in C([0, n_m])$ with $u_{n_k} \to z_m$ as $k \to \infty$ in $C([0, n_m])$ through $\mathbb{N}_m^*$. Let $\mathbb{N}_m = \mathbb{N}_m^* - \{m\}$.

Fix $t \in (0, \infty)$ and let $m \in \mathbb{N}$ with $t \leq n_m$. Define a function $u$ by $u(t) = z_m(t)$. Then $u \in C((0, \infty)),\ u(0) = u_0$ and $\|u(t)\| \leq R_n :\ for\ t \in [0, \infty)$.
Again fix $t \in (0, \infty)$ and let $m \in \mathbb{N}$ with $t \leq n_m$. Then for $n \in \mathbb{N}_m$ we have

$$u_{n_k}(t) = T(t)u_0 + \int_0^{n_m} T(t-s)\ f(s, u_{n_k}(s)) ds.$$

Let $n_k \to \infty$ through $\mathbb{N}_m$ to obtain

$$z_m(t) = T(t)u_0 + \int_0^{n_m} T(t-s)\ f(s, z_m(s)) ds.$$

We can use this method for each $t \in [0, n_m]$ and for each $m \in \mathbb{N}$. Thus

$$u'(t) = Au(t) + f(t, u(t));\ for\ t \in [0, n_m];$$

for each $m \in \mathbb{N}$ and the constructed function $u$ is a mild solution of problem (1.3). □

## 4 Bounded Integral Solutions

In this section, we present the main results for the existence of integral solutions for problem (1.3), in the case where $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator on the Banach space $E$.

**Definition 4.1.** ([7]). An integrated semigroup is a family of operators $(S(t))_{t \geq 0}$ of bounded linear operators $S(t)$ on $E$ with the following properties:

(i) $S(0) = 0$;

(ii) $t \to S(t)$ is strongly continuous;

(iii) $S(s)S(t) = \int_0^s (S(t+r) - S(r)) dr$; for all $t, s \geq 0$.

**Definition 4.2.** [22]. An operator $A$ is called a generator of an integrated semigroup if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ ($\rho(A)$, is the resolvent set of $A$) and there exists a strongly continuous exponentially bounded family $(S(t))_{t \geq 0}$ of bounded operators such that $S(0) = 0$ and $R(\lambda, A) := (\lambda I - A)^{-1} = \lambda \int_0^{\infty} e^{-\lambda t} S(t) dt$ exists for all $\lambda$ with $\lambda > \omega$.

**Definition 4.3.** We say that $u(\cdot) : \mathbb{R}_+ \to E$ is an integral solution of problem (1.3) if

(i) $u(t) = u_0 + A \int_0^t u(s) ds + \int_0^t f(s, u(s)) ds$; for each $t \in \mathbb{R}_+$;

(ii) $\int_0^t u(s) ds \in D(A)$; for each $t \in \mathbb{R}_+$.

From the above definition it follows that $u(t) \in \overline{D(A)}$, for each $t \in \mathbb{R}_+$, in particular $u_0 \in \overline{D(A)}$. Moreover, $u(\cdot)$ satisfies the following variation of constants formula:

$$u(t) = S'(t)u_0 + \frac{d}{dt} \int_0^t S(t-s) f(s, u(s)) ds;\ t \in \mathbb{R}_+. \tag{4.1}$$

Notice that, if $u(\cdot)$ satisfies (4.1), then

$$u(t) = S'(t)u_0 + \lim_{\lambda\to\infty} \int_0^t S'(t-s)B_\lambda f(s,u(s))ds;\ t \in \mathbb{R}_+.$$

Let us introduce the following hypothesis:

$(H_5)$ $A$ satisfies Hille-Yosida condition.

Let $B(t) = \{v(t) \in \overline{D(A)} : v \in B\};\ t \geq 0.$

**Theorem 4.4.** *Assume that the hypotheses* $(H_2) - (H_5)$ *are satisfied, and* $4L\widetilde{p}_n e^{\omega n} < 1$ *for each* $n \in \mathbb{N}$. *Then the problem (1.3) has at least one bounded integral solution defined on* $\mathbb{R}_+$.

*Proof.* The proof will be given in two parts. Fix $n \in \mathbb{N}$ and consider the problem (3.1).

**Part 1.** We begin by showing that (3.1) has a solution $u_n \in C(I_n)$ with $\|u_n\|_\infty \leq \rho_n$ where

$$\rho_n \geq \frac{Le^{\omega n}\|u_0\| + L\widetilde{p}_n e^{\omega n}}{1 - L\widetilde{p}_n e^{\omega n}}.$$

Consider the operator $G : C(I_n) \to C(I_n)$ defined by:

$$(Gu)(t) = u(t) = S'(t)u_0 + \frac{d}{dt}\int_0^t S(t-s)f(s,u(s))ds. \tag{4.2}$$

For any $u \in B_{\rho_n} := B(0,\rho_n) = \{w \in C(I_n) : \|w\|_\infty \leq \rho_n\}$, and each $t \in [0,n]$ we have

$$\begin{aligned}
\|(Gu)(t)\| &\leq \|S'(t)u_0 + \frac{d}{dt}\int_0^t S(t-s)f(s,u(s))ds\| \\
&\leq Le^{\omega n}\|u_0\| + Le^{\omega n}\left(\int_0^t e^{-\omega s}p_n(s)(1+\|u(s)\|)ds\right) \\
&\leq Le^{\omega n}\|u_0\| + Le^{\omega n}(1+\rho_n)\left(\int_0^n e^{-\omega s}p_n(s)ds\right) \\
&\leq Le^{\omega n}\|u_0\| + Le^{\omega n}\widetilde{p}_n(1+\rho_n) \\
&\leq \rho_n.
\end{aligned}$$

Thus

$$\|G(u)\|_\infty \leq \rho_n.$$

This proves that $G$ transforms the ball $B_{\rho_n}$ into itself. We shall show that the operator $G : B_{\rho_n} \to B_{\rho_n}$ satisfies all the assumptions of Theorem 2.6. The proof will be given in two steps.

**Step 1.** $G : B_{\rho_n} \to B_{\rho_n}$ *is continuous.*
Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence such that $u_k \to u$ in $B_{\rho_n}(w)$. Then, for each $t \in [0,n]$ and $w \in \Omega$, we have

$$\begin{aligned}
\|(Gu_k)(t) - (Gu)(t)\| &\leq \frac{d}{dt}\int_0^t \|S(t-s)\|_{B(E)}\|f(s,u_k(s)) - f(s,u(s))\|ds \\
&\leq Le^{\omega n}\int_0^t e^{-\omega s}\|f(s,u_k(s)) - f(s,u(s))\|ds.
\end{aligned}$$

Since $u_k \to u$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|G(u_k) - G(u)\|_n \to 0 \quad \text{as } k \to \infty.$$

Hence, we can conclude that $G : B_{\rho_n} \to B_{\rho_n}$ is a continuous.

**Step 2.** *For each bounded and equicontinuous set $B$ of $B_{\rho_n}$, $\mu(G(B)) \leq \widetilde{\ell}\mu(B)$.*
From Lemmas 2.4 and 2.5, for any $B \subset B_{\rho_n}$ and any $\epsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^{\infty} \subset B$, such that for all $t \in [0, n]$, we have

$$\begin{aligned}
\mu((GB)(t)) &= \mu\left(\left\{S'(t)u_0 + \frac{d}{dt}\int_0^t S(t-s)f(s,u(s))ds;\ u \in B\right\}\right)\\
&\leq 2\mu\left(\left\{\frac{d}{dt}\int_0^t S(t-s)f(s,u(s))ds\right\}_{k=1}^{\infty}\right) + \epsilon\\
&\leq 4Le^{\omega n}\int_0^t \mu\left(\left\{e^{-\omega s}f(s,u_k(s))\right\}_{k=1}^{\infty}\right) ds + \epsilon\\
&\leq 4Le^{\omega n}\int_0^t \mu\left(\{e^{-\omega s}f(s,u_k(s))\}_{k=1}^{\infty}\right) ds + \epsilon\\
&\leq 4Le^{\omega n}\int_0^t e^{-\omega s}p_n(s)\mu\left(\{u_k(s)\}_{k=1}^{\infty}\right) ds + \epsilon\\
&\leq 4Le^{\omega n}\widetilde{p}_n\int_0^t \mu\left(\{u_k(s)\}_{k=1}^{\infty}\right) ds + \epsilon\\
&\leq 4L\widetilde{p}_n e^{\omega n}\mu_c(B) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu_c((GB)(t)) \leq 4L\widetilde{p}_n e^{\omega n}\mu_c(B).$$

Hence, we can conclude that $G$ has at least one fixed point in $B_{\rho_n}$ which is an integral mild solution of problem (3.1).

**Part 2.** The diagonalization process.
Now, we use the following diagonalization process. For $k \in \mathbb{N}$ let

$$\begin{cases} w_k(t) = u_{n_k}(t);\ t \in [0, n_k],\\ w_k(t) = u_{n_k}(n_k);\ t \in [n_k, \infty). \end{cases}$$

Here $\{n_k\}_{k\in\mathbb{N}^*}$ is a sequence of numbers satisfying

$$0 < n_1 < n_2 < \ldots n_k < \ldots \uparrow \infty.$$

Let $S = \{w_k\}_{k=1}^{\infty}$. Notice that

$$\|w_{n_k}(t)\| \leq \rho_n :\ for\ t \in [0, n_1],\ k \in \mathbb{N}.$$

Also, if $k \in \mathbb{N}$ and $t \in [0, n_1]$, we have

$$w_{n_k}(t) = S'(t)u_0 + \frac{d}{dt}\int_0^t S(t-s)f(s, w_{n_k}(s))ds.$$

Thus, for $k \in \mathbb{N}$ and $t, x \in [0, n_1]$, we have

$$\begin{aligned}
\|w_{n_k}(t) - w_{n_k}(x)\| \leq\ & \|S'(t) - S'(s))\|_{B(E)}\|u_0\|\\
& + \frac{d}{dt}\int_0^{n_1} \|S(t-s) - S(x-s))\|_{B(E)}\|f(s, w_{n_k}(s))\|ds.
\end{aligned}$$

Hence

$$\begin{aligned}
\|w_{n_k}(t) - w_{n_k}(x)\| \leq\ & \|S'(t) - S'(s))\|_{B(E)}\|u_0\|\\
& + p_1^*\int_0^{n_1} \|S(t-s) - S(x-s))\|_{B(E)}ds.
\end{aligned}$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence $\mathbb{N}_1^*$ of $\mathbb{N}$ and a function $z_1 \in C([0,n_1])$ with $u_{n_k} \to z_1$ as $k \to \infty$ in $C([0,n_1])$ through $\mathbb{N}_1^*$. Let $\mathbb{N}_1 = \mathbb{N}_1^* - \{1\}$.
Notice that

$$|w_{n_k}(t)| \leq \rho_n : \ for\ t \in [0,n_2],\ k \in \mathbb{N}.$$

Also, if $k \in \mathbb{N}$ and $t, x \in [0,n_2]$, we have

$$\begin{aligned} \|w_{n_k}(t) - w_{n_k}(x)\| &\leq \|S'(t) - S'(s))\|_{B(E)} \|u_0\| \\ &\quad + p_2^* \int_0^{n_2} \|S(t-s) - S(x-s))\|_{B(E)} ds. \end{aligned}$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence $\mathbb{N}_2^*$ of $\mathbb{N}_1$ and a function $z_2 \in C([0,n_2])$ with $u_{n_k} \to z_2$ as $k \to \infty$ in $C([0,n_2])$ through $\mathbb{N}_2^*$. Note that $z_1 = z_2$ on $[0,n_1]$ since $\mathbb{N}_2^* \subset \mathbb{N}_1$. Let $\mathbb{N}_2 = \mathbb{N}_2^* - \{2\}$. Proceed inductively to obtain for $m = 3, 4, \ldots$ a subsequence $\mathbb{N}_m^*$ of $\mathbb{N}_{m-1}$ and a function $z_m \in C([0,n_m])$ with $u_{n_k} \to z_m$ as $k \to \infty$ in $C([0,n_m])$ through $\mathbb{N}_m^*$. Let $\mathbb{N}_m = \mathbb{N}_m^* - \{m\}$.

Define a function $y$ as follows. Fix $t \in (0,\infty)$ and let $m \in \mathbb{N}$ with $t \leq n_m$. Then define $u(t) = z_m(t)$. Then $u \in C((0,\infty)),\ u(0) = u_0$ and $\|u(t)\|_E < R_n : \ for\ t \in [0,\infty)$.
Again fix $t \in (0,\infty)$ and let $m \in \mathbb{N}$ with $t \leq n_m$. Then for $n \in \mathbb{N}_m$ we have

$$u_{n_k}(t) = S'(t)u_0 + \frac{d}{dt}\int_0^{n_m} S(t-s) f(s, u_{n_k}(s)) ds.$$

Let $n_k \to \infty$ through $\mathbb{N}_m$ to obtain

$$z_m(t) = S'(t)u_0 + \frac{d}{dt}\int_0^{n_m} S(t-s) f(s, z_m(s)) ds.$$

We can use this method for each $x \in [0,n_m]$ and for each $m \in \mathbb{N}$. Thus

$$u'(t) = Au(t) + f(t,u(t));\ for\ t \in [0,n_m];$$

for each $m \in \mathbb{N}$ and the constructed function $u$ is an integral solution of problem (1.3). □

## 5 Examples

**Example 1.** For a given a function $u \in L^2([0,\pi],\mathbb{R})$, we consider the following partial functional semilinear problem

$$\frac{\partial}{\partial t} z(t,x) = \frac{\partial^2}{\partial x^2} z(t,x) + Q(t, z(t,x));\ x \in [0,\pi],\ t \in \mathbb{R}_+, \tag{5.1}$$

$$z(t,0) = z(t,\pi) = 0;\ t \in \mathbb{R}_+, \tag{5.2}$$

$$z(0,x) = \phi(x);\ x \in [0,\pi],\ u \text{ is bounded on } \mathbb{R}_+, \tag{5.3}$$

where $\phi : [0,\pi] \to \mathbb{R},\ \ and\ \ Q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are the functions given by

$$\phi(x) = 1 + e^x,$$

and

$$Q(t,z) = \frac{e^{-t}}{1+z^2}(1+z).$$

Let

$$u(t)(x) = z(t,x);\ t \in \mathbb{R}_+,\ x \in [0,\pi],$$

$$f(t)(x) = Q(t, z(t,x));\ t \in \mathbb{R}_+,\ x \in [0,\pi],$$

$$u(0)(x) = \phi(x);\ x \in [0,\pi].$$

Take $E = L^2[0,\pi]$ and define $A : D(A) \subset E \to E$ by $Aw = w''$ with domain

$$D(A) = \{w \in E, w, w' \quad \text{are absolutely continuous, } w'' \in E, w(0) = w(\pi) = 0\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n,\ w \in D(A)$$

where $(\cdot,\cdot)$ is the inner product in $L^2$ and $w_n(s) = \sqrt{\frac{2}{\pi}}\, sin\, ns,\ n = 1,2,\dots$ is the orthogonal set of eigenvectors in $A$. It is well known (see [25]) that $A$ is the infinitesimal generator of an analytic semigroup $T(t),\ t \geq 0$ in $E$ and is given by

$$T(t)w = \sum_{n=1}^{\infty} exp(-n^2 t)(w, w_n) w_n,\ w \in E.$$

Since the analytic semigroup $T(t)$ is compact, there exists a constant $M \geq 1$ such that

$$\|T(t)\|_{B(E)} \leq M.$$

For any $n \in \mathbb{N}$ we have

$$|Q(t,z)| \leq e^{nt}(1 + |z|).$$

This means that for the locally integrable function $p_n : \mathbb{R}_+ \to \mathbb{R}_+$, with $p_n(t) = e^{nt}$, we have

$$\|f(t,u)\| \leq p_n(t)(1 + \|u\|),\ for\ a.e.\ t \in [0,n].$$

We can show that problem (1.3) is an abstract formulation of problem (5.1)-(5.3) . Since all the conditions of Theorem 2.6 are satisfied, the problem (5.1)-(5.3) has a bounded mild solution $z$ on $\mathbb{R}_+ \times [0,\pi]$.

**Example 2.** Consider now the following partial functional semilinear problem

$$\frac{\partial}{\partial t} z(t,x) = \frac{\partial^2}{\partial x^2} z(t,x) + Q(t, z(t,x));\ x \in [0,\pi],\ t \in \mathbb{R}_+, \tag{5.4}$$

$$z(t,0) = z(t,\pi) = 0;\ t \in \mathbb{R}_+, \tag{5.5}$$

$$z(0,x) = \phi(x);\ x \in [0,\pi],\ \ u \text{ is bounded on } \mathbb{R}_+, \tag{5.6}$$

where $\phi : [0,\pi] \to \mathbb{R},\ \ and\ \ Q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are the functions given by

$$\phi(x) = 1 + x^2,$$

and

$$Q(t,z) = \frac{e^{-t}}{2 + t^2}(1 + |z|).$$

Let

$$u(t)(x) = z(t,x);\ t \in \mathbb{R}_+,\ x \in [0,\pi],$$
$$f(t)(x) = Q(t, z(t,x));\ t \in \mathbb{R}_+,\ x \in [0,\pi],$$
$$u(0)(x) = \phi(x);\ x \in [0,\pi].$$

Take $E = C(\overline{\Omega})$, the Banach space of continuous function on $\overline{\Omega}$ with values in $\mathbb{R}$. Define the linear operator $A$ on $E$ by

$$Az = \frac{\partial^2}{\partial x^2} z, \quad \text{in} \quad D(A) = \{z \in C(\overline{\Omega}) : z = 0 \text{ on } \partial\Omega,\ \frac{\partial^2}{\partial x^2} z \in C(\overline{\Omega}\}.$$

Now, we have

$$\overline{D(A)} = C_0(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\} \neq C(\overline{\Omega}).$$

It is well known from [16] that $A$ is sectorial, $(0,+\infty) \subseteq \rho(A)$ and for $\lambda > 0$

$$\|R(\lambda, A)\|_{B(E)} \leq \frac{1}{\lambda}.$$

It follows that $A$ generates an integrated semigroup $(S(t))_{t\geq 0}$ and that

$$\|S'(t)\|_{B(E)} \leq e^{-\mu t};$$

for $t \geq 0$ and some constant $\mu > 0$, and $A$ satisfies the Hille-Yosida condition.

For any $n \in \mathbb{N}$, we have

$$|Q(t,z)| \leq e^{nt}(1+|z|).$$

This means that for the locally integrable function $p_n : \mathbb{R}_+ \to \mathbb{R}_+$, with $p_n(t) = e^{nt}$, we have

$$\|f(t,u)\| \leq p_n(t)(1+\|u\|),\ for\ a.e.\ t \in [0,n].$$

We can show that problem (1.3) is an abstract formulation of problem (5.4)-(5.6). Since all the conditions of Theorem 4.4 are satisfied, the problem (5.4)-(5.6) has an integral bounded solution $z$ defined on $\mathbb{R}_+ \times [0,\pi]$.

## References


[1] S. Abbas and M. Benchohra, *Advanced Functional Evolution Equations and Inclusions*, Developments in Mathematics, **39**, Springer, Cham, 2015.

[2] S. Abbas, W. Albarakati and M. Benchohra, Successive approximations for functional evolution equations and inclusions, *J. Nonlinear Funct. Anal.*, Vol. 2017 (2017), Article ID 39, pp. 1-13.

[3] S. Abbas, A. Arara, M. Benchohra and F. Mesri, Random evolution equations in Fréchet spaces, *Adv. Theory Nonlinear Anal. Appl.* **2** (3) (2018), 128-137.

[4] S. Abbas, M. Benchohra and F. Mesri, Second order functional evolution equations with infinite delay in Fréchet spaces, *Commun. Appl. Nonlinear Anal.* **25** (3) (2018), 1-12.

[5] A. Arara M. Benchohra, N. Hamidi and J.J. Nieto, Fractional order differential equations on an unbounded domain, *Nonlinear Analysis* **72** (2010) 580-586.

[6] N. U. Ahmed, *Semigroup Theory with Applications to Systems and Control*, Pitman Research Notes in Mathematics Series, 246. Longman Scientific & Technical, Harlow; John Wiley & Sons, New York, 1991.

[7] W. Arendt, Vector valued Laplace transforms and Cauchy problems, *Israel J. Math.* **59** (1987), 327-352.

[8] S. Baghli and M. Benchohra, Global uniqueness results for partial functional and neutral functional evolution equations with infinite delay, *Differential Integral Equations*, **23** (2010), 31–50.

[9] S. Baghli and M. Benchohra, Multivalued evolution equations with infinite delay in Fréchet spaces, *Electron. J. Qual. Theo. Differ. Equ.* 2008, No. 33, 24 pp.

[10] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[11] A. Baliki and M. Benchohra, Global existence and asymptotic behaviour for functional evolution equations, *J. Appl. Anal. Comput.* **4** (2) (2014), 129–138.

[12] A. Baliki and M. Benchohra, Global existence and stability for neutral functional evolution equations, *Rev. Roumaine Math. Pures Appl.* **LX** (1) (2015), 71-82.

[13] M. Benchohra, F. Berhoun and G.M. N'Guérékata, Bounded solutions for fractional order differential equations on the half line, *Bulletin Math. Anal. Appl.* **4** (1) (2012), 62-71.

[14] M. Benchohra, J. Henderson and F. Ouaar, Bounded solutions to an initial value problem for fractional differential equations on the half line, *PanAmer. Math. J.* **21** (2) (2011), 35-44.

[15] D. Bothe, Multivalued perturbation of m-accretive differential inclusions, *Isr. J. Math.* **108** (1998), 109-138.

[16] G. Da Prato and E. Sinestrari, Differential operators with non-dense domains, *Ann. Scuola. Norm. Sup. Pisa Sci.* **14** (1987), 285-344.

[17] G. Darbo, Punti uniti in transformazioni a condominio non compatto, *Rend Sem. Mat. Univ. Padova* **24** (1955), 84-92.

[18] K. Goebel, *Concise Course on Fixed Point Theorems*, Yokohama Publishers, Japan, 2002.

[19] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[20] D.J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces,* Kluwer Academic Publishers, Dordrecht, 1996.

[21] S. Heikkila and V. Lakshmikantham, *Monotone Iterative Technique for Nonlinear Discontinuous Differential Equations*, Marcel Dekker Inc., New York, 1994.

[22] H. Kellermann and M. Hieber, Integrated semigroup, *J. Funct. Anal*. **84** (1989), 160-180.

[23] W.A. Kirk and B. Sims, *Handbook of Metric Fixed Point Theory*, Springer-Science + Business Media, B.V, Dordrecht, 2001.

[24] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4** (1980), 985-999.

[25] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[26] K. Kuratowski, Sur les espaces complets, *Fund. Math.* **15** (1930), 301-309.

[27] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Applied Mathematical Sciences 119, Springer-Verlag, New York, 1996.

[28] K. Yosida, *Functional Analysis*, $6^{th}$ edn. Springer-Verlag, Berlin, 1980.

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