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Generating Primitive Triples Using a Matrix Upon Pythagorean Triples

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Abstract This paper aims to generate primitive triples using the matrix of components obtained upon the sides of right-angled triangles (Pythagorean triples), where sequence of primitive integers p, q, and h represents the right-angled triangle sides difference, sum, and hypotenuse, respectively and the sides of this triangle odd leg u, even leg e and the hypotenuse h are defined by a pair of positive integer indices (i, j) where i is an odd number, and j is an even number.

1 Introduction

The mathematical concept of the article is the primitive triples. That triple consists of three positive integers (a, b, c) such that a, b and c are coprime, that to say, they have no common divisor greater than one.

If these integers satisfy the relation $a^2 + b^2 = c^2$ then they represent lengths of right-angled triangle sides, such triangle is called the Pythagorean triangle, and these triples are called primitive Pythagorean triples, noticing that there exist Pythagorean triples which are not primitive.

The study of Pythagorean triples is a long history and a way to obtain primitive Pythagorean triples go back to Euclid, he obtained Pythagorean triples from pairs of positive integers (m, n) and primitive Pythagorean triples from pairs of co-prime positive integers (m, n), one being even. The idea of Bredenkamp [2] is to generate Pythagorean triples from pairs of positive integers (i, j) by other formula than that of Euclid's formula. We can find also in the literature that with three specific given 3 by 3 matrices and the primitive Pythagorean triple $(3, 4, 5)$ one obtains all the primitive Pythagorean triples. The use of matrices is at the core of many studies on Pythagorean triples.

So Bredenkamp [3] obtained them from the idea of indices. Where i is an odd number, and j is an even number, such that these indices (i, j) define the sides of every positive integer right-angled triangle as follows: an odd leg $u = i^2 + ij$, an even leg $e = ij + j^2/2$, and the hypotenuse $h = i^2 + ij + j^2/2$. Also, in [2] Bredenkamp found sequence of triangles that approach a right-angled triangle, i.e., that has one irrational side such as the triangle which allows for the creation of sequence of fractions that have as their limit an irrational number such $\sqrt{2}$ and $\sqrt{3}$.

Let (e, u, h) be a Pythagorean triple. Our aim is to study the triple (p, q, h) associated to (e, u, h) where $p = e - u$ and $q = e + u$ and this in particular cases given by Bredenkamp. To begin we have :

Lemma 1.1. *Let (e, u, h) be a primitive Pythagorean triple with e even and u odd. Set $p = e - u$ and $q = e + u$, then the triple (p, q, h) of integer numbers is primitive, with p and q odd and $p^2 + q^2 = 2h^2$.*

Proof. From the dentition, one has $p^2+q^2 = (e-u)^2+(u+v)^2 = u^2-2uv+v^2+u^2+2uv+v^2 = 2u^2 + 2v^2 = 2h^2$. Since e is even and u is odd, one sees that $p = e - u$ and $q = e + u$ are odd If (p, q, h) is not primitive, there exists a prime number ζ that is a common divisor of p, q and h.

Since p and q are odd, one has $\zeta \geq 3$. But $e = (p+q)/2$, and $u = (p-q)/2$, hence ζ is an odd divisor of $2e$ and $2u$, then it divides e and u. Therefore ζ is a common divisor of the primitive triple (e, u, h). A contradiction. □

Notice that, conversely giving primitive triples (p, q, h) of integer numbers such $p^2+q^2 = 2h^2$ with p and q are odd. Setting $\vartheta = (p+q)/2$,and $\varpi = (p-q)/2$, one obtains a primitive triple of integer numbers (ϑ, ϖ, h) such that $\vartheta^2+\varpi^2 = h^2$. Taking absolute value, one obtains a primitive Pythagorean triple $(|\vartheta|, |\varpi|, h)$. Notice also that h is odd with $h \leq q \leq [\sqrt{2}h]$, and $|p| < q$.

Referring to Bredenkamp [2], it is possible to find first a sequence of right-angled triangles defined on the subset of triangles triangles $(Tr(i, j))i, j = \{(i, j)|u = i^2+ij, e = j^2/2+ij, h^2 = u^2 + e^2, |e - u| = 1, j > i,$ where $(i + 1)/2$ and $j/2 \in \mathbb{N}\}$ obtained recursively from the set of pairs (i, j), starting an initial element $(i, j)_1 = (i_1,\ j_1)$ and for $(i, j)_n = (i_n,\ j_n)$ one sets $(i, j)_{n+1} = (i_n +\ j_n,\ 2i_n +\ j_n)$ that approximates the 45° right triangle, and second to find a sequence of right-angled triangles defined on the subset of triangles $(Tr(i, j))i, j = \{(i, j)|u = i^2+ij, e = j^2/2+ij, h^2 = u^2+e^2, |h-2u| = 1, j > i,$ where $(i+1)/2$ and $j/2 \in \mathbb{N}\}$ obtained recursively from the set of pairs (i, j) starting with an initial element $(i, j)_1 = (i_1,\ j_1)$ and for $(i, j)_n = (i_n,\ j_n)$ one sets $(i, j)_n :\ (i, j)_{n+1} = (i_n +\ j_n,\ 2i_n +\ 3j_n)$ this approximates the 30°/60° right triangle.

2 The 45° Triangle Case

Let us remind that according to Bredenkamp [1], one can obtain a sequence of triangles that approximates the 45° right triangle. This sequence is obtained recursively from a sequence of pairs of positive integers $(i, j)_n$ such that:
For the sequence $(i, j)_n$:

$$(i, j)_{n+1} = (i_n +\ j_n,\ 2i_n +\ j_n) \tag{2.1}$$

and

$$u_{n+1} = 2h_n +\ e_n +\ u_n \tag{2.2}$$
$$e_{n+1} = 2h_n +\ 2u_n \tag{2.3}$$
$$h_{n+1} = 3h_n +\ 2u_n \tag{2.4}$$

In this paper we define the primitive triples (q, p, h) by:

$$q = e + u \tag{2.5}$$
$$p = e - u \tag{2.6}$$
$$h = \sqrt{e^2 + u^2} \tag{2.7}$$

and in terms of the indices (i, j) defined by:

$$q = i^2 + j^2/2 + 2ij \tag{2.8}$$
$$p = j^2/2 - i^2 \tag{2.9}$$
$$h = i^2 + ij + j^2/2 \tag{2.10}$$

In the next steps we are going to find the next member $(q, p, h)_{n+1}$ in terms of $(i, j)_n$ and then in terms of $(q, p, h)_n, \forall\, n \in \mathbb{N}$:

$$e = \frac{1}{2}(q+p) \tag{2.11}$$

$$\text{and} \quad u = \frac{1}{2}(q-p) \tag{2.12}$$

So, $u_{n+1} = \frac{1}{2}(q_{n+1} - p_{n+1}) = 2h_n + 2(\frac{1}{2}(q_n + p_n)) + \frac{1}{2}(q_n - p_n)$ by (2.2), (2.11) and (2.12), hence

$$q_{n+1} - p_{n+1} = 4h_n + 3q_n + p_n \tag{2.13}$$

Also, $e_{n+1} = \frac{1}{2}(q_{n+1} + p_{n+1}) = 2h_n + \frac{1}{2}(q_n + p_n) + 2(\frac{1}{2}(q_n - p_n))$ by (2.3), (2.11) and (2.12), hence

$$q_{n+1} + p_{n+1} = 4h_n + 3q_n - p_n \tag{2.14}$$

and by (2.4), (2.11) and (2.12) we get:

$$h_{n+1} = 3h_n + 2(\frac{1}{2}(q_n + p_n)) + 2(\frac{1}{2}(q_n - p_n)) = 3h_n + 2q_n \tag{2.15}$$

By adding and subtracting the two equations (2.13) and (2.14) we get the following two formulas respectively:

$$q_{n+1} = 4h_n + 3q_n \tag{2.16}$$

$$p_{n+1} = -p_n \tag{2.17}$$

Rearrange the three formulas (2.17), (2.15) and (2.16) respectively taking into consideration the coefficients of p_n and q_n in these formulas, we get the system:

$$\begin{aligned} p_{n+1} &= -p_n + 0h_n + 0q_n \\ h_{n+1} &= 0p_n + 3h_n + 2q_n \\ q_{n+1} &= 0p_n + 4h_n + 3q_n \end{aligned}$$

Reformulate this system using a coefficient matrix $C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 4 & 3 \end{pmatrix}$ by the formula:

$$C(\, p_n \quad h_n \quad q_n)^T = (\, p_{n+1} \quad h_{n+1} \quad q_{n+1})^T$$

or

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} p_n \\ h_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n+1} \\ h_{n+1} \\ q_{n+1} \end{pmatrix} \tag{2.18}$$

which gives the forward iteration to generate the primitive triples in ascending order, retaining the order of (p, h, q) as ordered triples in the matrix multiplication formula (2.18) and using the $(p, h, q) = (1, 5, 7)$ triplet as the first member of the sequences obtained from the first ordered pair component numbers $(i, j) = (1, 2)$. To get the second member $(-1, 29, 41)$ in the sequence using the matrix multiplication:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 29 \\ 41 \end{pmatrix}$$

Applying the matrix equation to the second member produces the third triplet $(1, 239, 169)$:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 29 \\ 14 \end{pmatrix} = \begin{pmatrix} 1 \\ 169 \\ 239 \end{pmatrix}$$

Therefore, the third member of the sequence may be obtained by the square of the 3×3 matrix:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 4 & 3 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 17 & 12 \\ 0 & 24 & 17 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 169 \\ 239 \end{pmatrix}$$

So, the general formula for the equation to produce the triple n in the sequence is:

$$\begin{pmatrix} p_n \\ h_n \\ q_n \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 4 & 3 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \quad \forall n \geq 1 \tag{2.19}$$

Note that, if we rearrange the triples in the sequence in the order (p, q, h), then the general formula for the equation to get the member n is given by:

$$\begin{pmatrix} p_n & q_n & h_n \end{pmatrix} = \begin{pmatrix} 1 & 5 & 7 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 4 & 3 \end{pmatrix}^{n-1}, \quad \forall n \geq 1 \tag{2.20}$$

Table1 below shows the first ten forward iterations of the above matrix equation formula (2.19) that gives the sequence of the primitive triples $(\ p_n,\ h_n,\ q_n)$ values for the 45° triangle, this table was obtained by using Maple 18 algorithm.

Proposition 2.1. *Running the following algorithm [4] can generates infinitely many triples such as* $(\ p_n,\ q_n,\ h_n)$*, and* $p_{n+1} = -\ p_n,\ h_{n+1} =\ 3h_n +\ 2q_n,$ *and* $q_{n+1} =\ 4h_n +\ 3q_n$*, by using a Maple software ver18, we list some tables below (where the pair* $(\ h_n,\ q_n)$ *is coprime we can prove it by contradiction by assuming* $gcd(\ h_n,\ q_n) = d$*).*

```
> restart :
with(LinearAlgebra) :
> PM := proc(A :: Matrix, k)
option remember;
local i, j, m, r, q, n, d, f, P, F, C;
P := x− > CharacteristicPolynomial(A, x);
n := degree(P(x), x);
d := ldegree(P(x), x);
F := (i, j)− > rsolve(sum(coeff(P(x), x, m) ∗ f(m + q), m = 0..n) = 0, seq(f(r) = (A^r)[i, j], r = d + 1
q− > Matrix(n, n, F);
if type(k, integer) then return(simplify(A^k)) elif (Determinant(A) = 0
and not type(k, numeric)) then printf("The %a − th power of the matrix for %a >= %d :
", k, k, n)elif(Determinant(A) = 0andtype(k, numeric))thenreturn(simplify(A^k))fi; return(simplify
k, C(q)))) ; end.
```

Now since the 3×3 coefficient matrix C is nonsingular with nonzero determinant; det$C = -1$; then it is invertible, so that its inverse is:

$C^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -4 & 3 \end{pmatrix}$ which is also nonsingular with nonzero determinant; det$C^{-1} = -1$;

Table 1. $The\, forward\, sequence\, of\, the\, primitive\, triples\, (\ p_n,\ h_n,\ q_n)\, for$ 45° $triangle.$

n	p_n	h_n	q_n
gray!20 1	1	5	7
2	-1	29	41
gray!20 3	1	169	239
4	-1	985	1393
gray!20 5	1	5741	8119
6	-1	33461	47321
gray!20 7	1	195025	275807
8	-1	1136689	1607521
gray!20 9	1	6625109	9369319
10	-1	38613965	54608393

so using the matrix C^{-1} with any primitive triplet $(\ p_n,\ h_n,\ q_n)$ with order n will give the previous triplet $(\ p_{n-1},\ h_{n-1},\ q_{n-1})$ with order $n-1$ in a process called the backward iteration to generate the primitive triples in descending order, retaining the order of (p, h, q) as ordered triples in the matrix multiplication formula (2.21) below:

$$C^{-1}\begin{pmatrix} p_n & h_n & q_n \end{pmatrix}^T = \begin{pmatrix} p_{n-1} & h_{n-1} & q_{n-1} \end{pmatrix}^T$$

or

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -4 & 3 \end{pmatrix}\begin{pmatrix} p_n \\ h_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n-1} \\ h_{n-1} \\ q_{n-1} \end{pmatrix}, \quad \forall n \geq 2. \tag{2.21}$$

and using the $(p, h, q) = \ (p, h, q)_n = (\ p_n,\ h_n,\ q_n)$ to be triplet of order n as the $n-th$ member of the sequence, then we can generate the triplet of order r (the $r-th$ member of the sequence) where $r \leq n$ by using the matrix equation formula:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -4 & 3 \end{pmatrix}^{n-r}\begin{pmatrix} p_n \\ h_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_r \\ h_r \\ q_r \end{pmatrix}, \quad \forall\, n \geq 2 \text{ and } \forall\, r \leq n. \tag{2.22}$$

Table2 below shows the first ten iterations of the above matrix formula (2.21) or (2.22) that gives the sequence of the primitive triples $(\ p_n,\ h_n,\ q_n)$ values for the 45° triangle in a backward order starting continue with from the $10th$ member of the sequence $(-1, 38613965, 54608393)$.

Table 2. $The\, backward\, sequence\, of\, the\, primitive\, triples\, (\ p_n,\ h_n,\ q_n)\, for$ 45° $triangle.$

n	p_n	h_n	q_n
gray!20 10	-1	38613965	54608393
9	1	6625109	9369319
gray!20 8	-1	1136689	1607521
7	1	195025	275807
gray!20 6	-1	33461	47321
5	1	5741	8119
gray!20 4	-1	985	1393
3	1	169	239
gray!20 2	-1	29	41
1	1	5	7

Remark 2.2. Observe that the matrix C is a diagonal block matrix with entries the 1×1 matrix -1 and the 2×2 matrix S such that det$C$ = -1 and det$S = 1$. With this one can recovers the fact that each triple $(\ p_n,\ h_n,\ q_n)$ is primitive and the pair $(\ h_n,\ q_n)$ coprime.

3 The $30^\circ/60^\circ$ Triangle Case

In similar manner we'll get the formula to generate the sequence of the primitive triples for $30^\circ/60^\circ$ triangle. According to Bredenkamp [2], the subset of triangle describes the sequence of the $30^\circ/60^\circ$ triangle is $(Tr(i;j))i,j$ such that: $(Tr(i,j))i,j = \{(i,j)|u = i^2 + ij, e = j^2/2 + ij, h^2 = u^2 + e^2, |e-u| = 1, j > i,$ where $(i+1)/2$ and $j/2 \in \mathbb{N}\}$ on which the triangle has the component numbers u, e, h, i and j are arranged as the sequence progress to give the next member by the formulas:
For the sequence $(i,j)_n$:

$$(i,j)_{n+1} = (\ i_n +\ j_n,\ 2i_n +\ 3j_n) \tag{3.1}$$

and

$$u_{n+1} =\ 4h_n +\ 4e_n -\ u_n \tag{3.2}$$

$$e_{n+1} =\ 8h_n +\ 7e_n -\ 4u_n \tag{3.3}$$

$$h_{n+1} =\ 9h_n +\ 8e_n -\ 4u_n \tag{3.4}$$

Here also we define the primitive triples (q,p,h) by:

$$q = e + u \tag{3.5}$$

$$p = e - u \tag{3.6}$$

$$h^2 = e^2 + u^2 \tag{3.7}$$

Now to find the next member $(q,p,h)_{n+1}$ in terms of $(q,p,h)_n$ $\forall n \in \mathbb{N}$:
From (3.5) and (3.6) we get:

$$e = \frac{1}{2}(q+p) \tag{3.8}$$

and

$$u = \frac{1}{2}(q-p) \tag{3.9}$$

In the same manner;
$u_{n+1} = \frac{1}{2}(\ q_{n+1} -\ p_{n+1}) =\ 4h_n + 4(\frac{1}{2}(\ q_n +\ p_n)) - \frac{1}{2}(\ q_n -\ p_n)$ by (3.2), (3.8) and (3.9) hence

$$q_{n+1} -\ p_{n+1} =\ 8h_n +\ 5p_n +\ 3q_n \tag{3.10}$$

Also $e_{n+1} = \frac{1}{2}(\ q_{n+1} +\ p_{n+1}) =\ 8h_n + 7(\frac{1}{2}(\ q_n +\ p_n)) - 4(\frac{1}{2}(\ q_n -\ p_n))$ by (3.3), (3.8) and (3.9), so

$$q_{n+1} +\ p_{n+1} =\ 16h_n +\ 11p_n +\ 3q_n \tag{3.11}$$

and by (3.4), (3.8) and (3.9) we get:

$$h_{n+1} =\ 9p_n + 8(\frac{1}{2}(\ q_n +\ p_n)) - 4(\frac{1}{2}(\ q_n -\ p_n)) =\ 9h_n +\ 6p_n +\ 2q_n \tag{3.12}$$

By adding and subtracting the two equations (3.10) and (3.11) we get the following two formulas respectively:

$$q_{n+1} = 12h_n + 3q_n + 8p_n \tag{3.13}$$

$$p_{n+1} = 4h_n - 3p_n \tag{3.14}$$

Rearrange the three formulas (3.14), (3.12) and (3.13) respectively taking into consideration the coefficients of in these formulas, we get the system

$$\begin{aligned} p_{n+1} &= 3p_n + 4h_n + 0q_n \\ h_{n+1} &= 6p_n + 9h_n + 2q_n \\ q_{n+1} &= 8p_n + 12h_n + 3q_n \end{aligned}$$

Reformulate this system using a 3×3 coefficient matrix $D = \begin{pmatrix} 3 & 4 & 0 \\ 6 & 9 & 2 \\ 8 & 12 & 3 \end{pmatrix}$ by the formula:

$$D(\, p_n \quad h_n \quad q_n)^T = (\, p_{n+1} \quad h_{n+1} \quad q_{n+1})^T$$

or

$$\begin{pmatrix} 3 & 4 & 0 \\ 6 & 9 & 2 \\ 8 & 12 & 3 \end{pmatrix} \begin{pmatrix} p_n \\ h_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n+1} \\ h_{n+1} \\ q_{n+1} \end{pmatrix}, \tag{3.15}$$

retaining the order of (p, h, q) as ordered triples in the matrix multiplication and using the $(p, h, q) = (1, 5, 7)$ triplet as the first member of the sequence obtained from the first ordered pair component numbers $(i, j) = (1, 2)$. To get the second member $(23, 65, 89)$ in the sequence using the matrix multiplication:

$$\begin{pmatrix} 3 & 4 & 0 \\ 6 & 9 & 2 \\ 8 & 12 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 23 \\ 65 \\ 89 \end{pmatrix}$$

Applying the matrix equation to the second member produces the third triplet $(329, 901, 1231)$:

$$\begin{pmatrix} 3 & 4 & 0 \\ 6 & 9 & 2 \\ 8 & 12 & 3 \end{pmatrix} \begin{pmatrix} 23 \\ 65 \\ 89 \end{pmatrix} = \begin{pmatrix} 329 \\ 901 \\ 1231 \end{pmatrix}$$

Therefore, the third member of the sequence may be obtained by the square of the 3×3 matrix:

$$\begin{pmatrix} 3 & 4 & 0 \\ 6 & 9 & 2 \\ 8 & 12 & 3 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 33 & 48 & 8 \\ 88 & 129 & 24 \\ 120 & 176 & 33 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 329 \\ 901 \\ 1231 \end{pmatrix}$$

So, the general formula for the equation to produce the $n - th$ triple in the sequence is:

$$\begin{pmatrix} p_n \\ h_n \\ q_n \end{pmatrix} = \begin{pmatrix} 3 & 4 & 0 \\ 6 & 9 & 2 \\ 8 & 12 & 3 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \, \forall \, n \geq 1 \tag{3.16}$$

Also, if we rearrange the triples in the sequence in the order (p, q, h) , then the general formula for the equation to get the member n is given by:

$$\begin{pmatrix} p_n & q_n & h_n \end{pmatrix} = \begin{pmatrix} 1 & 7 & 5 \end{pmatrix} \begin{pmatrix} 3 & 4 & 0 \\ 6 & 9 & 2 \\ 8 & 12 & 3 \end{pmatrix}^{n-1}, \forall\, n \geq 1 \tag{3.17}$$

Table3 below shows the first ten iterations of the above matrix formula (3.16) that gives the sequence of the primitive triples $(\,p_n,\ h_n,\ q_n)$ values for the $30^\circ/60^\circ$ triangle.

Proposition 3.1. *Giving the recurrent linear system of the sequence of triples* (p_n, h_n, q_n) *and the fact that this recurrent linear system permits to recover the fact that each* (p_n, h_n, q_n) *is primitive.*

Table 3. $The\ forward\ sequence\ of\ the\ primitive\ triples\ (\,p_n,\ h_n,\ q_n)\ for$ $30^\circ/60^\circ$ $triangle.$

n	p_n	h_n	q_n
gray!20 1	1	5	7
2	23	65	89
gray!20 3	329	901	1231
4	4591	12545	17137
gray!20 5	63953	174725	238679
6	890759	2433601	3324361
gray!20 7	12406681	33895685	46302367
8	172802783	472105985	644908769
gray!20 9	2406832289	6575588101	8982420391
10	33522849271	91586127425	125108976697

Now since the 3×3 coefficient matrix D is nonsingular with nonzero determinant; det$D = 1$; then it is invertible, so that its inverse is:

$D^{-1} = \begin{pmatrix} 3 & -12 & 8 \\ -2 & 9 & -6 \\ 0 & -4 & 3 \end{pmatrix}$ which is also nonsingular with nonzero determinant det$D^{-1} = 1$,

so the using of the matrix D^{-1} with any primitive triplet $(\,p_n,\ h_n,\ q_n)$ with order n will give the previous triplet $(\,p_{n-1},\ h_{n-1},\ q_{n-1})$ with order $n-1$ in a process called the backward iteration to generate the primitive triples in descending order, retaining the order of (p, h, q) as ordered triples in the matrix multiplication formula (3.18) below:

$$D^{-1} \begin{pmatrix} p_n & h_n & q_n \end{pmatrix}^T = \begin{pmatrix} p_{n-1} & h_{n-1} & q_{n-1} \end{pmatrix}^T$$

or

$$\begin{pmatrix} 3 & -12 & 8 \\ -2 & 9 & -6 \\ 0 & -4 & 3 \end{pmatrix} \begin{pmatrix} p_n \\ h_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n-1} \\ h_{n-1} \\ q_{n-1} \end{pmatrix}, \quad \forall\, n \geq 2. \tag{3.18}$$

and using the $(p, h, q) = (p, h, q)_n = (p_n, h_n, q_n)$ to be triplet of order n as the $n - th$ member of the sequence, then we can generate the triplet of order r (the $r - th$ member of the sequence) where $r \leq n$ by using the matrix equation formula:

$$\begin{pmatrix} 3 & -12 & 8 \\ -2 & 9 & -6 \\ 0 & -4 & 3 \end{pmatrix}^{n-r} \begin{pmatrix} p_n \\ h_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_r \\ h_r \\ q_r \end{pmatrix}, \quad \forall\, n \geq 2 \text{ and } \forall\, r \leq n. \tag{3.19}$$

Table4 below shows the first ten iterations of the above matrix equation formula (3.18) or (3.19) that gives the sequence of the primitive triples (p_n, h_n, q_n) values for the $30^\circ/60^\circ$ triangle in a backward order starting from the 10th member of the sequence:
$(33522849271, 91586127425, 125108976697)$.

Table 4. $The\ backward\ sequence\ of\ the\ primitive\ triples\ (\ p_n,\ h_n,\ q_n)\ for$ $30^\circ/60^\circ$ $triangle.$

n	p_n	h_n	q_n
gray!20 10	33522849271	915861127425	125108976697
9	2406832289	6575588101	8982420391
gray!20 8	172802783	472105985	644908769
7	12406681	33895685	46302367
gray!20 6	890759	2433601	3324361
5	63953	174725	238679
gray!20 4	4591	12545	17137
3	329	901	1231
gray!20 2	23	65	89
1	1	5	7

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