Palestine Journal of Mathematics

Vol. 11(1)(2022) , 583–593

RICCI SOLITONS ON PARA-SASAKIAN MANIFOLDS SATISFYING PSEUDO-SYMMETRY CURVATURE CONDITIONS

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Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53C15, 53C25; Secondary 53C40.

Keywords and phrases: Para-Sasakian manifold, W_3-curvature tensor, pseudo-symmetric manifold, Ricci pseudo-symmetric manifold, Ricci generalized pseudo-symmetric manifold, Einstein manifold, Ricci soliton.

Abstract The paper deals with the study of Ricci solitons on para-Sasakian manifolds satisfying pseudo-symmetry curvature conditions. First, we investigate Ricci solitons in Ricci-pseudosymmetric para-Sasakian manifolds. Next, we consider Ricci solitons in W_3-Ricci-pseudo-symmetric para-Sasakian manifolds. Moreover, we investigate Ricci solitons in Ricci generalized pseudo-symmetric para-Sasakian manifold. Finally, we prove that Ricci solitons in para-Sasakian manifolds satisfying the curvature condition $Q \cdot R = 0$, is expanding and an example is given to verify the theorem.

1 Introduction

The concept of Ricci solitons was introduced by Hamilton [8]. They are natural generalizations of Einstein metrics, which have been a significant subject of intense study in differential geometry and geometric analysis. Ricci solitons also correspond to special solutions of Hamilton's Ricci flow [7] and often arise as limits of dilations of singularities in the Ricci flow. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one-parameter group of diffeomorphism and scaling. A Ricci soliton is a generalization of an Einstein metric and is defined on a Riemannian manifold (M, g). A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V is a potential vector field and λ a real scalar so that the following equation is satisfied:

$$L_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where L_V is the Lie derivative along the vector field V, S is the Ricci tensor of M. A Ricci soliton is said to be shrinking, steady or expanding according to λ negative, zero and positive, respectively. During the last two decades, the geometry of Ricci solitons has become a subject of growing interest for many mathematicians. The study of the Ricci solitons in contact geometry has begun with the work of Sharma [17], Nagaraja et al. [13] and others extensively studied Ricci solitons in contact metric manifolds. For details we refer to [1, 2, 3, 5, 9, 14, 21].

In 1976, Sato [16] introduced the notion of almost paracontact structure (ϕ, ξ, λ) on a differentiable manifold. This structure is an analogue of the almost contact structure. An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be of even dimension as well. Takahashi [18] defined almost contact manifolds (in particular, Sasakian manifolds) equipped with an associated pseudo-Riemannian metric.

In 1985, Kaneyuki and Williams [10] defined the notion of almost paracontact structure on a pseudo-Riemannian manifold of dimension $(2n + 1)$. Later, Zamkovoy [22] showed that any almost paracontact structure admits a pseudo-Riemannian metric with signature $(n + 1, n)$.

The paper is organized as follows: In Section 2, we recall some basic formulas on para Sasakian manifolds and we give some basic definitions of pseudo-symmetry curvature conditions and notions used in this study. In Section 3, we give a brief account on Ricci solitons on para-Sasakian manifolds. Next, in Section 4, we consider a Ricci soliton in Ricci-pseudo-symmetric para-Sasakian manifold and prove that the Ricci soliton is expanding provided $L_S \neq -1$. Section 5 deals with a Ricci soliton in W_3-Ricci-pseudo-symmetric para-Sasakian manifold and prove that the Ricci soliton is expanding provided $f \neq -2$. We discuss a Ricci soliton in W_3-pseudo-symmetric para-Sasakian manifold and prove that the Ricci soliton is expanding provided $L_{W_3} \neq -1$ in Section 6. In the next Section, we study Ricci solitons in Ricci generalized pseudo-symmetric para-Sasakian manifold, it is shown that the Ricci soliton is expanding provided $nL_R \neq 1$. Finally, we have pointed out that Ricci solitons in para-Sasakian manifolds satisfying the curvature condition $Q \cdot R = 0$, is expanding and we give an example of a Ricci soliton on a 5-dimensional para-Sasakian manifold to verify some results.

2 Preliminaries

An n-dimensional differentiable manifold M is called almost paracontact manifold with the almost paracontact structure (ϕ, ξ, η) consisting of a $(1,1)$-tensor field ϕ, a vector field ξ and an 1-form η satisfying the following conditions [10]:

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

where I denote the identity transformation. If an n-dimensional almost paracontact manifold M with an almost paracontact structure (ϕ, ξ, η) admits a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

then we say that M is an almost paracontact metric manifold with an almost paracontact metric structure (ϕ, ξ, η, g) and such a metric g is called compatible metric [22]. From (2.2), it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X). \tag{2.4}$$

The fundamental 2-form Φ of an almost paracontact structure (ϕ, ξ, η, g) is defined by

$$\Phi(X, Y) = g(X, \phi Y),$$

for all tangent vector fields X, Y. If $d\eta = \Phi$, then the manifold $M(\phi, \xi, \eta, g)$ is called a paracontact metric manifold associated to the metric g, where $X, Y, Z \in TM^n$; TM is the set of all differentiable vector fields on M. Here the paracontact metric structure is normal and the structure is called para-Sasakian [22]. Equivalently, a paracontact metric structure (ϕ, ξ, η, g) is para-Sasakian if

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.5}$$

for any $X, Y \in TM^n$, where ∇ is Levi–Civita connection of g. From the above equation, it follows that

$$\nabla_X \xi = -\phi X. \tag{2.6}$$

In an n-dimensional para-Sasakian manifold, the following relations hold:

$$R(X, Y, Z, W) = g(X, Z)g(Y, W) - g(Y, Z)g(X, W), \tag{2.7}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.8}$$

$$R(\xi.X)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.9}$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.10}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.11}$$

$$R(\xi,Y)\xi = Y - \eta(Y)\xi, \tag{2.12}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.13}$$

$$S(\xi,\xi) = -(n-1), \tag{2.14}$$

for any vector fields $X, Y, Z, W \in TM^n$, where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$. We define endomorphisms $R(X,Y)$ and $X \wedge_A Y$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.15}$$

and

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y, \tag{2.16}$$

respectively [6], where $X, Y, Z \in TM^n$, A is the symmetric $(0,2)$-tensor, R is the Riemannian curvature tensor of type $(1,3)$ and ∇ is the Levi-Civita connection. For a $(0,k)$-tensor field T, $k \geqslant 1$; on (M^n, g), we define the tensors $R \cdot T$ and $Q(g,T)$ by

$$\begin{aligned} & (R(X,Y)\cdot T)(X_1, X_2, X_3, ..., X_k) \\ = \; & -T(R(X,Y)X_1, X_2, X_3, ..., X_k) \\ & -T(X_1, R(X,Y)X_2, X_3, ..., X_k) \\ & -...... - T(X_1, X_2, X_3, ..., R(X,Y)X_k), \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} & Q(g,T)(X_1, X_2, X_3, ..., X_k; X, Y) \\ = \; & -T((X \wedge_g Y)X_1, X_2, X_3, ..., X_k) \\ & -T(X_1, (X \wedge_g Y)X_2, X_3, ..., X_k) \\ & -...... - T(X_1, X_2, X_3, ..., (X \wedge_g Y)X_k), \end{aligned} \tag{2.18}$$

respectively [20].

In 1973 Pokhariyal [15] introduced the notion of a new curvature tensor, denoted by W_3 and studied its relativistic significance. The W_3-curvature tensor of type $(1,3)$ on a para-Sasakian manifold is defined by:

$$W_3(X,Y)Z = R(X,Y)Z + \frac{1}{(n-1)}[g(Y,Z)QX - S(X,Z)Y], \tag{2.19}$$

Then in a para-Sasakian manifold, W_3 satisfies the following relations:

$$W_3(\xi,Y)Z = 2[\eta(Z)Y - g(Y,Z)\xi], \tag{2.20}$$

$$W_3(\xi,Y)\xi = 2[Y - \eta(Y)\xi], \tag{2.21}$$

$$W_3(\xi,\xi)Z = 0. \tag{2.22}$$

Definition 2.1. A para-Sasakian manifold (M^n, g) is said to be Ricci-pseudo-symmetric if the tensors $R \cdot S$ and $Q(g,S)$ are linearly dependent. This is equivalent to

$$R \cdot S = L_S Q(g,S), \tag{2.23}$$

holding on the set $U_S = \{x \in M : S \neq 0 \text{ at } x\}$, where L_S is some function on U_S. [20]

Definition 2.2. A para-Sasakian manifold (M^n, g) is said to be W_3-Ricci-pseudo-symmetric if the tensors $W_3 \cdot S$ and $Q(g, S)$ are linearly dependent. This is equivalent to

$$W_3 \cdot S = fQ(g, S), \tag{2.24}$$

holding on the set $U_S = \{x \in M : S \neq 0 \text{ at } x\}$, where f is some function on U_S.

Definition 2.3. A para-Sasakian manifold (M^n, g) is said to be Ricci generalized pseudo-symmetric if the tensors $R \cdot R$ and $Q(S, R)$ are linearly dependent. This is equivalent to

$$R \cdot R = L_R Q(S, R), \tag{2.25}$$

holding on the set $U_R = \{x \in M : R \neq 0 \text{ at } x\}$, where L_R is some function on U_R. [20]

A very important subclass of this class of manifolds realizing the condition is

$$R \cdot R = Q(S, R).$$

Every three dimensional manifold satisfies the above equation identically. Other examples are the semi-Riemannian manifolds (M, g) admitting a non-zero 1-form ω such that the equality $\omega(X)R(Y,Z)+\omega(Y)R(Z,X)+\omega(Z)R(X,Y) = 0$, holds on M. The condition $R\cdot R = Q(S,R)$ also appears in the theory of plane gravitational waves.

Furthermore, we define the tensors $R \cdot R$ and $R \cdot S$ on (M^n, g) by

$$\begin{aligned} & (R(X,Y) \cdot R)(U,V)W \\ = \; & R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ & -R(U,R(X,Y)V)W - R(U,V)R(X,Y)W, \end{aligned} \tag{2.26}$$

and

$$(R(X,Y) \cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V), \tag{2.27}$$

respectively [4]. Recently, Kowalczyk [11] studied semi-Riemannian manifolds satisfying $Q(S,R) = 0$ and $Q(S,g) = 0$, where S and R are the Ricci tensor and curvature tensor respectively.

Definition 2.4. A para-Sasakian manifold (M^n, g) is said to be W_3-pseudo-symmetric if the tensors $R \cdot W_3$ and $Q(g, W_3)$ are linearly dependent. This is equivalent to

$$R \cdot W_3 = L_{W_3} Q(g, W_3), \tag{2.28}$$

holding on the set $U_{W_3} = \{x \in M : W_3 \neq 0 \text{ at } x\}$, where L_{W_3} is some function on U_{W_3}.

A Riemannian manifold or pseudo-Riemannian manifold is said to be Ricci semi-symmetric if $R(X,Y) \cdot S = 0$, where S denotes the Ricci tensor of type $(0,2)$. A general classification of these manifolds has been worked out by Mirzoyan [12]. An example of a curvature condition of a semi-symmetry type is the following:

$$Q \cdot R = 0,$$

where Q is the Ricci operator of type $(1,1)$ and $S(X,Y) = g(QX,Y)$.

A natural extension of such curvature conditions from curvature conditions of pseudo-symmetry type. The curvature condition $Q \cdot R = 0$ have been studied by Verstraelen et al. in [19].

3 Ricci solitons in para-Sasakian manifolds

Let (g, ξ, λ) be a Ricci soliton in an n-dimensional para-Sasakian manifold M. From (1.1), we have

$$(L_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{3.1}$$

for any $X, Y \in TM^n$, where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci tensor field of the metric g and λ is real constant. On a para-Sasakian manifold M, from (2.6) and the skew-symmetric property of ϕ, we obtain

$$(L_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0. \tag{3.2}$$

By virtue of (3.2) in (3.1), we get

$$S(X,Y) = -\lambda g(X,Y). \tag{3.3}$$

Thus the pair (M, g, ξ, λ) is an Einstein one. Ricci soliton is called shrinking, steady or expanding according as λ is negative, zero or positive, respectively by [9].

4 Ricci solitons in Ricci pseudo-symmetric para-Sasakian manifolds

In this section, we consider a Ricci pseudo-symmetric para-Sasakian manifold. Then from the definition 2.1, we have

$$(R(X,Y)\cdot S)(U,V) = L_S Q(g,S)(X,Y;U,V),$$

which implies that

$$(R(X,Y)\cdot S)(U,V) = L_S((X\wedge_g Y)\cdot S)(U,V). \tag{4.1}$$

With the help of (2.27) and (2.18), we get from (4.1)

$$\begin{aligned} & -S(R(X,Y)U,V) - S(U,R(X,Y)V) \\ = \;& L_S[-S((X\wedge_g Y)U,V) - S(U,(X\wedge_g Y)V)]. \end{aligned} \tag{4.2}$$

Using (2.16) in (4.2), yields

$$\begin{aligned} & -S(R(X,Y)U,V) - S(U,R(X,Y)V) \\ = \;& L_S[-g(Y,U)S(X,V) + g(X,U)S(Y,V) \\ & -g(Y,V)S(U,X) + g(X,V)S(U,Y)]. \end{aligned} \tag{4.3}$$

Putting $X = U = \xi$ in (4.3) and using (2.4),(2.10), (2.12) and (2.13), we obtain

$$(1 + L_S)[S(Y,V) + (n-1)g(Y,V)] = 0. \tag{4.4}$$

We may conclude that either $L_S = -1$ or, the manifold is an Einstein manifold of the form

$$S(Y,V) = -(n-1)g(Y,V). \tag{4.5}$$

Hence, we state the following lemma:

Lemma 4.1. *A Ricci-pseudo-symmetric para-Sasakian manifold (M^n, g) is an Einstein manifold with $L_S \neq -1$.*

A Ricci-pseudo-symmetric para-Sasakian manifold (M^n, g) admits Ricci soliton. Then by virtue of (3.3) and (4.5), we obtain

$$\lambda = n - 1.$$

Therefore, λ is positive. Hence we can state the following result:

Theorem 4.2. *A Ricci soliton (g, ξ, λ) in a Ricci-pseudo-symmetric para-Sasakian manifold (M^n, g) is expanding provided $L_S \neq -1$.*

5 Ricci solitons in W_3-Ricci pseudo-symmetric para-Sasakian manifolds

Consider a W_3-Ricci pseudo-symmetric para-Sasakian manifold. Then from the definition 2.2, we have

$$(W_3(X,Y)\cdot S)(U,V) = fQ(g,S)(X,Y;U,V),$$

which implies that

$$(W_3(X,Y)\cdot S)(U,V) = f((X\wedge_g Y)\cdot S)(U,V). \tag{5.1}$$

This equation can be written as

$$\begin{aligned} & -S(W_3(X,Y)U,V) - S(U,W_3(X,Y)V) \\ = \; & f[-S((X \wedge_g Y)U,V) - S(U,(X \wedge_g Y)V)]. \end{aligned} \tag{5.2}$$

With the help of (2.27) and (2.18), we get from (5.2)

$$\begin{aligned} & -S(W_3(X,Y)U,V) - S(U,W_3(X,Y)V) \\ = \; & f[-g(Y,U)S(X,V) + g(X,U)S(Y,V) \\ & -g(Y,V)S(U,X) + g(X,V)S(U,Y)]. \end{aligned} \tag{5.3}$$

Putting $X = U = \xi$ in (5.3) and using (2.4), (2.13), (2.20) and (2.21), we obtain

$$(2+f)[S(Y,V) + (n-1)g(Y,V)] = 0. \tag{5.4}$$

We may conclude that either $f = -2$ or, the manifold is an Einstein manifold of the form

$$S(Y,V) = -(n-1)g(Y,V). \tag{5.5}$$

Hence, we state the following lemma:

Lemma 5.1. *A W_3-Ricci-pseudo-symmetric para-Sasakian manifold (M^n, g) is an Einstein manifold with $f \neq -2$.*

Let a W_3-Ricci-pseudo-symmetric para-Sasakian manifold (M^n, g) admits Ricci soliton. Then from (3.3) and (5.5), we obtain

$$\lambda = n - 1.$$

Therefore, λ is positive. Hence we can state the following:

Theorem 5.2. *A Ricci soliton (g, ξ, λ) in W_3-Ricci-pseudo-symmetric para-Sasakian manifold (M^n, g) is expanding provided $f \neq -2$.*

6 Ricci solitons in W_3-pseudo-symmetric para-Sasakian manifolds

Consider a W_3-pseudo-symmetric para-Sasakian manifold. Then from definition 2.4, we have

$$R \cdot W_3 = L_{W_3} Q(g, W_3),$$

which implies that

$$(R(X,Y) \cdot W_3)(U,V)W = L_{W_3}((X \wedge_g Y) \cdot W_3)(U,V)W. \tag{6.1}$$

Using (2.26) and (2.18) in (6.1), we have

$$\begin{aligned} & R(X,Y)W_3(U,V)W - W_3(R(X,Y)U,V)W - \\ & W_3(U,R(X,Y)V)W - R(U,V)W_3(X,Y)W \\ = \; & L_{W_3}[(X \wedge_g Y)W_3(U,V)W - W_3((X \wedge_g Y)U,V)W \\ & -W_3(U,(X \wedge_g Y)V)W - W_3(U,V)(X \wedge_g Y)W]. \end{aligned} \tag{6.2}$$

By virtue of (2.16) and (6.2), we find

$$\begin{aligned} & R(X,Y)W_3(U,V)W - W_3(R(X,Y)U,V)W \\ & -W_3(U,R(X,Y)V)W - W_3(U,V)R(X,Y)W \\ = \; & L_{W_3}[g(Y,W_3(U,V)W)X - g(X,W_3(U,V)W)Y \\ & -g(Y,U)W_3(X,Y)W + g(X,U)W_3(Y,V)W \\ & -g(Y,V)W_3(U,X)W + g(X,V)W_3(U,Y)W \\ & -g(Y,W)W_3(U,V)X + g(X,W)W_3(U,V)Y]. \end{aligned} \tag{6.3}$$

Putting $X = U = \xi$ in (6.3) and using (2.9), (2.12), (2.19), (2.20), (2.21) and (2.22), we obtain

$$\begin{aligned} & -2g(V,W)Y - W_3(Y,V)W + 2g(Y,W)V \\ = \;& L_{W_3}[2g(V,W)Y + W_3(Y,V)W - 2g(Y,W)V], \end{aligned} \tag{6.4}$$

i.e.,

$$(1 + L_{W_3})[W_3(Y,V)W + 2g(V,W)Y - 2g(Y,W)V] = 0. \tag{6.5}$$

Taking inner product with Z of (6.5) and using (2.19), we find

$$\begin{aligned} & (1 + L_{W_3})\{g(R(Y,V)W,Z) + \frac{1}{(n-1)}[g(V,W)S(Y,Z) \\ & -S(Y.W)g(V,Z)] + 2g(V,W)g(Y,Z) - 2g(Y,W)g(V,Z)\} \\ = \;& 0. \end{aligned} \tag{6.6}$$

Let $\{e_i\}$, $i = 1, 2, ..., n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $V = W = e_i$ in (6.6) and taking summation over $1 \leqslant i \leqslant n$, we have

$$(1 + L_{W_3})[S(Y,Z) + g(Y,Z)] = 0. \tag{6.7}$$

We may conclude that either $L_{W_3} = -1$ or, the manifold is an Einstein manifold of the form

$$S(Y,Z) = -g(Y,Z). \tag{6.8}$$

Hence, we state the following lemma:

Lemma 6.1. *A W_3-pseudo-symmetric para-Sasakian manifold (M^n, g) is an Einstein manifold with $L_{W_3} \neq -1$.*

Let a W_3-pseudo-symmetric para-Sasakian manifold (M^n, g) admits Ricci soliton. Then from equations (3.3) and (6.8), we obtain

$$\lambda = 1.$$

Therefore, λ is positive. Hence we have the following:

Theorem 6.2. *A Ricci soliton (g, ξ, λ) in W_3-pseudo-symmetric para-Sasakian manifold (M^n, g) is expanding provided $L_{W_3} \neq -1$.*

7 Ricci solitons in Ricci generalized pseudo-symmetric para-Sasakian manifolds

Consider a Ricci generalized pseudo-symmetric para-Sasakian manifold. Then from the definition 2.3, we have

$$R \cdot R = L_R Q(S,R),$$

which implies that

$$(R(X,Y) \cdot R)(U,V)W = L_R((X \wedge_S Y) \cdot R)(U,V)W. \tag{7.1}$$

Using (2.26) and (2.18) in (7.1), we have

$$\begin{aligned} & R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ & -R(U,R(X,Y)V)W - R(U,V)R(X,Y)W \\ = \;& L_R[(X \wedge_S Y)R(U,V)W - R((X \wedge_S Y)U,V)W \\ & -R(U,(X \wedge_S Y)V)W - R(U,V)(X \wedge_S Y)W], \end{aligned} \tag{7.2}$$

by virtue of (2.16) and (7.2), we obtain

$$\begin{aligned}&R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ &-R(U,R(X,Y)V)W - R(U,V)R(X,Y)W \\ =\ & L_R[S(Y,R(U,V)W)X - S(X,R(U,V)W)Y \\ &-S(Y,U)R(X,Y)W + S(X,U)R(Y,V)W \\ &-S(Y,V)R(U,X)W + S(X,V)R(U,Y)W \\ &-S(Y,W)R(U,V)X + S(X,W)R)(U,V)Y].\end{aligned} \tag{7.3}$$

Taking $X = U = \xi$ in (7.3) and using (2.9), (2.10), (2.11), (2.12) and (2.13), we have

$$\begin{aligned}&-g(V,W)Y + g(V,W)\eta(Y)\xi - R(Y,V)W \\ &+\eta(Y)\eta(W)V - g(V,W)\eta(Y)\xi \\ &-\eta(W)\eta(Y)V + g(Y,W)V \\ =\ & L_R[\eta(W)S(Y,V)\xi - (n-1)g(V,W)Y \\ &-(n-1)R(Y,V)W + (n-1)g(Y,W)\eta(V)\xi \\ &-S(Y,W)V + S(Y,W)\eta(V)\xi \\ &+(n-1)g(V,Y)\eta(W)\xi].\end{aligned} \tag{7.4}$$

Taking the inner product with Z of (7.4), we get

$$\begin{aligned}&-g(V,W)g(Y,W) - g(R(Y,V)W,Z) + g(Y,W)g(V,Z) \\ =\ & L_R[S(Y,V)\eta(W)\eta(Z) - (n-1)g(V,W)g(Y,Z) \\ &-(n-1)g(R(Y,V)W,Z) + (n-1)g(Y,W)\eta(V)\eta(Z) \\ &-S(Y,W)g(V,Z) + S(Y,W)\eta(V)\eta(Z) \\ &+(n-1)g(V,Y)\eta(W)\eta(Z)].\end{aligned} \tag{7.5}$$

Let $\{e_i\}$, $i = 1, 2, ...,n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $V = W = e_i$ in (7.5) and taking summation over $1 \leqslant i \leqslant n$, we obtain

$$S(Y,Z) + (n-1)g(Y,Z) = nL_R[S(Y,Z) + (n-1)g(Y,Z)], \tag{7.6}$$

i.e.,

$$(1 - nL_R)[S(Y,Z) + (n-1)g(Y,Z)] = 0. \tag{7.7}$$

We may conclude that either $L_R = \frac{1}{n}$ or, the manifold is an Einstein manifold of the form

$$S(Y,Z) = -(n-1)g(Y,Z). \tag{7.8}$$

Hence, we state the following lemma:

Lemma 7.1. *A Ricci generalized pseudosymmetric para-Sasakian manifold (M^n, g) is an Einstein manifold with $L_R \neq \frac{1}{n}$.*

Let a Ricci generalized pseudosymmetric para-Sasakian manifold (M^n, g) admits Ricci soliton. Then by virtue of (3.3) and (7.8), we have

$$\lambda = n - 1.$$

Therefore, λ is positive. Hence we can state the following result:

Theorem 7.2. *A Ricci soliton (g, ξ, λ) in Ricci generalized pseudosymmetric para-Sasakian manifold (M^n, g) is expanding provided $L_R \neq \frac{1}{n}$.*

8 Ricci solitons in para-Sasakian manifolds satisfying the curvature condition $Q \cdot R = 0$

This section is devoted to study Ricci solitons in para-Sasakian manifolds satisfying the curvature condition $Q \cdot R = 0$.

Let us consider a para-Sasakian manifold satisfying the curvature condition $Q \cdot R = 0$, i.e.,

$$(Q \cdot R)(X, Y)Z = 0,$$

for all vector fields X, Y and $Z \in TM^n$. This is equivalent to

$$Q(R(X, Y)Z) - R(QX, Y)Z - R(X, QY)Z - R(X, Y)QZ = 0. \tag{8.1}$$

Putting $X = Z = \xi$ in (8.1),we obtain

$$Q(R(\xi, Y)\xi) - R(Q\xi, Y)\xi - R(\xi, QY)\xi - R(\xi, Y)Q\xi = 0. \tag{8.2}$$

Using (2.12) in (8.2), we have

$$-\eta(Y)Q\xi - R(Q\xi, Y)\xi + \eta(QY)\xi - R(\xi, Y)Q\xi = 0. \tag{8.3}$$

Taking the inner product with ξ of (8.3), we get

$$-\eta(Y)S(\xi, \xi) - g(R(Q\xi, Y)\xi, \xi) + \eta(QY) - g(R(\xi, Y)Q\xi, \xi) = 0. \tag{8.4}$$

Now from (2.7), we obtain

$$g(R(Q\xi, Y)\xi, \xi) = 0, \tag{8.5}$$

and

$$g(R(\xi, Y)Q\xi, \xi) = -(n-1)\eta(Y) - S(Y, \xi). \tag{8.6}$$

Using (2.13), (8.5) and (8.6) in (8.4), we have

$$S(Y, \xi) = -(n-1)\eta(Y). \tag{8.7}$$

Taking $Y = \xi$ in (3.3), we obtain

$$S(X, \xi) = -\lambda\eta(Y), \tag{8.8}$$

by virtue of (8.7) and (8.8), we get

$$\lambda = n - 1.$$

Therefore, λ is positive. Hence we can state the following result:

Theorem 8.1. *A Ricci soliton (g, ξ, λ) in para-Sasakian manifold (M^n, g) satisfying the curvature condition $Q \cdot R = 0$, is expanding.*

9 Example

We consider 5-dimensional manifold M, where $M = \{(x_1, x_2, y_1, y_2, z) \in R^5\}$, where (x_1, x_2, y_1, y_2, z) are standard coordinates in R^5. Let e_1, e_2, e_3, e_4, e_5 be linearly independent frame fields on M given by

$$e_1 = \frac{\partial}{\partial x_1}, e_2 = \frac{\partial}{\partial x_2}, e_3 = \frac{\partial}{\partial y_1}, e_4 = \frac{\partial}{\partial y_2},$$

$$e_5 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + \frac{\partial}{\partial z}.$$

Let g be a Riemannian metric defined by

$$\begin{aligned} g(e_i, e_j) &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j,\ i, j = 1, 2, 3, 4, 5 \end{aligned}$$

Let η be the 1-form defined by $\eta(X) = g(X, e_5)\ \forall\ X \in \chi(M)$, where $\chi(M)$ be the set of all $C^{\infty}-$vector fields defined on M. Let ϕ be $(1,1)$ tensor field defined by

$$\phi e_1 = e_1, \phi e_2 = e_2, \phi e_3 = e_3, \phi e_4 = e_4, \phi e_5 = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_5) = 1, \phi^2 X = X - \eta(X)e_5,$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on $\chi(M)$. Thus for $e_5 = \xi$, the structure (φ, ξ, η, g) defines an almost contact metric structure on M. [4]

Let ∇ be the Levi-Civita connection with respect to metric g and R be the curvature tensor of the metric g. Then, we have

$$[e_1, e_2] = 0, [e_1, e_3] = 0, [e_1, e_4] = 0, [e_1, e_5] = e_1,$$

$$[e_2, e_3] = 0, [e_3, e_4] = 0, [e_2, e_4] = 0, [e_2, e_5] = e_2,$$

$$[e_3, e_5] = e_3, [e_4, e_5] = e_4.$$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X\{g(Y,Z)\} + Y\{g(Z,X)\} - Z\{g(X,Y)\} \\ &\quad -g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]). \end{aligned}$$

Taking $e_5 = \xi$ and using Koszul's formula, we obtain

$$\begin{aligned} \nabla_{e_1} e_1 &= -e_5, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = 0, \nabla_{e_1} e_4 = 0, \nabla_{e_1} e_5 = e_1, \\ \nabla_{e_2} e_1 &= 0, \nabla_{e_2} e_2 = -e_5, \nabla_{e_2} e_3 = 0, \nabla_{e_2} e_4 = 0, \nabla_{e_2} e_5 = e_2, \\ \nabla_{e_3} e_1 &= 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = -e_5, \nabla_{e_3} e_4 = 0, \nabla_{e_3} e_5 = e_3, \\ \nabla_{e_4} e_1 &= 0, \nabla_{e_4} e_2 = 0, \nabla_{e_4} e_3 = 0, \nabla_{e_4} e_4 = -e_5, \nabla_{e_4} e_5 = e_4, \\ \nabla_{e_5} e_1 &= 0, \nabla_{e_5} e_2 = 0, \nabla_{e_5} e_3 = 0, \nabla_{e_5} e_4 = 0, \nabla_{e_5} e_5 = 0. \end{aligned}$$

From the above, it can be easily seen that $e_5 = \xi$, (φ, ξ, η, g) is a para-Sasakian structure on M. Hence, $M(\varphi, \xi, \eta, g)$ is a 5-dimensional para-Sasakian manifold. [4]

Also, the Riemannian curvature tensor R is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

With the help of above results, we can verify the following results:

$$\begin{aligned} &R(e_1, e_2)e_1 = e_2,\ R(e_1, e_2)e_2 = -e_1,\ R(e_1, e_3)e_1 = e_3, R(e_1, e_3)e_3 = -e_1, \\ &R(e_1, e_4)e_1 = e_4,\ R(e_1, e_4)e_4 = -e_1,\ R(e_1, e_5)e_1 = e_5, R(e_1, e_5)e_5 = -e_1, \\ &R(e_2, e_3)e_2 = e_3,\ R(e_2, e_3)e_3 = -e_2,\ R(e_2, e_4)e_2 = e_4, R(e_2, e_4)e_4 = -e_2, \\ &R(e_2, e_5)e_2 = e_5,\ R(e_2, e_5)e_5 = -e_2,\ R(e_3, e_4)e_3 = e_4, R(e_3, e_4)e_4 = -e_3, \\ &R(e_3, e_5)e_3 = e_5,\ R(e_3, e_5)e_5 = -e_3,\ R(e_4, e_5)e_4 = e_5, R(e_4, e_5)e_5 = -e_4. \end{aligned}$$

The definition of Ricci tensor in 5-dimensional manifold implies that

$$S(X,Y) = \sum_{i=1}^{5} g(R(e_i, X)Y, e_i).$$

Using the components of the curvature tensor in the above equation, we get the following :

$$S(e_1, e_1) = -4,\ S(e_2, e_2) = -4,\ S(e_3, e_3) = -4, S(e_4, e_4) = -4,\ S(e_5, e_5) = -4.$$

That is,

$$S(X,Y) = -4g(X,Y).$$

Hence, the manifold is an Einstein manifold. With the help of the above expression of the Ricci tensor it can be easily verified that the manifold satisfies (3.3) for $\lambda = 4$, this implies that $\lambda > 0$, that is the Ricci soliton in 5-dimensional para-Sasakian manifold is expanding. Therefore, Theorem 8.1 is verified for 5-dimensional case.

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Received: July 23, 2020.
Accepted: October 12, 2020.