Palestine Journal of Mathematics

Vol. 11(1)(2022) , 594–607 © Palestine Polytechnic University-PPU 2022

On a generalized p-radical functional equation related to additive mappings in 2-Banach spaces

Sadeq AL-Ali, Mustapha E. Hryrou, Youssfi Elkettani and Mohammed A. Salman

Communicated by Ekrem Savas

MSC 2010 Classifications: Primary: 39B82; Secondary: 39B62, 47H10, 47H14, 47J20.

Keywords and phrases: stability , hyperstability, 2-Banach spaces, p-radical functional equations.

Abstract In this paper, we introduce and solve the following p-radical functional equation

$$f\left(\sqrt[p]{ax^p + by^p}\right) = af(x) + bf(y),$$

with $a, b \in \mathbb{Q}_+^*$ and $p \in \mathbb{N}_2$. We also investigate some stability and hyperstability results for this equation in 2-Banach spaces using the fixed point approach.

1 Introduction

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$ and $\mathbb{R}_+ = [0, \infty)$ the set of nonnegative real numbers. By $\mathbb{N}_m$, $m \in \mathbb{N}$, we will denote the set of all natural numbers greater than or equal to m.

The notion of linear 2-normed spaces was introduced by S. Gähler [23],[24] in the middle of 1960s. We need to recall some basic facts concerning 2-normed spaces and some preliminary results.

Definition 1.1. Let X be a real linear space with $dimX > 1$ and $\|.,.\| : X \times X \longrightarrow [0, \infty)$ be a function satisfying the following properties:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

(ii) $\|x, y\| = \|y, x\|$,

(iii) $\|\lambda x, y\| = |\lambda|\|x, y\|$,

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$,

for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$. Then the function $\|.,.\|$ is called *a 2-norm* on X and the pair $(X, \|.,.\|)$ is called a *linear 2-normed space*. Sometimes the condition (4) called the *triangle inequality*.

Example 1.2. For $x = (x_1, x_2)$, $y = (y_1, y_2) \in X = \mathbb{R}^2$, the Euclidean 2-norm $\|x, y\|_{\mathbb{R}^2}$ is defined by

$$\|x, y\|_{\mathbb{R}^2} = |x_1y_2 - x_2y_1| .$$

Lemma 1.3. *Let $(X, \|.,.\|)$ be a 2-normed space. If $x \in X$ and $\|x, y\| = 0$, for all $y \in X$, then $x = 0$.*

Definition 1.4. A sequence $\{x_k\}$ in a 2-normed space X is called a *convergent sequence* if there is an $x \in X$ such that

$$\lim_{k\to\infty} \|x_k - x, y\| = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to x, write $x_k \longrightarrow x$ with $k \longrightarrow \infty$ and call x the limit of $\{x_k\}$. In this case, we also write $\lim_{k\to\infty} x_k = x$.

Definition 1.5. A sequence $\{x_k\}$ in a 2-normed space X is said to be a *Cauchy sequence* with respect to the 2-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, y\| = 0,$$

for all $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be *complete* with respect to the 2-norm. Any complete 2-normed space is said to be a *2-Banach space*.

Now, we state the following results as lemma (See [29] for the details).

Lemma 1.6. *Let X be a 2-normed space. Then,*

(i) $\big|\|x, z\| - \|y, z\|\big| \le \|x - y, z\|$ *for all* $x, y, z \in X$,

(ii) if $\|x, z\| = 0$ *for all* $z \in X$, *then* $x = 0$,

(iii) for a convergent sequence x_n *in* X,

$$\lim_{n\longrightarrow\infty} \|x_n, z\| = \left\|\lim_{n\longrightarrow\infty} x_n, z\right\|$$

for all $z \in X$.

The concept of stability for a functional equation arises when defining, in some way, the class of approximate solutions of the given functional equation, one can ask whether each mapping from this class can be somehow approximated by an exact solution of the considered equation. Namely, when one replaces a functional equation by an inequality which acts as a perturbation of the considered equation. The first stability problem of functional equation was raised by S. M. Ulam [33] in 1940. This included the following question concerning the stability of group homomorphisms.
Let $(G_1, *_1)$ *be a group and let* $(G_2, *_2)$ *be a metric group with a metric* $d(.,.)$. *Given* $\varepsilon > 0$, *does there exists a* $\delta > 0$ *such that if a mapping* $h : G_1 \to G_2$ *satisfies the inequality*

$$d\big(h(x *_1 y), h(x) *_2 h(y)\big) < \delta$$

for all $x, y \in G_1$, *then there exists a homomorphism* $H : G_1 \to G_2$ *with*

$$d\big(h(x), H(x)\big) < \varepsilon$$

for all $x \in G_1$?
The first affirmative partial answer to the Ulam's problem for Banach spaces was provided by D. H. Hyers [26]. The result of Hyers was generalizable. Namely, it was generalized by T. Aoki [10] for additive mappings and by Th. M. Rassias [30] for linear mappings by considering an unbounded Cauchy difference. In 1994, P. Găvruţa [25] introduced the generalization of the Th. M. Rassias theorem was obtained by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.
It is trivial to prove the following lemma:

Lemma 1.7. *Let X and Y be two linear spaces and $f : X \to Y$ be a function satisfies the equation*

$$f(ax + by) = af(x) + bf(y) \tag{1.1}$$

for all $x, y \in X$ *where* $a, b \in \mathbb{R}_+^*$. *Then we have two cases as follows:*

(i) If $a + b = 1$, *then* f *is additive.*

(ii) If $a + b \neq 1$, *then* f *is Jensen-additive.*

In this paper, we achieve the general solutions of the following p-radical functional equation:

$$f\left(\sqrt[p]{ax^p + by^p}\right) = af(x) + bf(y), \quad p \in \mathbb{N}_2 \tag{1.2}$$

with $a, b \in \mathbb{Q}_+^*$. We also discuss the generalized Hyers-Ulam-Rassias stability problem in 2-Banach spaces by using Brzdęk's fixed point approach.

2 Brzdęk's fixed point theorem

Recently, J. Brzdęk [16] showed that Raissias' result [32] can be significantly improved and he presented and proved in [14] the fixed point theorem for a nonlinear operator in metric spaces. He used this result to study the Hyers-Ulam stability of some functional equations in non-Archimedean metric spaces. In 2011, J. Brzdęk and K. Ciepliński obtained the fixed point result in arbitrary metric spaces as follows:

Theorem 2.1. *[14] Let X be a nonempty set, (Y,d) be a complete metric space, and $\Lambda : Y^X \to Y^X$ be a non-decreasing operator satisfying the hypothesis*

$$\lim_{n\to\infty} \Lambda\delta_n = 0$$

for every sequence $\{\delta_n\}_{n\in\mathbb{N}}$ in Y^X with

$$\lim_{n\to\infty} \delta_n = 0.$$

Suppose that $\mathcal{T} : Y^X \to Y^X$ is an operator satisfying the inequality

$$d\big(\mathcal{T}\xi(x), \mathcal{T}\mu(x)\big) \leq \Lambda\big(\Delta(\xi,\mu)\big)(x), \quad \xi,\mu \in Y^X,\ x \in X, \tag{2.1}$$

where $\Delta : Y^X \times Y^X \to \mathbb{R}_+^X$ is a mapping which is defined by

$$\Delta(\xi,\mu)(x) := d\big(\xi(x), \mu(x)\big) \quad \xi,\mu \in Y^X,\ x \in X. \tag{2.2}$$

If there exist functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$d\Big((\mathcal{T}\varphi)(x), \varphi(x)\Big) \leq \varepsilon(x) \tag{2.3}$$

and

$$\varepsilon^*(x) := \sum_{n\in\mathbb{N}_0} \big(\Lambda^n\varepsilon\big)(x) < \infty \tag{2.4}$$

for all $x \in X$, then the limit

$$\lim_{n\to\infty} \big((\mathcal{T}^n\varphi)\big)(x) \tag{2.5}$$

exists for each $x \in X$. Moreover, the function $\psi \in Y^X$ defined by

$$\psi(x) := \lim_{n\to\infty} \big((\mathcal{T}^n\varphi)\big)(x) \tag{2.6}$$

is a fixed point of $\mathcal{T}$ with

$$d\big(\varphi(x), \psi(x)\big) \leq \varepsilon^*(x) \tag{2.7}$$

for all $x \in X$.

After that, J. Brzdęk [15] gave the fixed point result by applying Theorem 2.1 as follows:

Theorem 2.2. *[15] Let X be a nonempty set, (Y,d) be a complete metric space, $f_1,...,f_r : X \to X$ and $L_1,...,L_r : X \to \mathbb{R}_+$ be given mappings. Suppose that $\mathcal{T} : Y^X \to Y^X$ and $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ are two operators satisfying the conditions*

$$d\big(\mathcal{T}\xi(x), \mathcal{T}\mu(x)\big) \leq \sum_{i=1}^{r} L_i(x) d\Big(\xi\big(f_i(x)\big), \mu\big(f_i(x)\big)\Big) \tag{2.8}$$

for all $\xi,\mu \in Y^X$, $x \in X$ and

$$\Lambda\delta(x) := \sum_{i=1}^{r} L_i(x)\delta\big(f_i(x)\big), \quad \delta \in \mathbb{R}_+^X,\ x \in X. \tag{2.9}$$

If there exist functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$d\Big(\mathcal{T}\varphi(x), \varphi(x)\Big) \leq \varepsilon(x) \tag{2.10}$$

and

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \big(\Lambda^n \varepsilon\big)(x) < \infty \tag{2.11}$$

for all $x \in X$, then the limit (2.5) exists for each $x \in X$. Moreover, the function (2.6) is a fixed point of $\mathcal{T}$ with (2.7) for all $x \in X$.

Then by using this theorem, J. Brzdęk [15] improved, extended and complemented several earlier classical stability results concerning the additive Cauchy equation. Over the last few years, many mathematicians have investigated various generalizations, extensions and applications of the Hyers-Ulam stability of a number of functional equations (see, for instance, [1]-[5], [17], [18] and references therein); in particular, the stability problem of the radical functional equations in various spaces was proved in [7]-[9], [21, 22, 27, 28].

Recently, an analogue of Theorem 2.2 in 2-Banach spaces was given in [6].

Theorem 2.3. *[6] Let X be a nonempty set, $\big(Y, \|\cdot,\cdot\|\big)$ be a 2-Banach space, $g : X \to Y$ be a surjective mapping and let $f_1, ..., f_r : X \to X$ and $L_1, ..., L_r : X \to \mathbb{R}_+$ be given mappings. Suppose that $\mathcal{T} : Y^X \to Y^X$ and $\Lambda : \mathbb{R}_+^{X\times X} \to \mathbb{R}_+^{X\times X}$ are two operators satisfying the conditions*

$$\big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), g(z)\big\| \le \sum_{i=1}^{r} L_i(x)\Big\|\xi\big(f_i(x)\big) - \mu\big(f_i(x)\big), g(z)\Big\| \tag{2.12}$$

for all $\xi, \mu \in Y^X$, $x, z \in X$ and

$$\Lambda\delta(x,z) := \sum_{i=1}^{r} L_i(x)\delta\big(f_i(x), z\big), \quad \delta \in \mathbb{R}_+^{X\times X},\ x, z \in X. \tag{2.13}$$

If there exist functions $\varepsilon : X \times X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$\Big\|\mathcal{T}\varphi(x) - \varphi(x), g(z)\Big\| \le \varepsilon(x,z) \tag{2.14}$$

and

$$\varepsilon^*(x,z) := \sum_{n=0}^{\infty} \big(\Lambda^n \varepsilon\big)(x,z) < \infty \tag{2.15}$$

for all $x, z \in X$, then the limit

$$\lim_{n\to\infty} \big((\mathcal{T}^n \varphi)\big)(x) \tag{2.16}$$

exists for each $x \in X$. Moreover, the function $\psi : X \to Y$ defined by

$$\psi(x) := \lim_{n\to\infty} \big((\mathcal{T}^n \varphi)\big)(x) \tag{2.17}$$

is a fixed point of $\mathcal{T}$ with

$$\big\|\varphi(x) - \psi(x), g(z)\big\| \le \varepsilon^*(x,z) \tag{2.18}$$

for all $x, z \in X$.

3 Solution of equation (1.2)

In this section, we give the general solution of functional equation (1.2). The next theorem can be derived from [[20], Corollary 2.3 and Proposition 2.4(a)]. However, for the convenience of readers we present it with a direct proof.

Theorem 3.1. *Let Y be a linear space. A function $f : \mathbb{R} \to Y$ satisfies the functional equation (1.2) if and only if*

$$f(x) = F(x^p), \quad x \in \mathbb{R}, \tag{3.1}$$

with some additive function $F : \mathbb{R} \to Y$.

Proof. Indeed, It is not hard to check without any problem that if $f : \mathbb{R} \to Y$ satisfies (3.1), then it is a solution to (1.2). On the other hand, if $f : \mathbb{R} \to Y$ is a solution of (1.2), then we consider the following two cases:
Case 1: p is even:
We write $F_0(x) = f(\sqrt[p]{x})$, for $x \in [0, +\infty)$, then from (1.2) we obtain

$$\begin{aligned} F_0(ax+by) &= f(\sqrt[p]{ax+by}) \\ &= af(\sqrt[p]{x}) + bf(\sqrt[p]{y}) \\ &= aF_0(x) + bF_0(y) \end{aligned}$$

for all $x, y \in [0, +\infty)$. Substituting $x = y = 0$ in (1.2) to obtain $f(0) = 0$ since $a + b \neq 1$ and $f(0) \neq 0$ when $a + b = 1$. Setting $x = -x$ in (1.2), we obtain

$$f\left(\sqrt[p]{ax^p + by^p}\right) = af(-x) + bf(y) \tag{3.2}$$

for all $x, y \in \mathbb{R}$. If we compare (1.2) with (3.2), we obtain that f is even. So, $f(-x) = f(x) = F_0(x^p)$, for all $x \in [0, +\infty)$. Now, it is enough to observe that there is an additive $F : \mathbb{R} \to Y$ with $F(x) = F_0(x)$ for all $x \in [0, +\infty)$.
Case 2: p is odd:
By a similar method in case 1, we can write $F(x) = f(\sqrt[p]{x})$, for all $x \in \mathbb{R}$. Then, we get that there exists an additive $F : \mathbb{R} \to Y$ with $F(x) = F_0(x)$ for all $x \in \mathbb{R}$. This completes the proof. □

4 Approximation of the p-radical functional equation (1.2)

In the following two theorems, we use Theorem 2.3 to investigate the generalized Hyers-Ulam stability of the p-radical functional equation (1.2) in 2-Banach spaces.
Hereafter, we assume that $\big(Y, \|\cdot,\cdot\|\big)$ is a 2-Banach space.

Theorem 4.1. *Let $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}_+$ be two functions such that*

$$\mathcal{U} := \left\{ n \in \mathbb{N}\colon \alpha_n := \frac{1}{a}\lambda_1(a+bn^p)\lambda_2(a+bn^p) + \frac{b}{a}\lambda_1(n^p)\lambda_2(n^p) < 1 \right\} \neq \phi, \tag{4.1}$$

where

$$\lambda_i(n) := \inf\{t \in \mathbb{R}_+ \colon h_i(nx^p, z) \leq t\, h_i(x^p, z), \quad x, z \in \mathbb{R}\} \tag{4.2}$$

for all $n \in \mathbb{N}$, where $i = 1, 2$. Assume that $f : \mathbb{R} \to Y$ satisfies the inequality

$$\left\| f\left(\sqrt[p]{ax^p + by^p}\right) - af(x) - bf(y), g(z) \right\| \leq h_1(x^p, z)h_2(y^p, z) \tag{4.3}$$

for all $x, y, z \in \mathbb{R}$ where $g : X \to Y$ is a surjective mapping. Then there exists a unique function $F : \mathbb{R} \to Y$ that satisfies the equation (1.2) such that

$$\big\| f(x) - F(x), g(z) \big\| \leq \lambda_0 h_1(x^p, z)h_2(x^p, z) \tag{4.4}$$

for all $x, z \in \mathbb{R}$, where

$$\lambda_0 := \inf_{n \in \mathcal{U}} \left\{ \frac{\lambda_2(n^p)}{a\big(1 - \lambda_1(a+bn^p)\lambda_2(a+bn^p) - \lambda_1(n^p)\lambda_2(n^p)\big)} \right\}.$$

Proof. Replacing y with mx, where $x \in \mathbb{R}$ and $m \in \mathbb{N}$, in inequality (4.3) we get

$$\left\| f\left(\sqrt[p]{(a+bm^p)x^p}\right) - af(x) - bf(mx), g(z) \right\| \leq h_1(x^p, z)h_2(m^p x^p, z)$$

and

$$\left\| \frac{1}{a} f\left(\sqrt[p]{(a+bm^p)x^p}\right) - \frac{b}{a} f(mx) - f(x), g(z) \right\| \leq \frac{1}{a} h_1(x^p, z)h_2(m^p x^p, z) \tag{4.5}$$

for all $x, z \in \mathbb{R}$. For each $m \in \mathbb{N}$, we define the operator $\mathcal{T}_m : Y^{\mathbb{R}} \to Y^{\mathbb{R}}$ by

$$\mathcal{T}_m\xi(x) := \frac{1}{a}\xi\left(\sqrt[p]{(a+bm^p)x^p}\right) - \frac{b}{a}\xi(mx), \quad \xi \in Y^{\mathbb{R}}, x \in \mathbb{R}. \tag{4.6}$$

Further put

$$\varepsilon_m(x,z) := \frac{1}{a}h_1(x^p,z)h_2(m^px^p,z), \quad x, z \in \mathbb{R}, \tag{4.7}$$

and observe that

$$\varepsilon_m(x,z) = \frac{1}{a}h_1(x^p,z)h_2(m^px^p,z) \leq \frac{1}{a}\lambda_2(m^p)h_1(x^p,z)h_2(x^p,z), \quad x, z \in \mathbb{R}, m \in \mathbb{N}. \tag{4.8}$$

Then the inequality (4.5) takes the form

$$\left\|f(x) - \mathcal{T}_mf(x), g(z)\right\| \leq \varepsilon_m(x,z), \quad x, z \in \mathbb{R}. \tag{4.9}$$

Furthermore, for every $x, z \in \mathbb{R}$, $\xi, \mu \in Y^{\mathbb{R}}$, we obtain

$$\begin{aligned}\left\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x), g(z)\right\| &= \bigg\|\frac{1}{a}\xi\left(\sqrt[p]{(a+bm^p)x^p}\right) - \frac{b}{a}\xi(mx) \\ &\quad - \frac{1}{a}\mu\left(\sqrt[p]{(a+bm^p)x^p}\right) + \frac{b}{a}\mu(mx), g(z)\bigg\| \\ &\leq \frac{1}{a}\left\|(\xi-\mu)\left(\sqrt[p]{(a+bm^p)x^p}\right), g(z)\right\| + \frac{b}{a}\left\|(\xi-\mu)(mx), g(z)\right\|.\end{aligned}$$

This brings us to define the operator $\Lambda_m : \mathbb{R}_+^{\mathbb{R}\times\mathbb{R}} \to \mathbb{R}_+^{\mathbb{R}\times\mathbb{R}}$ by

$$\Lambda_m\delta(x,z) := \frac{1}{a}\delta\left(\sqrt[p]{(a+bm^p)x^p}, z\right) + \frac{b}{a}\delta(mx,z), \quad \delta \in \mathbb{R}_+^{\mathbb{R}\times\mathbb{R}}, x, z \in \mathbb{R}. \tag{4.10}$$

For each $m \in \mathbb{N}$, the above operator has the form described in (2.13) with $f_1(x) = \sqrt[p]{(a+bm^p)x^p}$, $f_2(x) = mx$ and $L_1(x) = \frac{1}{a}$, $L_2(x) = \frac{b}{a}$ for all $x \in \mathbb{R}$. By induction, we will show that for each $x, z \in \mathbb{R}$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$ we have

$$\left(\Lambda_m^n\varepsilon_m\right)(x,z) \leq \frac{1}{a}\lambda_2(m^p)\alpha_m^nh_1(x^p,z)h_2(x^p,z) \tag{4.11}$$

where

$$\alpha_m = \frac{1}{a}\lambda_1(a+bm^p)\lambda_2(a+bm^p) + \frac{b}{a}\lambda_1(m^p)\lambda_2(m^p).$$

From (4.7) and (4.8), we obtain that the inequality (4.11) holds for $n = 0$. Next, we will assume that (4.11) holds for $n = k$, where $k \in \mathbb{N}$. Then we have

$$\begin{aligned}\left(\Lambda_m^{k+1}\varepsilon_m\right)(x,z) &= \Lambda_m\Big(\left(\Lambda_m^k\varepsilon_m\right)(x,z)\Big) \\ &= \frac{1}{a}\left(\Lambda_m^k\varepsilon_m\right)\left(\sqrt[p]{(a+bm^p)x^p}, z\right) + \frac{b}{a}\left(\Lambda_m^k\varepsilon_m\right)(mx,z) \\ &\leq \frac{1}{a^2}\lambda_2(m^p)\alpha_m^kh_1((a+bm^p)x^p,z)h_2((a+bm^p)x^p,z) \\ &\quad + \frac{b}{a^2}\lambda_2(m^p)\alpha_m^kh_1(m^px^p,z)h_2(m^px^p,z) \\ &\leq \frac{1}{a}\lambda_2(m^p)\alpha_m^k\Big(\frac{1}{a}\lambda_1(a+bm^p)\lambda_2(a+bm^p) + \frac{b}{a}\lambda_1(m^p)\lambda_2(m^p)\Big)h_1(x^p,z)h_2(x^p,z) \\ &= \frac{1}{a}\lambda_2(m^p)\alpha_m^{k+1}h_1(x^p,z)h_2(x^p,z)\end{aligned}$$

for all $x, z \in \mathbb{R}$, $m \in \mathcal{U}$. This shows that (4.11) holds for $n = k + 1$. Now we can conclude that the inequality (4.11) holds for all $n \in \mathbb{N}_0$. Hence, we obtain

$$\begin{aligned}\varepsilon_m^*(x,z) &= \sum_{n=0}^{\infty} \left(\Lambda_m^n \varepsilon_m\right)(x,z) \\ &\leq \sum_{n=0}^{\infty} \frac{1}{a} \lambda_2(m^p) \alpha_m^n h_1(x^p, z) h_2(x^p, z) \\ &= \frac{\lambda_2(m^p) h_1(x^p, z) h_2(x^p, z)}{a(1-\alpha_m)} < \infty\end{aligned}$$

for all $x, z \in \mathbb{R}$, $m \in \mathcal{U}$. Therefore, according to Theorem 2.3 with $\varphi = f$ and $X = \mathbb{R}$ and using the surjectivity of g, we get that the limit

$$F_m(x) := \lim_{n\to\infty} \left(\mathcal{T}_m^n f\right)(x)$$

exists for each $x \in \mathbb{R}$ and $m \in \mathcal{U}$, and

$$\left\|f(x) - F_m(x), g(z)\right\| \leq \frac{\lambda_2(m^p) h_1(x^p, z) h_2(x^p, z)}{a(1-\alpha_m)}, \quad x, z \in \mathbb{R},\ m \in \mathcal{U}. \tag{4.12}$$

To prove that F_m satisfies the functional equation (1.2), just prove the following inequality

$$\left\|\mathcal{T}_m^n f\left(\sqrt[p]{ax^p + by^p}\right) - a\mathcal{T}_m^n f(x) - b\mathcal{T}_m^n f(y), g(z)\right\| \leq \alpha_m^n h_1(x^p, z) h_2(y^p, z) \tag{4.13}$$

for every $x, y, z \in \mathbb{R}$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$. Since the case $n = 0$ is just (4.3), take $k \in \mathbb{N}$ and assume that (4.13) holds for $n = k$ and every $x, y, z \in \mathbb{R}$, $m \in \mathcal{U}$. Then, for each $x, y, z \in \mathbb{R}$ and $m \in \mathcal{U}$, we get

$$\begin{aligned}&\left\|\mathcal{T}_m^{k+1} f\left(\sqrt[p]{ax^p + by^p}\right) - a\mathcal{T}_m^{k+1} f(x) - b\mathcal{T}_m^{k+1} f(y), g(z)\right\| = \Big\|\frac{1}{a}\mathcal{T}_m^k f\left(\sqrt[p]{(a+bm^p)(ax^p+by^p)}\right) \\ &\qquad -\frac{b}{a}\mathcal{T}_m^k f\left(m\sqrt[p]{ax^p+by^p}\right) - \mathcal{T}_m^k f\left(\sqrt[p]{(a+bm^p)x^p}\right) + b\mathcal{T}_m^k f(mx) \\ &\qquad -\frac{b}{a}\mathcal{T}_m^k f\left(\sqrt[p]{(a+bm^p)y^p}\right) + \frac{b^2}{a}\mathcal{T}_m^k f(my), g(z)\Big\| \\ &\leq \frac{1}{a}\Big\|\mathcal{T}_m^k f\left(\sqrt[p]{(a+bm^p)(ax^p+by^p)}\right) - a\mathcal{T}_m^k f\left(\sqrt[p]{(a+bm^p)x^p}\right) - b\mathcal{T}_m^k f\left(\sqrt[p]{(a+bm^p)y^p}\right), g(z)\Big\| \\ &\qquad +\frac{b}{a}\Big\|\mathcal{T}_m^k f\left(m\sqrt[p]{ax^p+by^p}\right) - a\mathcal{T}_m^k f(mx) - b\mathcal{T}_m^k f(my), g(z)\Big\| \\ &\leq \frac{1}{a}\alpha_m^k h_1((a+bm^p)x^p, z) h_2((a+bm^p)y^p, z) + \frac{b}{a}\alpha_m^k h_1(m^p x^p, z) h_2(m^p y^p, z) \\ &\leq \alpha_m^k \Big(\frac{1}{a}\lambda_1(a+bm^p)\lambda_2(a+bm^p) + \frac{b}{a}\lambda_1(m^p)\lambda_2(m^p)\Big) h_1(x^p, z) h_2(y^p, z) \\ &= \alpha_m^{k+1} h_1(x^p, z) h_2(y^p, z).\end{aligned}$$

Thus, by induction, we have shown that (4.13) holds for every $x, y, z \in \mathbb{R}$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$. Letting $n \to \infty$ in (4.13), we obtain the equality

$$F_m\left(\sqrt[p]{ax^p + by^p}\right) = aF_m(x) + bF_m(y), \quad x, y \in \mathbb{R}, m \in \mathcal{U}. \tag{4.14}$$

This implies that $F_m : \mathbb{R} \to Y$, defined in this way, is a solution of the equation

$$F(x) = \frac{1}{a} F\left(\sqrt[p]{(a+bm^p)x^p}\right) - \frac{b}{a} F(mx), \quad x \in \mathbb{R}, m \in \mathcal{U}. \tag{4.15}$$

Next, we will prove that each function $F : \mathbb{R} \to Y$ satisfiesnthe equation (1.2)and the following inequality

$$\big\|f(x) - F(x), g(z)\big\| \le L\, h_1(x^p, z)h_2(x^p, z), \quad x \in \mathbb{R} \tag{4.16}$$

with some $L > 0$, is equal to F_m for each $m \in \mathcal{U}$. To this end, we fix $m_0 \in \mathcal{U}$ and $F : \mathbb{R} \to Y$ satisfying (4.16). From (4.12), for each $x, z \in \mathbb{R}$, we get

$$\begin{aligned}
&\big\|F(x) - F_{m_0}(x), g(z)\big\| \\
&\qquad \le \big\|F(x) - f(x), g(z)\big\| + \big\|f(x) - F_{m_0}, g(z)\big\| \\
&\qquad \le L\, h_1(x^p, z)h_2(x^p, z) + \varepsilon^*_{m_0}(x, z) \\
&\qquad \le L_0\, h_1(x^p, z)h_2(x^p, z) \sum_{n=0}^{\infty} \alpha^n_{m_0},
\end{aligned} \tag{4.17}$$

where

$$L_0 := (1 - \alpha_{m_0})L + \frac{1}{a}\lambda_2(m_0^p) > 0$$

and we exclude the case that $h_1(x^p, z) \equiv 0$ or $h_2(x^p, z) \equiv 0$ which is trivial. Observe that F and F_{m_0} are solutions to equation (4.15) for all $m \in \mathcal{U}$. Next, we show that, for each $j \in \mathbb{N}_0$, we have

$$\big\|F(x) - F_{m_0}(x), g(z)\big\| \le L_0\, h_1(x^p, z)h_2(x^p, z) \sum_{n=j}^{\infty} \alpha^n_{m_0}, \quad x, z \in \mathbb{R}. \tag{4.18}$$

The case $j = 0$ is exactly (4.17). We fix $k \in \mathbb{N}$ and assume that (4.18) holds for $j = k$. Then, in view of (4.17), for each $x, z \in \mathbb{R}$, we get

$$\begin{aligned}
\big\|F(x) - F_{m_0}(x), g(z)\big\| &= \Big\|\frac{1}{a}F\left(\sqrt[p]{(a + bm_0^p)x^p}\right) - \frac{b}{a}F(m_0x) \\
&\quad - \frac{1}{a}F_{m_0}\left(\sqrt[p]{(a + bm_0^p)x^p}\right) + \frac{b}{a}F_{m_0}(m_0x), g(z)\Big\| \\
&\le \frac{1}{a}\Big\|F\left(\sqrt[p]{(a + bm_0^p)x^p}\right) - F_{m_0}\left(\sqrt[p]{(a + bm_0^p)x^p}\right), g(z)\Big\| \\
&\quad + \frac{b}{a}\big\|F(m_0x) - F_{m_0}(m_0x), g(z)\big\| \\
&\le L_0\, \frac{1}{a}h_1((a + bm_0^p)x^p, z)h_2((a + bm_0^p)x^p, z) \sum_{n=k}^{\infty} \alpha^n_{m_0} \\
&\quad + \frac{b}{a}L_0\, h_1(m_0^p x^p, z)h_2(m_0^p x^p, z) \sum_{n=k}^{\infty} \alpha^n_{m_0} \\
&= L_0\, \Big(\frac{1}{a}h_1((a + bm_0^p)x^p, z)h_2((a + bm_0^p)x^p, z) \\
&\quad + \frac{b}{a}h_1(m_0^p x^p, z)h_2(m_0^p x^p, z)\Big) \sum_{n=k}^{\infty} \alpha^n_{m_0} \\
&\le L_0\, \alpha_{m_0} h_1(x^p, z)h_2(x^p, z) \sum_{n=k}^{\infty} \alpha^n_{m_0} \\
&= L_0\, h_1(x^p, z)h_2(x^p, z) \sum_{n=k+1}^{\infty} \alpha^n_{m_0}.
\end{aligned}$$

This shows that (4.18) holds for $j = k+1$. Now we can conclude that the inequality (4.18) holds for all $j \in \mathbb{N}_0$. Now, letting $j \to \infty$ in (4.18), we get

$$F = F_{m_0}. \tag{4.19}$$

Thus, we have also proved that $F_m = F_{m_0}$ for each $m \in \mathcal{U}$, which (in view of (4.12)) yields

$$\|f(x) - F_{m_0}(x), g(z)\| \le \frac{\lambda_2(m^p)h_1(x^p, z)h_2(x^p, z)}{a(1-\alpha_m)}, \quad x, z \in \mathbb{R}, \ m \in \mathcal{U}. \tag{4.20}$$

This implies (2.2) with $F = F_{m_0}$ and (4.19) confirms the uniqueness of F. □

By a similar way we can prove the following theorem.

Theorem 4.2. *Let $h : \mathbb{R}^2 \to \mathbb{R}_+$ be a function such that*

$$\mathcal{U} := \{n \in \mathbb{N} \colon \beta_n := \lambda(a + bn^p) + \lambda(n^p) < 1\} \neq \phi, \tag{4.21}$$

where

$$\lambda(n) := \inf\{t \in \mathbb{R}_+ \colon h(nx^p, z) \le t\, h(x^p, z), \quad x, z \in \mathbb{R}\} \tag{4.22}$$

for all $n \in \mathbb{N}$. Assume that $f : \mathbb{R} \to Y$ satisfies the inequality

$$\left\|f\left(\sqrt[p]{ax^p + by^p}\right) - af(x) - bf(y), g(z)\right\| \le h(x^p, z) + h(y^p, z) \tag{4.23}$$

for all $x, y, z \in \mathbb{R}$ where $g : X \to Y$ is a surjective mapping. Then there exists a unique function $F : \mathbb{R} \to Y$ that satisfies the equation (1.2) such that

$$\|f(x) - F(x), g(z)\| \le \lambda_0 h(x^p, z) \tag{4.24}$$

for all $x, z \in \mathbb{R}$, where

$$\lambda_0 := \inf_{n \in \mathcal{U}} \left\{\frac{1 + \lambda(n^p)}{a\big(1 - \lambda(a + bn^p) - \lambda(n^p)\big)}\right\}.$$

Proof. Replacing in (4.23) x by mx, where $x \in \mathbb{R}$ and $m \in \mathbb{N}$, we get

$$\left\|\frac{1}{a}f\left(\sqrt[p]{(a + bm^p)x^p}\right) - \frac{b}{a}f(mx) - f(x), g(z)\right\| \le \frac{1}{a}\big(1 + \lambda(m^p)\big)h(x^p, z) \tag{4.25}$$

for all $x, z \in \mathbb{R}$. For each $m \in \mathbb{N}$, we define

$$\mathcal{T}_m\xi(x) := \frac{1}{a}\xi\left(\sqrt[p]{(a + bm^p)x^p}\right) - \frac{b}{a}\xi(mx), \quad \xi \in Y^{\mathbb{R}}, x \in \mathbb{R}, \tag{4.26}$$

$$\Lambda_m\delta(x, z) := \frac{1}{a}\delta\left(\sqrt[p]{(a + bm^p)x^p}, z\right) + \frac{b}{a}\delta(mx, z), \quad \delta \in \mathbb{R}_+^{\mathbb{R}\times\mathbb{R}}, x, z \in \mathbb{R}, \tag{4.27}$$

$$\varepsilon_m(x, z) := \frac{1}{a}\big(1 + \lambda(m^p)\big)h(x^p, z), \quad x, z \in \mathbb{R}. \tag{4.28}$$

As in Theorem 4.1 we observe that (4.25) takes form

$$\|f(x) - \mathcal{T}_m f(x), g(z)\| \le \varepsilon_m(x, z), \quad x, z \in \mathbb{R} \tag{4.29}$$

and Λ_m has the form described in (2.13) and (2.12) is valid for every $\xi, \mu \in Y^{\mathbb{R}}$, $x, z \in \mathbb{R}$. It is not hard to show that

$$\big(\Lambda_m^n \varepsilon_m\big)(x, z) \le \frac{1}{a}\big(1 + \lambda(m^p)\big)h(x^p, z)\Big(\lambda(m^p) + \lambda(a + bm^p)\Big)^n \tag{4.30}$$

for all $x, z \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Therefore

$$\begin{aligned}
\varepsilon_m^*(x,z) &= \sum_{n=0}^{\infty} \big(\Lambda_m^n \varepsilon_m\big)(x,z) \\
&\leq \sum_{n=0}^{\infty} \frac{1}{a}\big(1+\lambda(m^p)\big)h(x^p,z)\Big(\lambda(m^p)+\lambda(a+bm^p)\Big)^n \\
&= \frac{\big(1+\lambda(m^p)\big)h(x^p,z)}{a\big(1-\lambda(m^p)-\lambda(a+bm^p)\big)} < \infty
\end{aligned}$$

for all $x, z \in \mathbb{R}$ and $m \in \mathcal{U}$. Also the remaining reasonings are analogous as in the proof of Theorem 4.1. □

The following theorem concerns the η-hyperstability of (1.2) in 2-Banach spaces. Namely, We consider functions $f : \mathbb{R} \to Y$ fulfilling (1.2) approximately, i.e., satisfying the inequality

$$\Big\|f\left(\sqrt[p]{ax^p+by^p}\right) - af(x) - bf(y), g(z)\Big\| \leq \eta(x,y,z), \quad x,y,z \in \mathbb{R}, \tag{4.31}$$

with $\eta : \mathbb{R}^3 \to \mathbb{R}_+$ is a given mapping. Then we find a unique function $F : \mathbb{R} \to Y$ satisfies the equation (1.2) which is close to f. Then, under some additional assumptions on η, we prove that the conditional functional equation (1.2) is η-hyperstable in the class of functions $f : \mathbb{R} \to Y$, i.e., each $f : \mathbb{R} \to Y$ satisfying inequality (4.31), with such η, must fulfil equation (1.2).

Theorem 4.3. *Let h_1, h_2 and $\mathcal{U}$ be as in Theorem 4.1. Assume that*

$$\begin{cases} \lim_{n\to\infty} \lambda_1(n)\lambda_2(n) = 0, \\ \lim_{n\to\infty} \lambda_2(n) = 0. \end{cases} \tag{4.32}$$

Then every $f : \mathbb{R} \to Y$ satisfying (4.3) is a solution of (1.2).

Proof. Suppose that $f : \mathbb{R} \to Y$ satisfies (4.3). Then, by Theorem 4.1, there exists a mapping $F : \mathbb{R} \to Y$ satisfies (1.2) and

$$\|f(x) - F(x), g(z)\| \leq \lambda_0 h_1(x^p,z)h_2(x^p,z) \tag{4.33}$$

for all $x, z \in \mathbb{R}$, where $g : X \to Y$ is a surjective mapping and

$$\lambda_0 := \inf_{n\in\mathcal{U}} \left\{ \frac{\lambda_2(n^p)}{a\big(1-\lambda_1(a+bn^p)\lambda_2(a+bn^p)-\lambda_1(n^p)\lambda_2(n^p)\big)} \right\}.$$

Since, in view of (4.32), $\lambda_0 = 0$. This means that $f(x) = F(x)$ for all $x \in \mathbb{R}$, whence

$$f\left(\sqrt[p]{ax^p+by^p}\right) = af(x) + bf(y), \quad x,y \in \mathbb{R}$$

which implies that f satisfies the functional equation (1.2) on $\mathbb{R}$. □

5 Some particular cases

According to above theorems, we derive some particular cases from our main results.

Corollary 5.1. *Let $h_1, h_2 : \mathbb{R}^2 \to (0,\infty)$ be as in Theorem 4.1 such that*

$$\lim_{n\to\infty} \inf \sup_{x,z\in\mathbb{R}} \frac{h_1\big((a+bn^p)x^p,z\big)h_2\big((a+bn^p)x^p,z\big) + h_1(n^px^p,z)h_2(n^px^p,z)}{h_1(x^p,z)h_2(x^p,z)} = 0. \tag{5.1}$$

Assume that $f : \mathbb{R} \to Y$ satisfies (1.2). Then there exist a unique function $F : \mathbb{R} \to Y$ satisfies the equation (1.2) and a unique constant $\kappa \in \mathbb{R}_+$ with

$$\Big\|f(x) - F(x), g(z)\Big\| \leq \kappa\, h_1(x^p,z)h_2(x^p,z), \quad x,z \in \mathbb{R}, \tag{5.2}$$

where $g : X \to Y$ is a surjective mapping.

Proof. By the definition of $\lambda_i(n)$ in Theorem 4.1, we observe that

$$\begin{aligned}\lambda_1(n^p)\lambda_2(n^p) &= \sup_{x,z\in\mathbb{R}} \frac{h_1(n^p x^p, z)h_2(n^p x^p, z)}{h_1(x^p, z)h_2(x^p, z)} \\ &\le \sup_{x,z\in\mathbb{R}} \frac{h_1\big((a+bn^p)x^p, z\big)h_2\big((a+bn^p)x^p, z\big) + h_1(n^p x^p, z)h_2(n^p x^p, z)}{h_1(x^p, z)h_2(x^p, z)} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\lambda_1(a+bn^p)\lambda_2(a+bn^p) &= \sup_{x,z\in\mathbb{R}} \frac{h_1\big((a+bn^p x^p, z)\big)h_2\big((a+bn^p x^p, z)\big)}{h_1(x^p, z)h_2(x^p, z)} \\ &\le \sup_{x,z\in\mathbb{R}} \frac{h_1\big((a+bn^p)x^p, z\big)h_2\big((a+bn^p)x^p, z\big) + h_1(n^p x^p, z)h_2(n^p x^p, z)}{h_1(x^p, z)h_2(x^p, z)} \end{aligned} \tag{5.4}$$

Combining inequalities (5.3) and (5.5), we get

$$\begin{aligned}&\lambda_1(n^p)\lambda_2(n^p) + \lambda_1(a+bn^p)\lambda_2(a+bn^p) \\ &\quad\le 2\sup_{x,z\in\mathbb{R}} \frac{h_1\big((a+bn^p)x^p, z\big)h_2\big((a+bn^p)x^p, z\big) + h_1(n^p x^p, z)h_2(n^p x^p, z)}{h_1(x^p, z)h_2(x^p, z)}. \end{aligned} \tag{5.5}$$

Write

$$\gamma_n := \sup_{x,z\in\mathbb{R}} \frac{h_1\big((a+bn^p)x^p, z\big)h_2\big((a+bn^p)x^p, z\big) + h_1(n^p x^p, z)h_2(n^p x^p, z)}{h_1(x^p, z)h_2(x^p, z)}.$$

From (5.1), there is a subsequence $\{\gamma_{n_k}\}$ of a sequence $\{\gamma_n\}$ such that $\lim_{k\to\infty}\gamma_{n_k} = 0$, that is,

$$\lim_{k\to\infty} \sup_{x,z\in\mathbb{R}} \frac{h_1\big((a+bn_k^p)x^p, z\big)h_2\big((a+bn_k^p)x^p, z\big) + h_1(n_k^p x^p, z)h_2(n_k^p x^p, z)}{h_1(x^p, z)h_2(x^p, z)} = 0. \tag{5.6}$$

From (5.5) and (5.6), we find that

$$\lim_{k\to\infty} \lambda_1(a+bn_k^p)\lambda_2(a+bn_k^p) + \lambda_1(n_k^p)\lambda_2(n_k^p) = 0. \tag{5.7}$$

This implies that

$$\lim_{k\to\infty} \lambda_1(n_k^p)\lambda_2(n_k^p) = 0$$

and hence

$$\lim_{k\to\infty} \frac{\lambda_2(n_k^p)}{a\big(1-\lambda_1(a+bn_k^p)\lambda_2(a+bn_k^p) - \lambda_1(n_k^p)\lambda_2(n_k^p)\big)} = \lim_{k\to\infty}\frac{\lambda_2(n_k^p)}{a} := \kappa$$

which means that λ_0 defined in Theorem 4.1 is equal to κ. □

By a similar method, we can prove the following corollary where $\kappa = \frac{1}{a}$.

Corollary 5.2. *Let $h : \mathbb{R}^2 \to (0,\infty)$ be as in Theorem 4.2 such that*

$$\lim_{n\to\infty} \inf \sup_{x,z\in\mathbb{R}} \frac{h\big((a+bn^p)x^p, z\big) + h(n^p x^p, z)}{h(x^p, z)} = 0. \tag{5.8}$$

Assume that $f : \mathbb{R} \to Y$ satisfies (1.2). Then there exist a unique function $F : \mathbb{R} \to Y$ satisfies the equation (1.2) such that

$$\big\|f(x) - F(x), g(z)\big\| \le \frac{1}{a}h(x^p, z), \quad x, z \in \mathbb{R}, \tag{5.9}$$

where $g : X \to Y$ is a surjective mapping.

Corollary 5.3. *Let $\theta \geq 0$, $r \geq 0$, $s, t \in \mathbb{R}$ such that $s + t < 0$. Suppose that $f : \mathbb{R} \to Y$ such that $f(0) = 0$ and satisfies the inequality*

$$\left\| f\left(\sqrt[p]{ax^p + by^p}\right) - af(x) - bf(y), g(z)\right\| \leq \theta |x|^{ps} \, |y|^{pt} \, |z|^r, \quad x, y, z \in \mathbb{R}\backslash\{0\} \tag{5.10}$$

such that $g : X \to Y$ is a surjective mapping with $|a| > 1$ and $|b| > 1$, where p is odd, or f is even function satisfies (5.10) where p is even integer. Then f satisfies (1.2) on $\mathbb{R}$.

Proof. The proof follows from Theorem 4.1 by defining
$h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}_+$ by $h_1(x, z) = \theta_1 |x|^s |z|^{r_1}$ and $h_2(y, z) = \theta_2 |y|^t |z|^{r_2}$, with $\theta_1, \theta_2, r_1, r_2 \in \mathbb{R}_+$ and $s, t, \in \mathbb{R}$ such that $\theta_1\theta_2 = \theta$, $r_1 + r_2 = r$ and $s + t < 0$.
For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \lambda_1(n) &= \inf\left\{t \in \mathbb{R}_+ : h_1(nx^p, z) \leq t\, h_1(x^p, z), \ \ x, z \in \mathbb{R}\right\} \\ &= \inf\left\{t \in \mathbb{R}_+ : \theta_1 |nx^p|^s |z|^{r_1} \leq t\, \theta_1 |x^p|^s |z|^{r_1}, \ \ x, z \in \mathbb{R}\backslash\{0\}\right\} \\ &= n^s. \end{aligned}$$

Also, we have $\lambda_2(n) = n^t$ for all $n \in \mathbb{N}$. Clearly, we can find $n_0 \in \mathbb{N}$ such that

$$\lambda_1(a + bn^p)\lambda_2(a + bn^p) + \lambda_1(n^p)\lambda_2(n^p) = (a + bn^p)^{s+t} + (n^p)^{s+t} < 1, \quad n \geq n_0. \tag{5.11}$$

According to Theorem 4.1, there exists a unique function $F : \mathbb{R} \to Y$ satisfies the equation (1.2) such that

$$\left\| f(x) - F(x), g(z)\right\| \leq \theta\lambda_0 h_1(x^p, z) h_2(x^p, z) \tag{5.12}$$

for all $x, z \in \mathbb{R}$, where

$$\lambda_0 := \inf_{n \geq n_0} \left\{ \frac{\lambda_2(n^p)}{a\big(1 - \lambda_1(a + bn^p)\lambda_2(a + bn^p) - \lambda_1(n^p)\lambda_2(n^p)\big)} \right\}.$$

On the other hand, Since $s + t < 0$, one of s, t must be negative. Assume that $t < 0$. Then

$$\begin{cases} \lim_{n\to\infty} \lambda_1(n)\lambda_2(n) = \lim_{n\to\infty} n^{s+t} = 0, \\ \lim_{n\to\infty} \lambda_2(n) = \lim_{n\to\infty} n^t = 0. \end{cases} \tag{5.13}$$

Thus by Theorem 4.3, we get the desired results. □

The next corollary prove the hyperstability results for the inhomogeneous radical functional equation.

Corollary 5.4. *Let $\theta, s, t, r \in \mathbb{R}$ such that $\theta \geq 0$ and $s + t < 0$. Assume that $G : \mathbb{R}^2 \to Y$ and $f : \mathbb{R} \to Y$ satisfy the inequality*

$$\left\| f\left(\sqrt[p]{ax^p + by^p}\right) - af(x) - bf(y) - G(x, y), g(z)\right\| \leq \theta |x|^{ps} \, |y|^{pt} \, |z|^r, \quad x, y, z \in \mathbb{R}\backslash\{0\} \tag{5.14}$$

such that $g : X \to Y$ is a surjective mapping, where p is odd, or f is even function satisfies (5.14) where p is even integer. If the functional equation

$$f\left(\sqrt[p]{ax^p + by^p}\right) = af(x) + bf(y) + G(x, y), \quad x, y \in \mathbb{R}\backslash\{0\} \tag{5.15}$$

has a solution $f_0 : \mathbb{R} \to Y$, then f is a solution to (5.15).

Proof. From (5.14) we get that the function $K : \mathbb{R} \to Y$ defined by $K := f - f_0$ satisfies (5.10). Consequently, Corollary 5.3 implies that K is a solution to radical functional equation (1.2). Therefore,

$$\begin{aligned} f\left(\sqrt[p]{ax^p + by^p}\right) - af(x) - bf(y) - G(x, y) &= K\left(\sqrt[p]{ax^p + by^p}\right) + f_0\left(\sqrt[p]{ax^p + by^p}\right) \\ &\quad - aK(x) - af_0(x) - bK(y) - bf_0(y) - G(x, y) \\ &= 0, \quad x, y \in \mathbb{R}\backslash\{0\}. \end{aligned}$$

which means f is a solution to (5.15).

□

References

[1] M. Almahalebi, A. Charifi and S. Kabbaj, *Hyperstability of a monomial functional equation*, Journal of Scientific Research Reports, **3** (20), 2685-2693 (2014).

[2] M. Almahalebi and S. Kabbaj, *Hyperstability of a Cauchy-Jensen type functional equation*, Advances in Research, **2** (12), 1017-1025 (2014).

[3] M. Almahalebi, A. Charifi and S. Kabbaj, *Hyperstability of a Cauchy functional equation*, Journal of Nonlinear Analysis and Optimization: Theory Applications, Vol.6, No.2, 127-137 (2015).

[4] M. Almahalebi and C. Park, *On the hyperstability of a functional equation in commutative groups*, Journal of Computational Analysis Applications, **20** (1), 826-833 (2016).

[5] M. Almahalebi and A. Chahbi, Hyperstability of the Jensen functional equation in ultrametric spaces, Aequat. Math., **91** 4, 647661 (2017).

[6] M. Almahalebi, A. Chahbi, Approximate solution of p-radical functional equation in 2-Banach spaces, Acta Math. Scientia., **39** (2), 551-566 (2019).

[7] L. Aiemsomboon and W. Sintunavarat, *On a new type of stability of a radical quadratic functional equation using Brzdęks fixed point theorem*, Acta Math. Hungar. (2017) 151: 35. https://doi.org/10.1007/s10474-016-0666-2.

[8] L. Aiemsomboon and W. Sintunavarat, *On generalized hyperstability of a general linear equation*, Acta Math. Hungar., 149, 413-422 (2016).

[9] Z. Alizadeh and A. G. Ghazanfari, *On the stability of a radical cubic functional equation in quasi-β-spaces*, J. Fixed Point Theory Appl. (2016) 18: 843. https://doi.org/10.1007/s11784-016-0317-9.

[10] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** , 64-66 (1950).

[11] D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc., **57**, 223-237 (1951).

[12] J. Brzdęk, *A note on stability of additive mappings, in: Stability of Mappings of Hyers-Ulam Type, Rassias, T.M., Tabor, J. (eds.)*, Hadronic Press (Palm Harbor, 1994), 19-22 (1994).

[13] J. Brzdęk, J. Chudziak and Zs. Páles, *A fixed point approach to stability of functional equations*, Nonlinear Anal., **74**, 6728-6732 (2011).

[14] J. Brzdęk and K. Ciepliński, *A fixed point approach to the stability of functional equations in non-Archimedean metric spaces*, Nonlinear Anal., **74**, 6861-6867 (2011).

[15] J. Brzdęk, *Stability of additivity and fixed point methods*, Fixed Point Theory Appl., 2013, 2013:265, pp.9.

[16] J. Brzdęk, *Hyperstability of the Cauchy equation on restricted domains*, Acta Math. Hungar., **141**, 58-67 (2013).

[17] J. Brzdęk, L. Cadăriu and K. Ciepliński, *Fixed point theory and the Ulam stability*, J. Funct. Spaces, 2014 (2014), Article ID 829419, pp. 16.

[18] J. Brzdęk, W. Fechner, M. S. Moslehian and J. Sikorska, *Recent developments of the conditional stability of the homomorphism equation*, Banach J. Math. Anal., **9**, 278-327 (2015).

[19] J. Brzdęk,*Remark 3, In: Report of Meeting of 16th International Conference on Functional Equations and Inequalities (Będlewo, Poland, May 17-23, 2015), p. 196*, Ann. Univ. Paedagog. Crac. Stud. Math. **14**, 163-202 (2015).

[20] J. Brzdęk, *Remarks on solutions to the functional equations of the radical type*, Advances in the Theory of Nonlinear Analysis and its Application, **1**, 125-135 (2017).

[21] M. Eshaghi Gordji and M. Parviz, *On the HyersUlam stability of the functional equation* $f\left(\sqrt[2]{x^2+y^2}\right)=f(x)+f(y)$, Nonlinear Funct. Anal. Appl., **14**, 413-420 (2009).

[22] M. Eshaghi Gordji, H. Khodaei, A. Ebadian and G. H. Kim, *Nearly radical quadratic functional equations in p-2-normed spaces*, Abstr. Appl. Anal. 2012 (2012), Article ID 896032.

[23] S. Gähler, *2-metrische Räume und ihre topologische Struktur,* Math. Nachr. **26**, 115-148 (1963).

[24] S. Gähler, *Linear 2-normiete Räumen,* Math. Nachr. **28**, 1-43 (1964).

[25] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184**, 431-436 (1994).

[26] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A., **27**, 222-224 (1941).

[27] H. Khodaei, M. Eshaghi Gordji, S. S. Kim and Y. J. Cho, *Approximation of radical functional equations related to quadratic and quartic mappings*, J. Math. Anal. Appl. 395, 284-297 (2012).

[28] S. S. Kim, Y. J. Cho and M. Eshaghi Gordji, *On the generalized Hyers-Ulam-Rassias stability problem of radical functional equations*, J. Inequal. Appl. 2012, **2012**:186.

[29] W. -G. Park, *Approximate additive mappings in 2-Banach spaces and related topics,* J. Math. Anal. Appl. **376**, 193-202 (2011).

[30] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**, 297-300 (1978).

[31] Th. M. Rassias, *Problem 16; 2. Report of the 27th international symposium on functional equations*, Aequationes Math., **39**, 292-293 (1990).

[32] Th. M. Rassias, *On a modified Hyers-Ulam sequence*, J. Math. Anal. Appl., **158**, 106-113 (1991).

[33] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, John-Wiley & Sons Inc., New York, (1964).

Author information

Sadeq AL-Ali, Mustapha E. Hryrou, Youssfi Elkettani, Department of Mathematics, Faculty of Sciences, University of Ibn Tofail, BP 133 Kenitra, Morocco.
E-mail: `sadeqalali2018@gmail.com; hryrou.mustapha@hotmail.com; elkettani@ui-ibntofail.ac.ma`

Mohammed A. Salman, Department of Mathematics and statistics, Education and Languages College, Amran University, Yemen.
E-mail: `masalman@gmail.com`

Received: August 16, 2020.
Accepted: October 7, 2020.