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Matrix characterization of Asymptotically Probability Equivalent for Measurable Functions

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Abstract In this article, we put forward new ideas of asymptotically equivalent functions in probability and asymptotic regular matrices in probability. Two nonnegative real-valued function $f(s)$ and $g(s)$, measurable on $(1,\infty)$ are said to be asymptotically probability equivalent if for every $\varepsilon > 0$,

$$\lim_s P\left(\left|\frac{f(s)}{g(s)} - 1\right| < \varepsilon\right) = 1$$

Also, we examine bivariate function transformation of asymptotically equivalent in probability for measurable functions.

1 Introduction

The definitions of asymptotically equivalent sequences and asymptotic regular matrices which preserve the asymptotic equivalence of two nonnegative number sequences were presented by Pobyvanets [3] and new ways of comparing rates of convergence was introduced by Fridy [1]. Furthermore, the definitions for asymptotically statistical equivalent sequences and asymptotically statistical regular matrices was presented by Patterson [4] and this concept was deal with probability theory by Patterson and Savas [5].

In order to present the main results, we first consider some definitions.

Definition 1.1. [1] For $y = (y_r)$ in $\acute{l}$, where $\acute{l} = \left\{ y_r : \sum_{r=1}^{\infty} |y_r| < \infty \right\}$, the "remainder sequence" $[Ry]$ is defined to be $(R_m y)$ where, for each $m \in \mathbb{N}$,

$$R_m y := \sum_{m}^{\infty} |y_r| .$$

Definition 1.2. [5] Two nonnegative sequences $y = (y_r)$ and $z = (z_r)$ are said to be asymptotically equivalent in probability if

$$\lim_r P\left(\left|\frac{y_r}{z_r} - 1\right| < \varepsilon\right) = 1.$$

(denoted by $y \overset{probability}{\sim} z$)

The main goal of this paper is to introduce the ideas of asymptotically probability equivalent and asymptotically regular in probability by using $g(s)$ and $h(s)$ two nonnegative real-valued Lebesgue measurable functions on $(1,\infty)$ instead of sequences. In addition, we shall also prove some of inclusion theorems.

2 Main Definitions

In this section, we will present useful main definitions. Let us note the following notations:

$L^f = \left\{ g(s) : \int_1^\infty |g(s)|\, ds < \infty \right\},$
$D_A = \left\{ g(s) : \lim_t \int_1^\infty \tau(t,s) g(s) ds \text{ exists, where } \tau(t,s) \text{ is a bivariate function} \right\},$
$P_\delta^f = \{$The class of all real valued lebesgue measurable functions such that $g(s) \geq \delta > 0$ for all $s\}$,
and
$P_0^f = \{$The class of all real valued measurable functions which have at most a finite number of zero values$\}$.

Definition 2.1. Let $g(s)$ be a real-valued function which is measurable on $(1, \infty)$. If $g(s)$ is a L^f, then we denote the remainder function given by

$$R_t(g) := \int_t^\infty |g(s)|\, ds.$$

Definition 2.2. Let $g(s)$ be a real-valued function which is measurable on $(1, \infty)$, if $R_t(\tau(t,s)g(s))$ is a L^f, then we denote the remainder function given by

$$R_t(\tau(t,s)g(s)) := \int_t^\infty |\varphi g(s)|\, ds$$

when $(\varphi g)_t = \int_1^\infty \tau(t,s) g(s) ds$, in short $(\varphi g)_t$.

Definition 2.3. Two nonnegative real-valued function $g(s)$ and $h(s)$, measurable on $(1, \infty)$ are said to be asymptotically equivalent if

$$\lim_s \frac{g(s)}{h(s)} = 1$$

(denoted by $g \overset{F}{\sim} h$).

Definition 2.4. Two nonnegative real-valued function $g(s)$ and $h(s)$, measurable on $(1, \infty)$ are said to be asymptotically probability equivalent of multiple ξ provided that for every $\varepsilon > 0$,

$$\lim_s P\left(\left| \frac{g(s)}{h(s)} - \xi \right| < \varepsilon \right) = 1$$

(denoted by $g \overset{probability}{\sim} h$), and simply asymptotically probability equivalent if $\xi = 1$.

Definition 2.5. A bivariate function $\varphi = \tau(t,s)$ is asymptotically regular in probability provided that $\varphi g \overset{probability}{\sim} \varphi h$ whenever $g \overset{probability}{\sim} h$, $g(s) \in P_0^f$, and $h(s) \in P_\delta^f$ for $\delta > 0$.

3 Main Results

In the following theorem, we present necessary and sufficient conditions on the entries of a summability bivariate functions to ensure that the bivariate function transformation will conserve asymptotically equivalent in probability of multiple ξ for Lebesgue measurable functions.

Theorem 3.1. *If $\varphi = \tau(t,s)$ is a bivarite function that maps bounded functions $g(s)$ into L^f, then the followings are equivalent:*

(i) If $g(s)$ and $h(s)$ are bounded functions such that $g \overset{probability}{\sim} h$, $g(s) \in P_0^f$, and $h(s) \in P_\delta^f$ for some $\delta > 0$, then

$$R_t(\varphi g) \overset{probability}{\sim} R_t(\varphi h).$$

(ii)

$$\lim_{t\to\infty} P\left(\left|\frac{\int\limits_{m=t}^{\infty}\tau(m,p)dm}{\int\limits_{m=t}^{\infty}\int\limits_{n=1}^{\infty}\tau(m,n)dndm} < \varepsilon\right|\right) = 1 \text{ for each } p \text{ and } \varepsilon > 0.$$

Proof. The definition for asymptotically equivalent in probability of multiple ξ can be defined as the following

$$\lim_{s} P\left(\left|\frac{g(s)}{h(s)} - \xi\right| < \varepsilon\right) = 1$$

This implies that

$$\lim_{s} P\left(\left(\xi-\varepsilon\right)h(m) < g(m) \leq (\xi+\varepsilon)h(m)\right) = 1. \tag{3.1}$$

We now consider the following

$$\frac{R_t(\varphi g)}{R_t(\varphi h)} \leq \frac{\int\limits_{1}^{N-1} g(n) \int\limits_{m=t}^{\infty} \sup\limits_{0\leq n\leq N-1} \tau(m,n)dmdn}{\int\limits_{t}^{\infty}\int\limits_{1}^{\infty}\tau(m,n)h(n)dndm} + \frac{\int\limits_{t}^{\infty}\int\limits_{N}^{\infty}\tau(m,n)g(n)dndm}{\int\limits_{t}^{\infty}\int\limits_{1}^{\infty}\tau(m,n)h(n)dndm}$$

and we consider lower bound as follows:

$$\begin{aligned} R_t(\varphi g) &= \int\limits_{t}^{\infty}\int\limits_{1}^{\infty}\tau(m,n)dndm \\ &\leq \int\limits_{m=t}^{\infty} \inf_{0\leq n\leq N-1}\tau(m,n)dmdn \int\limits_{1}^{N-1} g(n) + \int\limits_{m=t}^{\infty}\int\limits_{N}^{\infty}\tau(m,n)dndm. \end{aligned}$$

From (3.1) we are granted the following

$$\frac{R_t(\varphi g)}{R_t(\varphi h)} \leq \frac{\int\limits_{n=1}^{N-1} g(n) \int\limits_{m=t}^{\infty} \sup\limits_{0\leq n\leq N-1} \tau(m,n)dmdn}{\delta\int\limits_{m=t}^{\infty}\int\limits_{1}^{\infty}\tau(m,n)dndm} + (\xi+\varepsilon) \text{ in probability}$$

and

$$\begin{aligned} \frac{R_t(\varphi g)}{R_t(\varphi h)} \geq &\left(\frac{\int\limits_{n=1}^{N-1} g(n)dn}{\sup\limits_{n} h(n)}\right)\left(\frac{\int\limits_{m=t}^{\infty}\left(\inf\limits_{0\leq n\leq N-1}\tau(m,n)\right)dm}{\int\limits_{m=t}^{\infty}\int\limits_{n=1}^{\infty}\tau(m,n)dndm}\right) + (\xi+\varepsilon) \\ &- \left(\frac{(\xi-\varepsilon)\int\limits_{1}^{N-1} h(n)dn}{\delta}\right)\left(\frac{\int\limits_{m=t}^{\infty}\left(\max\limits_{0\leq n\leq N-1}\tau(m,n)\right)dm}{\int\limits_{m=t}^{\infty}\int\limits_{1}^{\infty}\tau(m,n)dndm}\right) \text{ in probability.} \end{aligned}$$

Therefore, from the last two equations we get the following

$$\lim_{t} P\left(\left|\frac{R_t(\varphi g)}{R_t(\varphi h)} - \xi\right| < \varepsilon\right) = 1$$

Hence,

$$R_t(\varphi g) \overset{probability}{\sim} R_t(\varphi h).$$

For the second part of this theorem, we consider the following two functions,

$$g(m) := \begin{cases} 0, m \leq p; \\ 1, \text{otherwise} \end{cases}$$

where $p \in \mathbb{Z}^+$ and $h(m) = 1$ for all m. These two functions imply as follows:

$$\begin{aligned} R_t(\varphi g) &= \int_{m=t}^{\infty} \int_{n=p+1}^{\infty} \tau(m,n)dndm \\ &= \int_{m=t}^{\infty} \int_{n=1}^{\infty} \tau(m,n)dndm - \int_{m=t}^{\infty} \int_{n=1}^{p} \tau(m,n)dndm. \end{aligned}$$

Therefore

$$\liminf_t \frac{R_t(\varphi g)}{R_t(\varphi h)} \leq 1 - \lim \sup_{t\to\infty} \frac{\int_{m=t}^{\infty} \tau(m,p)dm}{\int_{m=t}^{\infty} \int_{n=1}^{\infty} \tau(m,n)dndm}$$

Since each nonconstant element of the last inequality has statistically limit zero, We have the following

$$\lim_t P\left(\left|\frac{R_t(\varphi g)}{R_t(\varphi h)} - 1\right| < \varepsilon\right) = 0.$$

□

Theorem 3.2. *In order for bivariate function $A = \tau(t,s)$ to be asymptotically probability regular it is necessary and sufficient that for each fixed positive integer s_0*

1. $\int_1^{s_0} \tau(m,t)dm$ is bounded for each m.
2. $$\lim_t P\left(\left|\frac{\int_{t=1}^{s_0} \tau(m,t)dm}{\int_1^{\infty} \tau(m,t)dm} < \varepsilon\right|\right) = 1$$

Proof. The necessary part of above theorem is similarly established as necessary part of the previous theorem. To construct the sufficient part, let $\varepsilon > 0$, $g \overset{probability}{\sim} h$, $g(s) \in P_0^f$ and $h(s) \in P_\delta^f$ for some $\delta > 0$, we can write

$$P\left((\xi - \varepsilon)h(m+\alpha) \leq g(m+\alpha) \leq (\xi + \varepsilon)h(m+\alpha)\right) = 1. \tag{3.2}$$

Let us consider the following

$$\begin{aligned} \frac{(\varphi g)_t}{(\varphi h)_t} &= \frac{\int_{t=1}^{\alpha} \tau(m,t)g(t)dt + \int_{t=\alpha+1}^{\infty} \tau(m,t)g(t)dt}{\int_{t=1}^{\alpha} \tau(m,t)h(t)dt + \int_{t=\alpha+1}^{\infty} \tau(m,t)h(t)dt} \\ &= \frac{\frac{\int_{t=1}^{\alpha} \tau(m,t)g(t)dt}{\int_{t=\alpha+1}^{\infty} \tau(m,t)h(t)dt} + \frac{\int_{t=\alpha+1}^{\infty} \tau(m,t)g(t)dt}{\int_{t=\alpha+1}^{\infty} \tau(m,t)h(t)dt}}{\frac{\int_{t=1}^{\alpha} \tau(m,t)h(t)dt}{\int_{t=\alpha+1}^{\infty} \tau(m,t)h(t)dt} + 1} \end{aligned}$$

Inequality (3.2) implies that

$$\lim_{m=t} P\left(\left|\frac{\int\limits_{t=\alpha+1}^{\infty}\tau(m,t)g(t)dt}{\int\limits_{t=\alpha+1}^{\infty}\tau(m,t)h(t)dt}-\xi\right|<\varepsilon\right)=1.$$

Since $g(s)\in P_0^f$, $h(s)\in P_\delta^f$, and condition (2) holds we obtain the following

$$\lim_{t} P\left(\left|\frac{\int\limits_{t=1}^{\alpha}\tau(m,t)g(t)dt}{\int\limits_{t=\alpha+1}^{\infty}\tau(m,t)h(t)dt}\right|<\varepsilon\right)=1,$$

and

$$\lim_{t} P\left(\frac{\int\limits_{t=1}^{\infty}\tau(m,t)h(t)dt}{\int\limits_{t=\alpha+1}^{\infty}\tau(m,t)h(t)dt}<\varepsilon\right)=1.$$

Thus

$$\lim_{t} P\left(\left|\frac{(\varphi g)_t}{(\varphi h)_t}-\xi\right|<\varepsilon\right)=1.$$

This implies that $(\varphi g)\overset{probability}{\sim}(\varphi h)$ whenever $g\overset{probability}{\sim}h$, $g\in P_0^f$ and $h\in P_\delta^f$ for some $\delta>0$. □

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