Palestine Journal of Mathematics

Vol. 11(1)(2022) , 613–622

SOME LINEAR PROCESSES FOR FOURIER SERIES AND BEST APPROXIMATIONS OF FUNCTIONS IN MORREY SPACES

Sadulla Z. JAFAROV

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 41A10, 41A25; Secondary 42A05, 42A10.

Keywords and phrases: Morrey space, best approximation, trigonometric polynomials, Fejér mean, Zygmund mean, Abel-Poisson mean.

The author would like to thank the referee for his/her valuable comments and suggestions.

Abstract In the present work we estimate of deviations of periodic functions from linear operators constructed on basis of its Fourier series in terms of the best approximation of these functions in Morrey space. Specifically, we study the problem of the effect of metric of space on order of change of deviations.

1 Introduction

Let $\mathbb{T}$ denote the interval $[0, 2\pi]$. Let $L^p(\mathbb{T})$, $1 \leq p < \infty$ be the Lebesgue space of all measurable $2\pi-$periodic functions defined on $\mathbb{T}$ such that

$$\|f\|_p := \left(_{\mathbb{T}} |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty.$$

The *Morrey spaces* $L_0^{p,\lambda}(\mathbb{T})$ for a given $0 \leq \lambda \leq 2$ and $p \geq 1$,we define as the set of functions $f \in L_{loc}^p(\mathbb{T})$ such that

$$\|f\|_{L_0^{p,\lambda}(\mathbb{T})} := \left\{ \sup_I \frac{1}{|I|^{1-\frac{\lambda}{2}}} {}_I |f(t)|^p \, dt \right\}^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all intervals $I \subset [0, 2\pi]$. Note that $L_0^{p,\lambda}(\mathbb{T})$ becomes a Banach spaces, $\lambda = 2$ coincides with $L^p(\mathbb{T})$ and for $\lambda = 0$ with $L^\infty(\mathbb{T})$. If $0 \leq \lambda_1 \leq \lambda_2 \leq 2$, then $L_0^{p,\lambda_1}(\mathbb{T}) \subset L_0^{p,\lambda_2}(\mathbb{T})$. Also, if $f \in L_0^{p,\lambda}(\mathbb{T})$, then $f \in L^p(\mathbb{T})$ and hence $f \in L^1(\mathbb{T})$.The Morrey spaces, were introduced by C. B. Morrey in 1938. The properties of these spaces have been investigated intensively by several authors and together with weighted Lebesgue spaces L_ω^p play an important role in the theory of partial equations, in the study of local behavior of the solitions of elliptic differential equations and describe local reqularity more precisely than Lebesgue spaces L^p. The detailed information about properties of the Morrey spaces can be found in [11-13], [17], [22], [31], [32], [40], [42], [44]. and [47].

In what follows by $L^{p,\lambda}(\mathbb{T})$ we denote the closure of the linear supspace of $L_0^{p,\lambda}(\mathbb{T})$ functions, whose shifts are continuous in $L_0^{p,\lambda}(\mathbb{T})$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x; f), \ A_k(x; f) := a_k(f) \cos kx + b_k(f) \sin kx \tag{1.1}$$

be the Fourier series of the function $f \in L_1(\mathbb{T})$, where $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function f. The nth *partial sums* of the series (1) is defined as:

$$S_n(x;\ f) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k(x; f).$$

We consider the sequence of the functions $\{\lambda_k(r)\}$ defined in the set E of the number line, satisfying the conditions that

$$\lambda_0(r) = 1,\ \lim_{r \longrightarrow r_0} \lambda_\nu(r) = 1$$

for an arbitrary fixed $\nu = 0, 1, 2, ...$

For an arbitrary $r \in E$ and for every function $f \in L^{p,\lambda}(\mathbb{T})$, $0 \leq \lambda \leq 2$ and $p \geq 1$ the series

$$U(\ f;\ x\,;\lambda) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \lambda_k(r)\ A_k(x;\ f) \tag{1.2}$$

converges in the space $L^{p,\lambda}(\mathbb{T})$, $0 \leq \lambda \leq 2$ and $p \geq 1$

For each linear operator $U_r(\ f;\ x\,;\ \lambda)$ we set

$$R_r(f;\ \lambda)_{L^{p,\lambda}(\mathbb{T})} := \|f - U_r(f;\ x;\ \lambda)\|_{L^{p,\lambda}(\mathbb{T})}.$$

Let $r = 0, 1, 2, ...,$ if we substitute the followings

$$\lambda_\nu(r) = \begin{cases} 1 - \frac{\nu}{r+1}, & 0 \leq \nu \leq r, \\ 0, & \nu > r, \end{cases} \tag{1.3}$$

$$\lambda_\nu(r) = \begin{cases} 1 - \frac{\nu^k}{(r+1)^k}, & 0 \leq \nu \leq r, \\ 0, & \nu > r, \end{cases} \tag{1.4}$$

where $k \geq 1$,

$$\lambda_\nu(r) = r^\nu,\ (\nu = 0, 1, 2, ...),\ (0 \leq r < 1) \tag{1.5}$$

into (1.2) we obtain *Fejér means, Zygmund means of order* k *and Abel-Poisson means* of the series (1.1) respectively.

We denote by $E_n(f)_{L^{p,\lambda}(\mathbb{T})}$ the best approximation of $f \in L^{p,\lambda}(\mathbb{T})$, $0 \leq \lambda \leq 2$ and $p \geq 1$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n(f)_{L^{p,\lambda}(\mathbb{T})} = \inf\left\{\|f - T_n\|_{L^{p,\lambda}(\mathbb{T})} : T_n \in \Pi_n\right\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

We use the constants $c, c_1, c_2, ...$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest.

Note that the density of polynomials is an indispensable condition in approximation problems. Therefore, the polynomials are dense in Morrey spaces $L^{p,\lambda}(\mathbb{T})$, $0 < \lambda \leq 2$ *and* $1 < p < \infty$.

The problems of approximation theory in the weighted and non-weighted Morrey spaces have been investigated by several authors (see, for example, [3-7], [17], [18], [20] and [33]).

In the present paper we investigate the problems of estimating the deviation of the functions from the linear operators constructed on the basis of its Fourier series in terms of the best approximation of these functions in Morrey spaces. Obtained results show that the estimates of $R_r(f\,;\,\lambda)_{L^{p,\lambda}(\mathbb{T})}$ depends on both the rate of decrease of the sequence $\left\{E_n(f)_{L^{p,\lambda}(\mathbb{T})}\right\}$ and in some cases the metric of the considered space. This is valid for the upper and lower estimates of the quantity $R_r(f\,;\,\lambda)_{L^{p,\lambda}(\mathbb{T})}$. The similar problems of the approximation theory in the different spaces were investigated in [1], [2], [8-10],[14-16], [23-29],[34-36], [38], [39], [41], [43], [45], [46] and [48].

2 Main Results

Using the proof method in Marcinkiewicz- Multiplier Therem in weighted Lebesgue spaces [30, Theorem 2] , the following theorem can be prowed in Morrey spaces $L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$ *and* $1<p<\infty$.

Theorem 2.1. *Let a sequence χ_k satisfy the conditions*

$$|\chi_k|\leq c_1,\ \sum_{k=2^{j-1}}^{2^j-1}|\chi_k-\chi_{k+1}|\leq c_2 \tag{2.1}$$

where $A>0$ does not depend on k and j. If $f\in L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$ and $1<p<\infty$ has the Fourier series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}A_k(x;f)$$

there exists a function $F\in L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$ and $1<p<\infty$ with the Fourier series

$$\frac{\lambda_0 a_0}{2}+\sum_{k=1}^{\infty}\lambda_k A_k(x;f)$$

and

$$\| F\|_{L^{p,\lambda}(\mathbb{T})}\leq c_3\| f\|_{L^{p,\lambda}(\mathbb{T})}, \tag{2.2}$$

where $c_3>0$ does not depend on $f\in L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$ and $1<p<\infty$.

Also, using the proof scheme developed in [30, Theorem 1] and [37] we can prove the following theorem related to the Littlewood -Paley inequality in the Morrey Spaces $L^{p,\lambda}(0,2\pi)$, $0<\lambda\leq 2$ and $1<p<\infty$.

Theorem 2.2. *Let $f\in L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$ and $1<p<\infty$. Then there exist constants $c_4>0$ and $c_5>0$ such that*

$$c_4\| f\|_{L^{p,\lambda}(\mathbb{T})}\leq \left\| \left(\sum_{j=0}^{\infty}\left|\sum_{k=2^{j-1}}^{2^j-1}A_k(x,f)\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p,\lambda}(\mathbb{T})}\leq c_5\| f\|_{L^{p,\lambda}(\mathbb{T})}\,. \tag{2.3}$$

Our main results are the following.

Theorem 2.3. *Let $\{\lambda_\nu(r)\}$ be an arbitrary triangular matrix*
$(r=0,1,2,3,...;\ \lambda_0(r)=1;\ \lambda_\nu(r)=0,\ \nu>r)$. Let $f\in L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$ *and* $1<p<\infty$ *then the following inequality holds*:

$$R_r(f;\ \lambda)_{L^{p,\lambda}(\mathbb{T})}\leq c_6\{(1+K_r)E_r(f)_{L^{p,\lambda}(\mathbb{T})}+\sum_{v=0}^{m-1}\delta(2^{\nu+1};r)\ E_{2^\nu-1}(f)_{L^{p,\lambda}(\mathbb{T})}$$

$$+\delta(r;\ r)\ E_{2^m}(f)_{L^{p,\lambda}(\mathbb{T})},\} \tag{2.4}$$

where $2^m\leq r<2^{m+1}$, c_6 is a constant not depending on r,

$$K_r=\frac{2}{\pi}\int_0^{\pi}\left|\frac{1}{2}+\sum_{\nu=1}^{r}\lambda_\nu(r)\cos\nu\theta\right|d\theta,$$

$$\delta(\mu;r)=\int_0^{\pi}\left|\frac{1-\lambda_\mu(r)}{2}+-\sum_{\nu=1}^{\mu-1}\{1-\lambda_\nu(r)\}\cos\nu\theta\right|d\theta,\ \mu\leq r. \tag{2.5}$$

Proof. We consider the trigonometric polynomial

$$T_r(x) = \sum_{\nu=o}^{r}(\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x).$$

We obtain

$$R_r(f;\ \lambda)_{L^{p,\lambda}(\mathbb{T})} = \left\| f(x) - \sum_{\nu=0}^{r}\lambda_\nu(r)A_\nu(x;\ f) \right\|_{L^{p,\lambda}(\mathbb{T})}$$

$$\leq \|f(x) - T_r(x)\|_{L^{p,\lambda}(\mathbb{T})} + \left\| T_r(x) - \sum_{\nu=0}^{r}\lambda_\nu(r)(\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x) \right\|_{L^{p,\lambda}(\mathbb{T})}$$

$$+ \left\| \sum_{\nu=0}^{r}\lambda_\nu(r)A_\nu(x;\ f) - \sum_{\nu=0}^{r}(\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x)\lambda_\nu(r) \right\|_{L^{p,\lambda}(\mathbb{T})}$$

$$= \|f(x) - T_r(x)\|_{L^{p,\lambda}(\mathbb{T})} + R_r(T_r;\lambda)_{L^{p,\lambda}(\mathbb{T})}$$

$$+ \left\| \frac{1}{\pi}\int_0^{2\pi} \{f(x+\theta) - T_r(x+\theta\} \left\{ \frac{1}{2} + \sum_{\nu=1}^{r}\lambda_\nu(r)\cos\nu\theta \right\} d\theta \right\|_{L^{p,\lambda}(\mathbb{T})}.$$

Then from the last inequality we conclude that

$$R_r(f,\lambda)_{L^{p,\lambda}(\mathbb{T})} \leq \|f(x) - T_r(x)\|_{L^{p,\lambda}(\mathbb{T})}(1 + K_r) + R_r(T_r;\lambda \tag{2.6}$$

where

$$K_r = \frac{2}{\pi}\int_0^{\pi} \left| \frac{1}{2} + \sum_{\nu=1}^{r}\lambda_\nu(r)\cos\nu\theta \right| d\theta.$$

By [46] the identity

$$\sum_{\nu=1}^{n}\{1 - \lambda_\nu(r)\}(\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x) = \frac{2}{\pi}T_n(x+\theta)\cos n\theta B_n(r,\theta)d\theta, \tag{2.7}$$

holds, where $\lambda_0(r) = 1$ and

$$T_n(x) = \sum_{\nu=o}^{n}(\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x).$$

$$B_n(r,\theta) = \frac{1 - \lambda_n(r)}{2} + \sum_{\nu=0}^{n-1}(1 - \lambda_{n-\nu}(r))\cos\nu\theta.$$

Let $f \in L^{p,\lambda}(\mathbb{T})$, $0 < \lambda \leq 2$, $1 < p < \infty$ and let $T_n \in \Pi_n$ $(n = 0,1,2,...)$ be the polynomial of best approximation to f , i. e.

$$E_n(f)_{L^{p,\lambda}(\mathbb{T})} = \|f(x) - T_n(x)\|_{L^{p,\lambda}(\mathbb{T})}.$$

We set

$$\rho_k(\nu;r;x) = \frac{1}{\pi}\int_0^{2\pi} T_k(x+\theta)\sum_{\mu=1}^{\nu}\{1 - \lambda_\mu(r)\}\cos\mu\theta,\ (0 \leq k \leq \nu \leq r), \tag{2.8}$$

The following equalities holds:

$$R_r(T_r;\lambda)_{L^{p,\lambda}(\mathbb{T})} = \|\rho_r(r;r;x)\|_{L^{p,\lambda}(\mathbb{T})},$$

$$\rho_0(2;r;x) = 0, \rho_k(\nu;r;x) = 0, \rho_k(k;r;x) = 0,\ (\nu > k).$$

We suppose that the number $m \in N$ satisfies condition $2^m \leq r < 2^{m+1}$. Then we obtain

$$\begin{aligned} R_r(T_r;\lambda)_{L^{p,\lambda}(\mathbb{T})} \quad &\leq \quad \|\rho_2(2;r;x) - \rho_0(2;r;x)\|_{L^{p,\lambda}(\mathbb{T})} \\ &+\sum_{\mu=1}^{m-1} \left\|\rho_{2^{\mu+1}}(2^{\mu+1};r;x) - \rho_{2^\mu}(2^{\mu+1};r;x)\right\|_{L^{p,\lambda}(\mathbb{T})} \\ &+ \|\rho_r(r;r;x) - \rho_{2^m}(r;r;x)\|_{L^{p,\lambda}(\mathbb{T})}. \end{aligned} \tag{2.9}$$

Use of (2.7) and (2.8) gives us

$$\begin{aligned} &\left\|\rho_{2^{\mu+1}}(2^{\mu+1};r;x) - \rho_{2^\mu}(2^{\mu+1};r;x)\right\|_{L^{p,\lambda}(\mathbb{T})} \\ &= \left\|\frac{1}{\pi}\int_0^{2\pi} \{T_{2^{\mu+1}}(x+\theta) - T_{2^\mu}(x+\theta)\} \sum_{j=1}^{2^{m+1}} \{1-\lambda_j(r)\}\cos j\theta d\theta\right\|_{L^{p,\lambda}(\mathbb{T})} \\ &= \left\|\frac{2}{\pi}\int_0^{2\pi} \{T_{2^{\mu+1}}(x+\theta) - T_{2^\mu}(x+\theta)\}\cos 2^{\mu+1}\theta B_{2^{\mu+1}}(r;\theta)\right\|_{L^{p,\lambda}(\mathbb{T})} \\ &\leq c_7\delta(2^{\mu+1};r)E_{2^\mu}(f)_{L^{p,\lambda}(\mathbb{T})}. \end{aligned} \tag{2.10}$$

The relations (2.9) and (2.10) imply that

$$\begin{aligned} R_r(T_r;\lambda)_{L^{p,\lambda}(\mathbb{T})} &\leq c_8\delta(2;r)E_0(f)_{L^{p,\lambda}(\mathbb{T})} + \sum_{\mu=1}^{m-1}\delta(2^{\mu+1};r)E_{2^\mu}(f)_{L^{p,\lambda}(\mathbb{T})} \\ &+\delta(r;r)E_{2m}(f)_{L^{p,\lambda}(\mathbb{T})}. \end{aligned} \tag{2.11}$$

According to [46] $K_r \leq c_9$. The inequalities (2.6) and (2.11) immediately yield (2.4).□

Corollary 2.4. *Suppose that the conditions of Theorem 2.3. are satisfied.*

1. Let $\lambda_\nu(r)$, $\nu = 0,1,2,...$ *be a system of numbers defined by relations* (1.3). *Then the following inequality holds*:

$$R_r(f;\ \lambda)_{L^{p,\lambda}(\mathbb{T})} \leq \frac{c_{10}}{r+1}\sum_{\nu=0}^{r} E_\nu(f)_{L^{p,\lambda}(\mathbb{T})}. \tag{2.12}$$

2. Let $\lambda_\nu(r)$, $\nu = 0,1,2,..$*be a system of numbers defined by relations* (1.4). *Then the following inequality holds*:

$$R_r(f;\ \lambda)_{L^{p,\lambda}(\mathbb{T})} \leq \frac{c_{11}}{(r+1)^k}\sum_{\nu=0}^{r}(\nu+1)^{k-1}E_\nu(f)_{L^{p,\lambda}(\mathbb{T})}, \tag{2.13}$$

Proof. If we put

$$\lambda_\nu(r) = 1 - \frac{\nu^k}{(\nu+1)^k},\ (0 \leq \nu \leq r) \text{ and } \lambda_\nu(r) = 0,\ \nu > r$$

in the inequality (2.7) we obtain

$$\begin{aligned} &\sum_{\nu=1}^{n}\nu^k(\alpha_\nu\cos\nu x + \beta_\nu\sin\nu x) \\ &= \frac{2n^k}{\pi}\int_0^{2\pi} T_n(x+\theta)\cos n\theta\left\{\frac{1}{2} + \sum_{\nu=1}^{n-1}(1-\frac{\nu}{n})^k\cos\nu\theta\right\}d\theta. \end{aligned} \tag{2.14}$$

Taking account of (2.14) we have

$$\left\|\sum_{\nu=1}^{n}\nu^k(\alpha_\nu\cos\nu x+\beta_\nu\sin\nu x)\right\|_{L^{p,\lambda}(\mathbb{T})}\le c_{12}n^k\left\|T_n(x)\right\|_{L^{p,\lambda}(\mathbb{T})}.$$

If we put

$$\lambda_{2^{\mu+1}}(r)=1-\frac{2^{(\mu+1)}}{(r+1)^k}$$

in (2.5) we get

$$\begin{aligned}\delta(2^{\mu+1};r)&={}_0^{\pi}\left|\frac{1-\lambda_{2^{\mu+1}}(r)}{2}+\sum_{\nu=1}^{2^{\mu+1}}\{1-\lambda_{2^{\mu+1}-\nu}(r)\}\cos\nu\theta\right|d\theta\\&=\frac{2^{(\mu+1)k}}{(r+1)^k}{}_0^{\pi}\left|\frac{1}{2}+\sum_{\nu=1}^{2^{\mu+1}-1}(1-\frac{\nu}{2^{\mu+1}})^k\cos\nu\theta\right|d\theta\\&\le c_{13}\frac{2^{(\mu+1)k}}{(r+1)^k}.\end{aligned}\tag{2.15}$$

Consideration of (2.15) and (2.4) gives us the inequalities (2.12) and (2.13) of Corollary 2.4.□

Theorem 2.5. *Let* $f\in L^{p,\lambda}(\mathbb{T})$, $0<\lambda\le 2$, $1<p<\infty$ *and* $\gamma=\max\{2,\ p\}$. *then for the system of numbers defined by (4) the following inequality holds:*

$$R_r(f;\ \lambda)_{L^{p,\lambda}(\mathbb{T})}\ge\frac{c_{14}}{(r+1)^k}\left\{{}_{\nu=1}^{r}\nu^{k\gamma-1}E_\nu^\gamma(f)_{L^{p,\lambda}(\mathbb{T})}\right\}^{\frac{1}{\gamma}},$$

where c_{13} *is a constant depending on* $p,\ \lambda$ and k.

Proof. We suppose that the number $m\in N$ satisfies condition $2^m\le n<2^{m+1}$. From $E_n(f)_{L^{p,\lambda}(\mathbb{T})}\downarrow 0$ we get

$$\begin{aligned}\sigma_{n,k,\gamma}:=\left\{\frac{\nu^{k\gamma-1}}{(n+1)^{k\gamma}}\sum_{\nu=1}^{n}E_\nu^\gamma(f)_{L^{p,\lambda}(\mathbb{T})}\right\}^{\frac{1}{\gamma}}&\le\left\{\sum_{\nu=0}^{m+1}\sum_{\mu=2^\nu}^{2^{\nu+1}-1}\frac{\mu^{k\gamma-1}}{(n+1)^{k\gamma}}E_n^\gamma(f)_{L^{p,\lambda}(\mathbb{T})}\right\}^{\frac{1}{\gamma}}\\&\le\left\{\sum_{\nu=0}^{m+1}\frac{2^{\nu\gamma k}}{(n+1)^{k\gamma}}E_{2^\nu}^\gamma(f)_{L^{p,\lambda}(\mathbb{T})}\right\}^{\frac{1}{\gamma}}.\end{aligned}$$

By [21] the inequality

$$\|f(x)-S_n(x,f)\|_{L^{p,\lambda}(\mathbb{T})}\le c_{15}E_n(f)_{L^{p,\lambda}(\mathbb{T})}\tag{2.16}$$

holds. Then taking account of (2.3) we obtain

$$\begin{aligned}\sigma_{n,k,\gamma}&\le\left\{\sum_{\nu=0}^{m+1}\frac{2^{\nu\gamma k}}{(n+1)^{k\gamma}}\left\|\sum_{\mu=2^\nu}^{\infty}A_\mu(x;f)\right\|_{L^{p,\lambda}(\mathbb{T})}^{\gamma}\right\}^{\frac{1}{\gamma}}\\&\le\left\{\sum_{\nu=0}^{m+1}\frac{2^{\nu\gamma k}}{(n+1)^{k\gamma}}\left\|\left(\sum_{\mu=\nu}^{\infty}\Delta_{\mu+1}^2\right)^{\frac{1}{2}}\right\|_{L^{p,\lambda}(\mathbb{T})}^{\gamma}\right\}^{\frac{1}{\gamma}}\end{aligned}\tag{2.17}$$

Let $\gamma = 2$ and $1 < p \le 2$. Using Minkowski's inequality we find that

$$\begin{aligned}\sigma_{n,k,2} &\le c_{16}\left\{\sum_{\nu=0}^{m+1}\frac{2^{\nu\gamma k}}{(n+1)^{k\gamma}}\left\|\left(\sum_{\mu=\nu}^{\infty}\Delta_{\mu+1}^{2}\right)^{\frac{1}{2}}\right\|_{L^{p,\lambda}(\mathbb{T})}^{2}\right\}^{\frac{1}{2}}\\ &\le \left\|\left(\sum_{\nu=0}^{m+1}\frac{2^{2\nu k}}{(n+1)^{2k}}\sum_{\mu=\nu}^{\infty}\Delta_{\mu+1}^{2}\right)^{\frac{1}{2}}\right\|_{L^{p,\lambda}(\mathbb{T})}.\end{aligned}$$

By Abel's transformation we obtain

$$\begin{aligned}\sigma_{n,k,2} &\le c_{17}\left\|\left(\sum_{\nu=0}^{m}\frac{2^{2\nu k}}{(n+1)^{2k}}\Delta_{\nu+1}^{2}+\frac{2^{2(m+1)k}}{(n+1)^{2k}}\sum_{\mu=m+1}^{\infty}\Delta_{\mu+1}^{2}\right)^{\frac{1}{2}}\right\|_{L^{p,\lambda}(\mathbb{T})}\\ &\le c_{18}\left\|\left(\sum_{\nu=0}^{m}\frac{2^{\nu k}}{(n+1)^{2k}}\Delta_{\nu+1}^{2}\right)^{\frac{1}{2}}\right\|_{L^{p,\lambda}(\mathbb{T})}+c_{19}\left\|\left(\sum_{\mu=m+1}^{\infty}\Delta_{\mu+1}^{2}\right)^{\frac{1}{2}}\right\|_{L^{p,\lambda}(\mathbb{T})}. \qquad (2.18)\end{aligned}$$

Taking the relations (2.3) and (2.16) into account we get

$$\begin{aligned}\left\|\left(\sum_{\mu=m+1}^{\infty}\Delta_{\mu+1}^{2}\right)^{\frac{1}{2}}\right\|_{L^{p,\lambda}(\mathbb{T})} &\le c_{20}\left\|\sum_{\mu=2^{m+1}}^{\infty}A_{\mu}(x;f)\right\|_{L^{p,\lambda}(\mathbb{T})}\\ &\le c_{21}E_{n}(f)_{L^{p,\lambda}(\mathbb{T})}. \qquad (2.19)\end{aligned}$$

Then the inequalities (2.18) and (2.19) imply that

$$\sigma_{n,k,\gamma} \le c_{22}\left\|\left(\sum_{\nu=0}^{m}\frac{2^{2\nu k}}{(n+1)^{2k}}\Delta_{\nu+1}^{2}\right)^{\frac{1}{2}}\right\|_{L^{p,\lambda}(\mathbb{T})}+c_{23}E_{n}(f)_{L^{p,\lambda}(\mathbb{T})}.$$

Note that system of multipliers

$$\lambda_{\mu}=\frac{2^{\nu k}}{\mu^{k}(n+1)^{k}}\ (2^{\nu}\le\mu\le 2^{\nu+1}-1,\ \nu=1,2,...,\ 2^{m+1}-1),\ \lambda_{\mu}=0\ (\mu\ge 2^{m+1})$$

satisfies the conditions (2.1). Then from inequality (2.2) we conclude that

$$\sigma_{n,k}\le c_{24}\left\|\left|\sum_{\mu=0}^{n}\frac{\mu^{k}}{(n+1)^{k}}A_{\mu}(x;f)\right|\right\|_{L^{p,\lambda}(\mathbb{T})}+c_{25}E_{n}(f)_{L^{p,\lambda}(\mathbb{T})}\le c_{26}R_{n}(f;\lambda)_{L^{p,\lambda}(\mathbb{T})}.$$

Let $2\le p<\infty$ and $\gamma=p$. Using (2.17) we get

$$\begin{aligned}
\sigma_{n,k,p} &\leq \left\{ \sum_{\nu=0}^{m+1} \frac{2^{\nu pk}}{(n+1)^{kp}} \left\| \sum_{\mu=2^{\nu}}^{\infty} A_{\mu}(x;f) \right\|_{L^{p,\lambda}(\mathbb{T})}^{p} \right\}^{\frac{1}{p}} \\
&\leq \left\{ \sum_{\nu=0}^{m+1} \frac{2^{\nu pk}}{(n+1)^{kp}} \left\| \left(\sum_{\mu=\nu}^{\infty} \Delta_{\mu+1}^{2} \right)^{\frac{1}{2}} \right\|_{L^{p,\lambda}(\mathbb{T})}^{p} \right\}^{\frac{1}{p}} \\
&\leq c_{27} \left\{ \left\| \sum_{\nu=0}^{m+1} \frac{2^{\nu k}}{(n+1)^{k}} \left(\sum_{\mu=\nu}^{\infty} \Delta_{\mu+1}^{2} \right)^{\frac{1}{2}} \right\|_{L^{p,\lambda}(\mathbb{T})}^{p} \right\}^{\frac{1}{p}} \\
&\leq c_{28} \left\{ \left\| \left(\sum_{\nu=0}^{m+1} \frac{2^{2\nu k}}{(n+1)^{2k}} \sum_{\mu=\nu}^{\infty} \Delta_{\mu+1}^{2} \right)^{\frac{1}{2}} \right\|_{L^{p,\lambda}(\mathbb{T})} \right\}.
\end{aligned}$$

Further, using the same Abel's transformation and reasoning as in the case $1 < p \leq 2$ we have

$$\sigma_{n,k,p} \leq c_{29} R_n(f;\lambda)_{L^{p,\lambda}(\mathbb{T})}.$$

Proof of Theorem 2.5 is completed.□

Theorem 2.6. *Let* $f \in L^{p,\lambda}(\mathbb{T})$, $0 < \lambda \leq 2$, $1 < p < \infty$ *and* $\gamma = \max\{2, p\}$, *then for the system of numbers defined by (1.5) the following inequality holds:*

$$R_r(f;\ \lambda)_{L^{p,\lambda}(\mathbb{T})} \geq c_{30}(1-r) \left\{ \sum_{\nu=0}^{\infty} r^{\nu} (\nu+1)^{\gamma-1} E_{\nu}^{\gamma}(f)_{L^{p,\lambda}(\mathbb{T})} \right\}^{\frac{1}{\gamma}},$$

where c_{30} *is a constant depending on* p and λ.

Proof of Theorem 2.6 is similar to proof of Theorem 2.5.

References

[1] R. Akgun, *Jackson and converse theorems of trigonometric approximation in weighted Lebesgue spaces*, Proc. A. Razmadze Math. Inst. **152** (2010), 1-18.

[2] R. Akgun, *Realization and characterization of modulus of smoothness in weighted Lebesgue spaces*, St. Petersburg Math. J. **26** (2015), 741-756.

[3] A. Almeida, S. Samko, *Approximation in Morrey spaces,* J. Funct. Anal. **272** (6) (2017), 2392-2411.

[4] A. Almeida, S. Samko, *Appropximation in generalized Morrey spaces,* Georgian Math. J. **25 (2)** (2018), 155-168.

[5] V. I. Burenkov, M. Laza De Crutoforis, N. A. Kydyrmina, *Approximation by C^{∞} functions in Morrey spaces*, Math. J. **14** (52) 2 (2014), 66-75.

[6] Z. Cakir, C. Aykol, D. Soylemez, A. Serbetci, Approximation by trigonıometric polynomials in Morrey spaces, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Math. Sci. Mathematics, **39** (1), (2019), 24-37.

[7] Z. Cakir, C. Aykol, D. Soylemez, A. Serbetci, Approximation by trigonometric polynomials in weighted Morrey spaces, Tbilisi Mathematical Journal, **13** (1), (2020), 123-138.

[8] U. Deger, I. Dagadur and M. Kücükaslan, *Approximation by trigonometric polynomials to functions in L_p norm,* Proc. Jangjeon Math. Soc. **15** (2012), No. 2, 203-213

[9] U. Deger, M. Kaya, *On the approximation by Cesàro submethod*, Palest. J. Math. **4** (2015), No. 1, 44-56.

[10] U. Deger, *On approximation by Nörlund and Riesz submethod in variable exponent Lebesgue spaces*, Commun. Fac. Sci. Univ. Ank. Series A1 **67** (2018), No. 1, 46-59.

[11] D. Fan, S. Lu, D. Yang, *Regularity in Morrey spaces of strang solitions to nondivergence elliptic wquations with VMO coefficients* , Georgian Math. J. **5** (1998), 425-440.

[12] M. Giaquinta, Multiple Integrals in the Calculus of Variations and Non-linear Elliptic Systems, Princeton Univ. Oress, Princeton (1983).

[13] Y. Giga, T. Miyakawa, *Navier-Stokes flow in R^3 with measures as initial vorticity and Morrey spaces*, Commun. Pert. Difer. Eqs. **14** (5) (1989), 577-618.

[14] A. Guven, *Trigonometric,approximation of functions in weighted L^p spaces*,Sarajevo J. Math., **5** 17 (2009), 99-108.

[15] A. Guven, D. M. Israfilov, *Approximation by Means of Fourier trigonometric series in weighted Orlicz space*s, Adv. Stud. Contemp. Math. (Kyundshang).,**19** (2009), No: 2, 283-295.

[16] A. Guven, D. M. Israfilov, *Polynomial approximation in Smirnov -Orlicz classes*, Comput. Methods Funct. Theory **2** (2002), No. 2, 509-517.

[17] V. S. Guliyev, J. Hasanov, S. Samko, *Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey Spaces,* Math. Scand. **107** (2010), 285-304.

[18] V. S. Guliyev, A. Ghorbanalizadeh, Y. Sawano, *Approximation by trigonometric polynomials in variable exponent Morrey spaces,* Anal. Math. Phys. **9** (3) (2019), 1265-1285.

[19] D. M. Israfilov, Ahmet Testici, *Approximation by matrix transformation in weighted Lebesgue spaces, with variable exponent,* Results Math. **73**: 8, (2018), 1-25.

[20] D. M. Israfilov and N. P. Tozman, *Approximation by polynomials in Morrey- Smirnov classes*, East J. Approx, **14** (3) (2008), 255-269.

[21] D. M. Israfilov, N. P. Tozman, *Approximation in Morrey-Smirnov classes*, Azerbaijan J. Math. **1** (2) (2011), 99-113.

[22] M. Izuki, E. Nakai, Y. Sawano, *Function spaces with variable exponent -an introduction*, Sci. Math. Jpn. **77** (2) (2014), 187-305.

[23] S. Z. Jafarov, *Approximation by Fejér sums of Fourier trigonometric series in weighted Orlicz spaces,* Hacet, J. Math. Stat., **42** (2013), no. 3, 259-268.

[24] S. Z. Jafarov, *Approximation by linear summability neans in Orlicz spaces,* Novi Sad J. Math., **44** 2 (2014), 161-172.

[25] S. Z. Jafarov, *Linear methods of summing Fourier series and approximation in weighted variable exponent Lebesgue spaces*, Ukrainian Math. J., **66** 10 (2015), 1509-1518.

[26] S. Z. Jafarov, *Approximation of periodic functions by Zygmund means in Orlicz spaces*, J. Classical Anal, **9** (1) (2016), 43-52.

[27] S. Z. Jafarov, Approximation by matrix transforms in weighted Orlicz spaces, Turk. J. Math. **44** (2020), 179-193.

[28] S. Z. Jafarov, *Approximation by Rational Functions in Smirnov-Orlicz Classes*, J. Math. Anal. Appl. **379** (2011) 870-877.

[29] S. Z. Jafarov, *Approximation of functions by de la Vallee-Poissin sums in weighted Orlicz spaces,* Arab J. Math. **5** (2016), 125-137.

[30] D. S. Kurtz, Littlewood -Paley and Multiplier Theorems on Weighted L^p spaces, Trans. Amer. Mat. Soc. 259 (1980), 235-254.

[31] V. Kokilashvili, A. Meskhi, *Boundedness of maximal and singular operators in Morrey spaces with variable exponent* , Armen. J. Math. **1** (1) (2008), 18-28.

[32] A. Kufner, O. John, S. Fucik, Function Spaces,p. 454+XV. Noordhoff International Publishing, London (1977).

[33] A. Kinj, M. Ali and S. Mahmoud, *Approximation properties of de la Vallée- Poussin sums in Morrey spaces*, SQU J. Sci. **22** (2) (2017), 89-95.

[34] X. Z. Krasniqi, *On trigonometric approximation in the spaces, $L^{p(x)}$,* TWMS J. App. Eng. Math. **4** (2), (2014), 147-154.

[35] X. Z. Krasniqi, *Trigonometric approximation of (signals) functions by Nörlund type means in the variable space $L^{p(x)}$,* Palesti. J. Math. **6** (1), (2017), 84-93

[36] X. Z. Krasniqi, *Approximation of periodic functions sub-matrix means of their Fourier series*, TWMS J. App. Eng. Math. **10** (1), (2020), 279–287.

[37] J. Littlewood, R. Paley, *Theorems on Fourier series and power series,* Proc. London Math. Soc., **42** (1937), 52-89

[38] M. I. Mittal, B. F. Rhoades, *On degree of approximation of contuous functions by using linear operators on their Fourier series,* International Journal of Mathematics, Game Theory, and Algebra., **9** (4) (1999), 259-267

[39] M. L. Mittal, Mradul Veer Singh, *Approximation of signals (functions) by trigonometric polynomials in L_p-norm,* Hindawi Publishing Corporation, International Journal of Mathematics and Mathematical Sciences., Volume 2014, Article ID 267383, 6 pages,

[40] A. L. Mazzucato, *Decomposition of Besov- Morrey spaces,* In: Proceerings of the Conference on Harmonic Analysis, in : Contemp. Math. Amer. Math. Soc. Providence, RI, **320** (2003), 279-294.

[41] E. S. Quade, *Trigonometric approximation in the mean*, Duke Math. J., **3** (3) (1937), 529-542.

[42] H. Raferio N. Samko, S. Samko, *Morrey- Campanato spaces: an overviev,* (English summary), Operator theory, pscudo-differtential equations, and mathematical physics, Oper. Theory Adv. Appl. **228** (2013), 293-323.

[43] S. B. Stechkin, *The approximation of periodic functions by Fejér sums,* Trudy Math. Inst. Steklov., G2 (1961), 522-523 (in Russian).

[44] N. Samko, *Weighted Hardy and singular operators in Morrey spaces*, J. Math. Anal. Appl. **350** (1) (2009), 56-72.

[45] S. Sonker, U. Singh, *Approximation of signals (functions)belonging to* $Lip(\alpha, p, \omega)-$*class using trigonometric polynomials*, Procedia Engineering, **38** (2012), 1575-1585.

[46] M. F. Timan, *Best approximation of a function and linear methods of summing Fourier series*, Izv. Akad. Nauyk SSSR Ser: Math., **29** (1965), 587-604.

[47] M. E. Taylor, Tools for PDE, volume 81 of Mathematical Surveys and Monographs, Pseudodifferential Operators, Paradifferential Operators and Layer Potentials, American Mathematical Society, Providence (2000).

[48] A. Testici, *Approximation by Nörlund and Riesz means in weighted Lebesgue space with variable exponent*, Commun. Fac. Sci. Univ. Ank. Series A1 **68** (2019), No. 2, 2014-2025.

Author information

Sadulla Z. JAFAROV, Department of Mathematics and Science Education, Faculty of Education, Muş Alparslan University, 49250, Muş, TURKEY; Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, 9 B. Vahabzadeh str., AZ 1141, Baku, AZERBAIJAN.
E-mail: `s.jafarov@alparslan.edu.tr`

Received: August 10, 2020.
Accepted: May 7, 2021.