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SOME PROPERTIES OF MEROMORPHICALLY UNIFORMLY CONVEX FUNCTIONS DEFINED BY HURWITZ -LERCH ZETA FUNCTION

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Abstract. In this paper we introduce and study a new subclass of meromorphically uniformly convex functions with positive coefficients defined by a differential operator and obtain coefficient estimates, growth and distortion theorem, radius of convexity, integral transforms, convex linear combinations, convolution properties and $\delta-$neighborhoods for the class $\sigma(\alpha, s, b)$.

1 Introduction

Let S be denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{1.1}$$

which are analytic and univalent in $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Denote by $S^*(\gamma)$ and $K(\gamma)$, $0 \le \gamma < 1$ the subclasses of functions in S that are starlike and convex functions of order α respectively. Analytically $f \in S^*(\gamma)$ if and only if f is of the form (1.1) and satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma, \quad z \in U.$$

Similarly, $f \in K(\gamma)$ if and only if f is of the form (1.1) and satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma, \quad z \in U.$$

Also denote by T the subclasses of S consisting of functions of the form

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^m, a_m \ge 0 \tag{1.2}$$

introduced and studied by Silverman [19], let $T^*(\gamma) = T \cap S^*(\gamma), CV(\gamma) = T \cap K^*(\gamma)$. The classes $T^*(\gamma)$ and $K^*(\gamma)$ posses some interesting properties and have been extensively studied by Silverman [19] and others. In 1991, Goodman [10, 11] introduced an interesting subclass uniformly convex (uniformly starlike) of the class CV of convex functions (ST starlike functions) denoted by UCV (UST). A function $f(z)$ is uniformly convex (uniformly starlike) in U if $f(z)$ in CV (ST) has the property that for every circular arc γ contained in U with center ξ also in U, the arc $f(\gamma)$ is a convex arc (starlike arc) with respect to $f(\xi)$.

Motivated by Goodman [10, 11], Ronning [16, 17] introduced and studied the following subclasses of S. A function $f \in S$ is said to be in the class $S_p(\gamma, k)$ uniformly $k-$starlike functions if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)} - \gamma\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad 0 \le \gamma < 1, k \ge 0, z \in U \tag{1.3}$$

and is said to be in the class UCV (γ, k), uniformly $k-$convex functions if it satisfies the condition

$$\Re\left(1+\frac{zf''(z)}{f'(z)}-\gamma\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \quad 0 \leq \gamma < 1, k \geq 0, z \in U. \tag{1.4}$$

Indeed it follows from (1.3) and (1.4) that

$$f \in UCV(\gamma, k) \Leftrightarrow zf' \in S_p(\gamma, k). \tag{1.5}$$

Further Ahuja et al. [1], Bharathi et al. [4], Murugusundaramoorthy et al. [15] and others have studied and investigated interesting properties for the classes $S_p(\gamma, k)$ and $UCV(\gamma, k)$.

Let Σ denote the class of functions of the form

$$f(z) = z^{-1} + \sum_{m=1}^{\infty} a_m z^m, \quad a_m \geq 0 \tag{1.6}$$

which are analytic in the punctured open disk $U^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\} = U \setminus \{0\}$.

Let $f, g \in \Sigma$, where f is given by (1.6) and g is defined by

$$g(z) = z^{-1} + \sum_{m=1}^{\infty} b_m z^m.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = z^{-1} + \sum_{m=1}^{\infty} a_m b_m z^m = (g * f)(z).$$

Let $\Sigma_s, \Sigma^*(\gamma)$ and $\Sigma_k(\gamma)$ $(0 \leq \gamma < 1)$ denote the subclasses of Σ that are meromorphic univalent, meromorphically starlike functions of order γ and meromorphically convex functions of order γ respectively. Analytically, $f \in \Sigma^*(\gamma)$ if and only if f is of the form (1.6) and satisfies

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \quad z \in U.$$

Similarly, $f \in \Sigma_k(\gamma)$ if and only if f is of the form (1.6) and satisfies

$$-\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \gamma, \quad z \in U$$

and similar other classes of meromorphically univalent functions have been extensively studied by (for example) Altintas et al. [2], Aouf [3] and Mogra et al. [14].

The following we recall a general Hurwitz-Lerch Zeta function $\phi(z, s, a)$ defined by (see [21], p. 121)

$$\phi(z, s, a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^s}$$

for $a \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}$ when $|z| < 1$; $\Re(s) > 1$ when $|z| = 1$, where $\mathbb{Z}_0^- = \mathbb{Z} \setminus \{\mathbb{N}\}$, $\mathbb{Z} = \{0, \pm1, \pm2, \cdots\}$, $\mathbb{N} = \{1, 2, 3, \cdots\}$.

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\phi(z, s, a)$ can be found in the recent investigation by (for example) Choi and Srivastava [5], Ferreira and Lopez [6], Garg et al. [7], Lin and Srivastava [12], Luo and Srivastava [13], Srivastava et al. [22], Ghanim [8] and others.

By making use of Hurwitz-Lerch Zeta function $\phi(z, s, a)$, Srivastava and Attiya [20] recently introduced and investigated the integral operator

$$\mathcal{J}_{s,b} f(z) = z + \sum_{m=2}^{\infty} \left(\frac{1+b}{k+b}\right)^s a_m z^m, (b \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}, z \in U).$$

Motivated essentially by the above mentioned Srivastava-Atiya operator $\mathcal{J}_{s,b}$, we now introduce the linear operator

$$\mathcal{W}_{s,b} : \Sigma \to \Sigma$$

defined, in terms of the Hardmard product (or convolution), by

$$\mathcal{W}_{s,b}f(z) = \Theta_{s,b}(z) * f(z), \quad (b \in \mathbb{C} \setminus \mathbb{Z}_0^- \cup \{1\}, s \in \mathbb{C}, f \in \Sigma, z \in U^*), \tag{1.7}$$

where for convenience,

$$\Theta_{s,b}(z) = (b-1)^s \left[\phi(z,s,b) - b^{-s} + \frac{1}{z(b-1)^s}\right], \quad z \in U^*.$$

It can be easily be seen from (1.7) that

$$\mathcal{W}_{s,b}f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} L(m,s,b)a_m z^m, \tag{1.8}$$

$$\text{where } L(m,s,b) = \left(\frac{b-1}{b+m}\right)^s.$$

Indeed, the operator $\mathcal{W}_{s,b}$ can be defined for $b \in \mathbb{C} \setminus \mathbb{Z}_0^- \cup \{1\}$, where

$$\mathcal{W}_{s,0}f(z) = \lim_{b \to 0}\{\mathcal{W}_{s,b}f(z)\}.$$

We observe that

$$\mathcal{W}_{0,b}f(z),$$

and

$$\mathcal{W}_{1,\gamma} = \frac{\gamma-1}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt, \quad \Re(\gamma) > 1.$$

Furthermore, from the definition (1.8), we find that

$$\mathcal{W}_{s+1,b}f(z) = \frac{b-1}{z^b}\int_0^z t^{b-1}\mathcal{W}_{s,b}f(t)dt, \quad \Re(b) > 1. \tag{1.9}$$

Differentiating both sides of (1.9) with respect to z, we get the following useful relationship:

$$z\left(\mathcal{W}_{s+1,b}f\right)'(z) = (b-1)\mathcal{W}_{s,b}f(z) - b\mathcal{W}_{s+1,b}f(z).$$

Now, we define a new subclass $\sigma(\alpha,s,b)$ of Σ.

Definition 1.1. For $-1 \le \alpha < 1$, we let $\sigma(\alpha,s,b)$ be the subclass of Σ consisting of the form (1.6) and satisfying the analytic criterion

$$-\Re\left\{\frac{z(\mathcal{W}_{s,b}f(z))'}{\mathcal{W}_{s,b}f(z)} + \alpha\right\} > \left|\frac{z(\mathcal{W}_{s,b}f(z))'}{\mathcal{W}_{s,b}f(z)} + 1\right|, \tag{1.10}$$

$\mathcal{W}_{s,b}f(z)$ is given by (1.8) .

The main object of this paper is to study some usual properties of the geometric function theory such as coefficient bounds, growth and distortion properties, radius of convexity, convex linear combination and convolution properties, integral operators and $\delta-$neighbourhoods for the class $\sigma(\alpha,s,b)$.

2 Coefficient inequality

In this section we obtain the coefficient bounds of function $f(z)$ for the class $\sigma(\alpha,s,b)$.

Theorem 2.1. *A function $f(z)$ of the form (1.6) is in $\sigma(\alpha,s,b)$ if*

$$\sum_{m=1}^{\infty} L(m,s,b)[2m+3-\alpha]\,|a_m| \;\le\; (1-\alpha), \; -1 \le \alpha < 1. \tag{2.1}$$

Proof. It sufficient to show that

$$\left|\frac{z(\mathcal{W}_{s,b}f(z))'}{\mathcal{W}_{s,b}f(z)}+1\right|+\Re\left\{\frac{z(\mathcal{W}_{s,b}f(z))'}{\mathcal{W}_{s,b}f(z)}+1\right\}\leq(1-\alpha).$$

We have $\left|\frac{z(\mathcal{W}_{s,b}f(z))'}{\mathcal{W}_{s,b}f(z)}+1\right|+\Re\left\{\frac{z(\mathcal{W}_{s,b}f(z))'}{\mathcal{W}_{s,b}f(z)}+1\right\}$

$$\leq 2\left|\frac{z(\mathcal{W}_{s,b}f(z))'}{\mathcal{W}_{s,b}f(z)}+1\right|$$

$$\leq\frac{2\sum_{m=1}^{\infty}L(m,s,b)(m+1)|a_m||z^m|}{\frac{1}{|z|}-\sum_{m=1}^{\infty}L(m,s,b)|a_m||z^m|}.$$

Letting $z\to1$ along the real axis, we obtain

$$\leq\frac{2\sum_{m=1}^{\infty}L(m,s,b)(m+1)|a_m|}{1-\sum_{m=1}^{\infty}L(m,s,b)|a_m|}.$$

The above expression is bounded by $(1-\alpha)$ if

$$\sum_{m=1}^{\infty}L(m,s,b)[2m+3-\alpha]\;|a_m|\;\leq\;(1-\alpha).$$

Hence the theorem is completed. □

Corollary 2.2. *Let the function $f(z)$ defined by (1.6) be in the class $\sigma(\alpha,s,b)$. Then*

$$a_m\leq\frac{(1-\alpha)}{\sum_{m=1}^{\infty}(L(m,s,b)[2m+3-\alpha]},\quad(m\geq1).$$

Equality holds for the function of the form

$$f_m(z)=\frac{1}{z}+\frac{(1-\alpha)}{L(m,s,b)[2m+3-\alpha]}\,z^m.$$

3 Distortion Theorems

In this section we obtain the sharp for the distortion theorems of the form (1.6).

Theorem 3.1. *Let the function $f(z)$ defined by (1.6) be in the class $\sigma(\alpha,s,b)$. Then for $0<|z|=r<1$,*

$$\frac{1}{r}-\frac{(1-\alpha)}{L(1,s,b)[5-\alpha]}\,r\leq|f(z)|\leq\frac{1}{r}+\frac{(1-\alpha)}{L(1,s,b)[5-\alpha]}r \tag{3.1}$$

with equality for the function

$$f(z)=\frac{1}{z}+\frac{(1-\alpha)}{L(1,s,b)[5-\alpha]}\,z,\ at\ \ z=r,\ ir. \tag{3.2}$$

Proof. Suppose $f(z)$ is in $\sigma(\alpha,s,b)$. In view of Theorem 2.1, we have

$$L(1,s,b)[5-\alpha]\sum_{m=1}^{\infty}a_m\leq\sum_{m=1}^{\infty}L(m,s,b)[2m+3-\alpha]\leq(1-\alpha)$$

which evidently yields $\sum\limits_{m=1}^{\infty} a_m \leq \frac{1-\alpha}{L(1,s,b)[5-\alpha]}$. Consequently, we obtain

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right| \leq \left|\frac{1}{z}\right| + \sum_{m=1}^{\infty} a_m |z|^m \\ &\leq \frac{1}{r} + r\sum_{m=1}^{\infty} a_m \\ &\leq \frac{1}{r} + \frac{1-\alpha}{L(1,s,b)[5-\alpha]}\, r.\end{aligned}$$

$$\begin{aligned}\text{Also, } |f(z)| &= \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right| \geq \left|\frac{1}{z}\right| - \sum_{m=1}^{\infty} a_m |z|^m \\ &\geq \frac{1}{r} - r\sum_{m=1}^{\infty} a_m \\ &\geq \frac{1}{r} - \frac{1-\alpha}{L(1,s,b)[5-\alpha]}\, r.\end{aligned}$$

Hence the result (3.1) follows. □

Theorem 3.2. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $\sigma(\alpha, s, b)$. *Then for* $0 < |z| = r < 1$,

$$\frac{1}{r^2} - \frac{1-\alpha}{L(1,s,b)[5-\alpha]} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{1-\alpha}{L(1,s,b)[5-\alpha]}.$$

The result is sharp, the extremal function being of the form (3.2).

Proof. From Theorem 2.1, we have

$$L(1,s,b)[5-\alpha]\sum_{m=1}^{\infty} m a_m \leq \sum_{m=1}^{\infty} L(m,s,b)[2m+3-\alpha] \leq (1-\alpha)$$

which evidently yields $\sum\limits_{m=1}^{\infty} m a_m \leq \frac{1-\alpha}{L(1,s,b)[5-\alpha]}$.
Consequently, we obtain

$$\begin{aligned}|f'(z)| &\leq \left|\frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m r^{m-1}\right| \\ &\leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m \\ &\leq \frac{1}{r^2} + \frac{(1-\alpha)}{L(1,s,b)[5-\alpha]}.\end{aligned}$$

$$\begin{aligned}\text{Also, } |f'(z)| &\geq \left|\frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m r^{m-1}\right| \\ &\geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m \\ &\geq \frac{1}{r^2} + \frac{(1-\alpha)}{L(1,s,b)[5-\alpha]}.\end{aligned}$$

This completes the proof. □

4 Class preserving integral operators

In this section we consider the class preserving integral operator of the form (1.6) .

Theorem 4.1. *Let the function $f(z)$ defined by (1.6) be in the class $\sigma(\alpha, s, b)$. Then*

$$f(z) = cz^{-c-1} \int_0^z t^c f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{c}{c+m+1} a_m z^m, \quad c > 0 \tag{4.1}$$

belongs to the class $\sigma[\delta(\alpha, s, b, c)]$, where

$$\delta(\alpha, s, b, c) = \frac{L(1, s, b)(5-\alpha)(c+2) - c(1-\alpha)}{L(1, s, b)(5-\alpha)(c+2) + c(1-\alpha)}. \tag{4.2}$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{(1-\alpha)}{L(1,s,b)[5-\alpha]} z$.

Proof. Suppose $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ is in $\sigma(\alpha, s, b)$. We have

$$f(z) = cz^{-c-1} \int_0^z t^c f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{c}{c+m+1} a_m z^m, \quad c > 0.$$

It is sufficient to show that

$$\sum_{m=1}^{\infty} \frac{m+\delta}{1-\delta} \frac{c}{c+m+1} a_m \leq 1. \tag{4.3}$$

Since $f(z)$ is in $\sigma_p(\alpha)$, we have

$$\sum_{m=1}^{\infty} \frac{L(m, s, b)[2m+3-\alpha]}{1-\alpha} |a_m| \leq 1. \tag{4.4}$$

Thus (4.3) will be satisfied if

$$\frac{(m+\delta)}{(1-\delta)} \frac{c}{(c+m+1)} \leq \frac{L(m, s, b)[2m+3-\alpha]}{1-\alpha}, \text{ for each } m.$$

Solving for δ, we obtain

$$\delta \leq \frac{L(m, s, b)[2m+3-\alpha](c+m+1) - mc(1-\alpha)}{L(m, s, b)[2m+3-\alpha](c+m+1) + c(1-\alpha)} = G(m). \tag{4.5}$$

Then $G(m+1) - G(m) > 0$, for each m.
Hence $G(m)$ is increasing function of m, since $G(1) = \frac{L(1,s,b)(5-\alpha)(c+2)-c(1-\alpha)}{L(1,s,b)(5-\alpha)(c+2)+c(1-\alpha)}$.
The result follows. □

Theorem 4.2. *If the function $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ is in $\sigma(\alpha, s, b)$ then $f(z)$ is meromorphically convex of order δ $(0 \leq \delta < 1)$ in $|z| < r = r(\alpha, m, s, b, \delta)$, where*

$$r(\alpha, m, s, b, \delta) = \inf_{n \geq 1} \left\{ \frac{(1-\delta)L(m, s, b)[2m+3-\alpha]}{(1-\alpha)m(m+2-\delta)} \right\}^{\frac{1}{m+1}}.$$

The result is sharp.

Proof. Let $f(z)$ be in $\sigma(\alpha, s, b)$. Then, by Theorem 2.1, we have

$$\sum_{m=1}^{\infty} L(m, s, b)[2m+3-\alpha]|a_m| \leq (1-\alpha). \tag{4.6}$$

It is sufficient to show that $\left|2+\frac{zf''(z)}{f'(z)}\right| \leq (1-\delta)$ for $|z|<r=r(\alpha,\delta)$, where $r(\alpha,\delta)$ is specified in the statement of the theorem. Then

$$\left|2+\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum\limits_{m=1}^{\infty} m(m+1)a_m z^{m-1}}{\frac{-1}{z^2}+\sum\limits_{m=1}^{\infty} m a_m z^{m-1}}\right| \leq \frac{\sum\limits_{m=1}^{\infty} m(m+1)a_m|z|^{m+1}}{1-\sum\limits_{m=1}^{\infty} m a_m |z|^{m+1}}.$$

This will be bounded by $(1-\delta)$ if

$$\sum_{m=1}^{\infty} \frac{m(m+2-\delta)}{1-\delta} a_m |z|^{m+1} \leq 1. \tag{4.7}$$

By (4.6), it follows that (4.7) is true if

$$\frac{m(m+2-\delta)}{1-\delta}|z|^{m+1} \leq \frac{L(m,s,b)[2m+3-\alpha]}{1-\alpha}|a_m|, \quad m \geq 1$$

$$\text{or } \; |z| \leq \left\{\frac{(1-\delta)L(m,s,b)[2m+3-\alpha]}{(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}. \tag{4.8}$$

Setting $|z| = r(\alpha,m,s,b,\delta)$ in (4.8), the result follows.

The result is sharp for the function

$$f_m(z) = \frac{1}{z} + \frac{(1-\alpha)}{L(m,s,b)[2m+3-\alpha]} z^m, \quad m \geq 1.$$

□

5 Convex linear combinations and convolution properties

In this section we obtain sharp for $f(z)$ is meromorphically convex of order δ and necessary and sufficient condition for $f(z)$ is in the class $\sigma(\alpha,s,b)$. And also proved that convolution is in the class $\sigma(\alpha,s,b)$.

Theorem 5.1. *Let* $f_0(z) = \frac{1}{z}$ *and* $f_m(z) = \frac{1}{z} + \frac{(1-\alpha)}{L(m,s,b)[2m+3-\alpha]} z^m, \quad m \geq 1$. *Then* $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ *is in the class* $\sigma(\alpha,s,b)$ *if and only if it can be expressed in the form* $f(z) = \omega_0 f_0(z) + \sum\limits_{m=1}^{\infty} \omega_m f_m(z)$, *where* $\omega_0 \geq 0, \omega_m \geq 0, m \geq 1$ *and* $\omega_0 + \sum\limits_{m=1}^{\infty} \omega_m = 1$.

Proof. Let $f(z) = \omega_0 f_0(z) + \sum\limits_{m=1}^{\infty} \omega_m f_m(z)$ with $\omega_0 \geq 0, \omega_m \geq 0, m \geq 1$ and $\omega_0 + \sum\limits_{m=1}^{\infty} \omega_m = 1$. Then

$$\begin{aligned} f(z) &= \omega_0 f_0(z) + \sum_{m=1}^{\infty} \omega_m f_m(z) \\ &= \frac{1}{z} + \sum_{m=1}^{\infty} \omega_m \frac{(1-\alpha)}{L(m,s,b)[2m+3-\alpha]} z^m. \end{aligned}$$

$$\begin{aligned} \text{Since } \sum_{m=1}^{\infty} \frac{L(m,s,b)[2m+3-\alpha]}{(1-\alpha)} &\times \omega_m \frac{(1-\alpha)}{L(m,s,b)[2m+3-\alpha]} \\ &= \sum_{m=1}^{\infty} \omega_m = 1-\omega_0 \leq 1. \end{aligned}$$

By Theorem 2.1, $f(z)$ is in the class $\sigma(\alpha,s,b)$.

Conversely suppose that the function $f(z)$ is in the class $\sigma(\alpha, s, b)$. Then

$$a_m \leq \frac{(1-\alpha)}{L(m,s,b)[2m+3-\alpha]} z^m, m \geq 1.$$

$$\text{Now } \omega_m = \sum_{m=1}^{\infty} \frac{L(m,s,b)[2m+3-\alpha]}{(1-\alpha)} a_m \text{ and } \omega_0 = 1 - \sum_{m=1}^{\infty} \omega_m.$$

It follows that $f(z) = \omega_0 f_0(z) + \sum_{m=1}^{\infty} \omega_m f_m(z)$.
This completes the proof of the theorem. □

For the functions $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ and $g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m$ belongs to Σ, we denoted by $(f * g)(z)$ the convolution of $f(z)$ and $g(z)$ and defined as

$$(f * g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m.$$

Theorem 5.2. *If the function $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ and $g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m$ are in the class $\sigma(\alpha, s, b)$ then $(f * g)(z)$ is in the class $\sigma(\alpha, s, b)$.*

Proof. Suppose $f(z)$ and $g(z)$ are in $\sigma(\alpha, s, b)$. By Theorem 2.1, we have

$$\sum_{m=1}^{\infty} \frac{L(m,s,b)[2m+3-\alpha]}{(1-\alpha)} a_m \leq 1$$

$$\text{and } \sum_{m=1}^{\infty} \frac{L(m,s,b)[2m+3-\alpha]}{(1-\alpha)} b_m \leq 1 .$$

Since $f(z)$ and $g(z)$ are regular are in U, so is $(f * g)(z)$. Further more

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{L(m,s,b)[2m+3-\alpha]}{(1-\alpha)} a_m b_m \\
&\leq \sum_{m=1}^{\infty} \left\{ \frac{L(m,s,b)[2m+3-\alpha]}{(1-\alpha)} \right\}^2 a_m b_m \\
&\leq \left(\sum_{m=1}^{\infty} \frac{L(m,s,b)[2m+3-\alpha]}{(1-\alpha)} a_m \right) \left(\sum_{m=1}^{\infty} \frac{L(m,s,b)[2m+3-\alpha]}{(1-\alpha)} b_m \right) \\
&\leq 1.
\end{aligned}$$

Hence, by Theorem 2.1, $(f * g)(z)$ is in the class $\sigma(\alpha, s, b)$. □

6 Neighborhoods for the class $\sigma(\alpha, s, b, \gamma)$

In this section we define the $\delta-$neighborhood of a function $f(z)$ and establish a relation between $\delta-$neighborhood and $\sigma(\alpha, s, b, \gamma)$ class of a function.

Definition 6.1. A function $f \in \Sigma$ is said to in the class $\sigma(\alpha, s, b, \gamma)$ if there exists a function $g \in \sigma(\alpha, s, b)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < (1-\gamma), \quad z \in U, \; 0 \leq \gamma < 1. \tag{6.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [9] and Ruschweyh [18], we defined the $\delta-$neighborhood of a function $f \in \Sigma$ by

$$N_\delta(f) = \left\{ g \in \Sigma \ | \ g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m \ : \ \sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta \right\}. \tag{6.2}$$

Theorem 6.2. *If* $g \in \sigma(\alpha, s, b)$ *and*

$$\gamma = 1 - \frac{\delta L(1,s,b)[5-\alpha]}{L(1,s,b)[5-\alpha] + \alpha - 1} \tag{6.3}$$

then $N_\delta(g) \subset \sigma(\alpha, s, b, \gamma)$.

Proof. Let $f \in N_\delta(g)$. Then we find from (6.2) that

$$\sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta \tag{6.4}$$

which implies the coefficient of inequality $\sum\limits_{m=1}^{\infty} |a_m - b_m| \leq \delta, \ \ m \in \mathbb{N}$.

Since $g \in \sigma(\alpha, s, b)$, we have $\sum\limits_{m=1}^{\infty} b_m = \frac{1-\alpha}{L(1,s,b)(5-\alpha)}$.

So that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum\limits_{m=1}^{\infty} |a_m - b_m|}{1 - \sum\limits_{m=1}^{\infty} b_m} \leq \frac{\delta L(1,s,b)[5-\alpha]}{L(1,s,b)[5-\alpha] + \alpha - 1} = 1 - \gamma,$$

provided γ is given by (6.3).
Hence, by Definition, $f \in \sigma(\alpha, s, b, \gamma)$ for γ given by (6.3), which completes the proof of theorem. □

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