Palestine Journal of Mathematics

Vol. 11(1)(2022) , 633–650

SUMMATION IDENTITIES INVOLVING PADOVAN AND PERRIN NUMBERS

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Communicated by Ziyad Sharawi

MSC 2010 Classifications: Primary 11B37; Secondary 65B10, 11B65.

Keywords and phrases:Padovan sequence, Perrin sequence, generating function, binomial sum, summation identity.

Abstract. Unlike in the case of Fibonacci and Lucas numbers, there is a paucity of literature dealing with summation identities involving the Padovan and Perrin numbers. In this paper, we derive various summation identities for these numbers, including binomial and double binomial identities. Our results derive from the rich algebraic properties exhibited by the zeros of the characteristic polynomial of the Padovan/Perrin sequence.

1 Introduction

The Padovan numbers, P_n, are defined by

$$P_n = P_{n-2} + P_{n-3}\ (n \geq 3)\,, \quad P_0 = P_1 = P_2 = 1\,, \tag{1.1}$$

and the Perrin numbers, Q_n, by

$$Q_n = Q_{n-2} + Q_{n-3}\ (n \geq 3)\,, \quad Q_0 = 3,\ Q_1 = 0,\ Q_2 = 2\,. \tag{1.2}$$

Both sequences (P_n) and (Q_n) can be extended to negative indices by writing the recurrence relations as $P_n = P_{n+3} - P_{n+1}$ and $Q_n = Q_{n+3} - Q_{n+1}$ and replacing n with $-n$, thus obtaining

$$P_{-n} = P_{-(n-3)} - P_{-(n-1)}, \quad Q_{-n} = Q_{-(n-3)} - Q_{-(n-1)}\,. \tag{1.3}$$

It is possible to access the negative subscript numbers without using the recurrence relation (1.3). We have (see Theorem 1.5)

$$P_{-n} = P_{n-7}^2 - P_{n-6}P_{n-8}$$

and

$$2Q_{-n} = Q_n^2 - Q_{2n}\,.$$

Compared to the related Fibonacci and Lucas sequences, there is a dearth of literature on Padovan and Perrin sequences. We mention Shannon et. al. [3] and Yilmaz and Taskara [6]. Useful information is contained in the Mathworld articles [4, 5] and the Mathpages [2] article on these numbers. The purpose of this paper is to present binomial summation identities such as

$$\sum_{j=0}^{\lfloor n/2\rfloor} (-1)^j \frac{n}{n-j}\binom{n-j}{j} P_{p+n-3j} = (-1)^n (Q_n P_p - P_{n+p})\,,$$

and

$$\sum_{j=0}^{\lfloor n/2\rfloor} (-1)^j \binom{n-j}{j} P_{p+n-3j} = (-1)^{n-1}(P_{p+1}P_{n-3} - P_p P_{n-2})\,.$$

We will also derive double binomial summation identities such as

$$\sum_{j=0}^{\lfloor n/2\rfloor}\sum_{k=0}^{n-2j} (-1)^{j+k} \frac{n}{n-j}\binom{n-j}{j}\binom{n-2j}{k} Q_p^{n-2j-k} P_{q-pj+pk} = Q_{pn}P_q - P_{pn+q}\,.$$

Finally, we will derive ordinary summation identities such as the sum of Padovan numbers and Perrin numbers with subscripts in arithmetic progression:

$$\sum_{j=0}^{n} P_{pj+q} = \frac{\begin{vmatrix} P_{pn+p+q} - P_q & P_{p-3} & P_{p-4} \\ P_{pn+p+q+1} - P_{q+1} & P_{p-2}-1 & P_{p-3} \\ P_{pn+p+q-1} - P_{q-1} & P_{p-4} & P_{p-5}-1 \end{vmatrix}}{\begin{vmatrix} P_{p-2}-1 & P_{p-3} & P_{p-4} \\ P_{p-1} & P_{p-2}-1 & P_{p-3} \\ P_{p-3} & P_{p-4} & P_{p-5}-1 \end{vmatrix}},$$

$$\sum_{j=0}^{n} Q_{pj+q} = \frac{\begin{vmatrix} Q_{pn+p+q} - Q_q & P_{p-3} & P_{p-4} \\ Q_{pn+p+q+1} - Q_{q+1} & P_{p-2}-1 & P_{p-3} \\ Q_{pn+p+q-1} - Q_{q-1} & P_{p-4} & P_{p-5}-1 \end{vmatrix}}{\begin{vmatrix} P_{p-2}-1 & P_{p-3} & P_{p-4} \\ P_{p-1} & P_{p-2}-1 & P_{p-3} \\ P_{p-3} & P_{p-4} & P_{p-5}-1 \end{vmatrix}},$$

and the generating function of Padovan numbers with subscripts in arithmetic progression:

$$\sum_{j=0}^{\infty} P_{pj+q} y^j = \frac{\begin{vmatrix} P_q & -yP_{p-3} & -yP_{p-4} \\ P_{q+1} & -yP_{p-2}+1 & -yP_{p-3} \\ P_{q-1} & -yP_{p-4} & -yP_{p-5}+1 \end{vmatrix}}{\begin{vmatrix} -yP_{p-2}+1 & -yP_{p-3} & -yP_{p-4} \\ -yP_{p-1} & -yP_{p-2}+1 & -yP_{p-3} \\ -yP_{p-3} & -yP_{p-4} & -yP_{p-5}+1 \end{vmatrix}}.$$

Here and throughout this paper, $|\cdot|$ denotes matrix determinant.

Throughout this paper, we denote by α, β and γ, the zeros of the characteristic polynomial, $x^3 - x - 1$, of the Padovan sequence.

1.1 Algebraic properties of α, β and γ

Vieta's formulas give

$$\alpha + \beta = -\gamma, \quad \alpha\beta = 1/\gamma, \quad \alpha\beta + \alpha\gamma + \beta\gamma = -1\,, \tag{1.4}$$

from which we also infer

$$\alpha^2 + \beta^2 = \gamma^2 - 2/\gamma = -\gamma^2 + 2\,, \tag{1.5}$$

$$(\alpha - \beta)^2 = 1 - 3/\gamma = 4 - 3\gamma^2\,, \tag{1.6}$$

$$\alpha^2\beta + \beta^2\alpha = -1 = \alpha\beta(\alpha + \beta)\,, \tag{1.7}$$

$$(\alpha^2 - \beta^2)^2 = \gamma^2 - 3\gamma \tag{1.8}$$

and

$$\alpha/\beta + \beta/\alpha = \gamma - 1\,. \tag{1.9}$$

The following result is readily established by mathematical induction.

Lemma 1.1. *The following identities hold for any integer n:*

$$\alpha^n = \alpha^2 P_{n-4} + \alpha P_{n-3} + P_{n-5}\,, \tag{P1}$$

$$\beta^n = \beta^2 P_{n-4} + \beta P_{n-3} + P_{n-5} \tag{P1a}$$

and

$$\gamma^n = \gamma^2 P_{n-4} + \gamma P_{n-3} + P_{n-5}\,. \tag{P1c}$$

Proof. To prove identity P1, we first note that using the recurrence relations (1.1) and (1.3), P1 holds, straightforwardly, for $n = 0$ and $n = 1$. Now assume that P1 holds for a positive integer k, so that

$$I_k := \alpha^k = \alpha^2 P_{k-4} + \alpha P_{k-3} + P_{k-5}\,.$$

We wish to prove that $I_k \implies I_{k+1}$ and $I_{-k} \implies I_{-(k+1)}$.

Assuming I_k, multiplying both sides by α, we have

$$\begin{aligned}\alpha^{k+1} &= \alpha^3 P_{k-4} + \alpha^2 P_{k-3} + \alpha P_{k-5}\\ &= (\alpha+1)P_{k-4} + \alpha^2 P_{k-3} + \alpha P_{k-5}\\ &= \alpha^2 P_{k-3} + \alpha(P_{k-4} + P_{k-5}) + P_{k-4}\\ &= \alpha^2 P_{k-3} + \alpha P_{k-2} + P_{k-4}\\ &= \alpha^2 P_{(k+1)-4} + \alpha P_{(k+1)-3} + P_{(k+1)-5}\,.\end{aligned}$$

Thus, $I_k \implies I_{k+1}$. Now, for any non-negative integer k,

$$I_{-k} := \alpha^{-k} = \alpha^2 P_{-k-4} + \alpha P_{-k-3} + P_{-k-5}\,.$$

Assuming I_{-k} and multiplying both sides by α^{-1} gives

$$\alpha^{-(k+1)} = \alpha P_{-k-4} + P_{-k-3} + \alpha^{-1} P_{-k-5}\,.$$

Since $\alpha^{-1} = \alpha^2 - 1$, identity (1.20), we have

$$\begin{aligned}\alpha^{-(k+1)} &= \alpha P_{-k-4} + P_{-k-3} + (\alpha^2 - 1)P_{-k-5}\\ &= \alpha^2 P_{-k-5} + \alpha P_{-k-4} + (P_{-k-3} - P_{-k-5})\\ &= \alpha^2 P_{-k-5} + \alpha P_{-k-4} + P_{-k-6}\\ &= \alpha^2 P_{-(k+1)-4} + \alpha P_{-(k+1)-3} + P_{-(k+1)-5}\,.\end{aligned}$$

Thus, $I_{-k} \implies I_{-(k+1)}$. The proof of P1 is complete. □

Standard techniques for solving difference equations give

$$Q_n = \alpha^n + \beta^n + \gamma^n\,, \tag{1.10}$$

which we shall often employ in the useful form

$$\alpha^n + \beta^n = Q_n - \gamma^n\,. \tag{1.11}$$

Adding the identities in Lemma 1.1, we have

$$Q_n = \alpha^n + \beta^n + \gamma^n = Q_2 P_{n-4} + Q_1 P_{n-3} + 3P_{n-5}\,,$$

which allows us to express the Perrin numbers in terms of the Padovan numbers:

$$Q_n = 2P_{n-4} + 3P_{n-5}\,. \tag{1.12}$$

From identity (1.5) and Lemma 1.1, we also have

$$\alpha^n + \beta^n = -\gamma^2 P_{n-4} - \gamma P_{n-3} + 2P_{n-2}\,. \tag{1.13}$$

Lemma 1.1 gives

$$\alpha^n - \beta^n = (\alpha^2 - \beta^2)P_{n-4} + (\alpha - \beta)P_{n-3}\,; \tag{1.14}$$

so that

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = -\gamma P_{n-4} + P_{n-3}\,. \tag{1.15}$$

Squaring (1.14) and making use of (1.6) and (1.8) gives

$$(\alpha^n - \beta^n)^2 = (P_{n-4}^2 - 3P_{n-3}^2)\gamma^2 - (3P_{n-4}^2 + 2P_{n-4}P_{n-3})\gamma + 4P_{n-3}^2 + 6P_{n-3}P_{n-4}\,. \tag{1.16}$$

Since $P_{n-4} = P_{n-2} - P_{n-5}$, identity (1.12) can also be written as

$$Q_n = 2P_{n-2} + P_{n-5}\,. \tag{1.17}$$

From (1.17) and Lemma 1.1 we have

Lemma 1.2. *The following identities hold for any integer n:*

$$2\alpha^{n+2} + \alpha^{n-1} = \alpha^2 Q_n + \alpha Q_{n+1} + Q_{n-1}\,, \tag{P1d}$$

$$2\beta^{n+2} + \beta^{n-1} = \beta^2 Q_n + \beta Q_{n+1} + Q_{n-1}$$

and

$$2\gamma^{n+2} + \gamma^{n-1} = \gamma^2 Q_n + \gamma Q_{n+1} + Q_{n-1}\,.$$

Presently, we derive more algebraic properties of α, β and γ.
Since α is a zero of $x^3 - x - 1$, we have

$$\alpha + 1 = \alpha^3\,; \tag{1.18}$$

so that

$$\alpha^3 - \alpha = 1 \Rightarrow \alpha(\alpha - 1)(\alpha + 1) = 1\,,$$

from which we get

$$\alpha - 1 = 1/\alpha^4, \quad \text{by (1.18)}\,, \tag{1.19}$$

and

$$\alpha^2 - 1 = 1/\alpha\,. \tag{1.20}$$

We also write identity (1.19) as

$$\alpha^4 + 1 = \alpha^5\,. \tag{1.21}$$

On account of identity (1.19), identity (1.9) is

$$\alpha/\beta + \beta/\alpha = 1/\gamma^4\,, \tag{1.22}$$

which also implies

$$1/\alpha^2 + 1/\beta^2 = 1/\gamma^3\,. \tag{1.23}$$

Addition and subtraction of (1.18) and (1.19) give

$$\alpha^7 + 1 = 2\alpha^5 \tag{1.24}$$

and

$$\alpha^7 - 1 = 2\alpha^4\,, \tag{1.25}$$

while, upon multiplication, (1.24) and (1.25) give

$$\alpha^{14} - 1 = 4\alpha^9. \tag{1.26}$$

Identities (1.18) — (1.26), excluding (1.22) and (1.23), can be collected into the following lemma.

Lemma 1.3. *The following identities hold for any integer m, for each set of values of a, b, c, d, e, f and g given in Table 1:*

$$a\alpha^{m+c} + b\alpha^{m+d} = f\alpha^{m+e}\,, \tag{1.27}$$

$$a\beta^{m+c} + b\beta^{m+d} = f\beta^{m+e} \tag{1.28}$$

and

$$a\gamma^{m+c} + b\gamma^{m+d} = f\gamma^{m+e}\,. \tag{1.29}$$

The observation that if a, b, c, d, e and f are rational numbers and λ and λ^2 are *linearly independent* irrational numbers, then $a\lambda^2 + b\lambda + c = d\lambda^2 + e\lambda + f$ if and only if $a = d$, $b = e$ and $c = f$ leads to the following properties.

	a	b	c	d	e	f
Set 1:	1	−1	1	−1	−2	1
Set 2:	1	−1	1	−2	−1	1
Set 3:	1	1	−1	−2	1	1
Set 4:	1	1	2	−2	3	1
Set 5:	1	−1	3	2	−2	1
Set 6:	1	−1	3	−2	2	1
Set 7:	2	−1	−1	1	−6	1
Set 8:	2	−1	−1	−6	1	1
Set 9:	1	1	1	−6	−1	2
Set 10:	2	1	2	−2	5	1
Set 11:	1	−1	5	−2	2	2
Set 12:	1	−2	5	2	−2	1
Set 13:	1	−1	7	−7	2	4
Set 14:	1	−4	7	2	−7	1
Set 15:	1	4	−7	2	7	1

Table 1. Coefficients of the algebraic equations satisfied by the roots of the Padovan-Perrin characteristic equation.

Lemma 1.4. *If a, b, c, d, e and f are rational numbers, then:*

$$a\alpha^2 + b\alpha + c = d\alpha^2 + e\alpha + f \iff a = d,\ b = e \text{ and } c = f\,, \tag{P2}$$

$$a\beta^2 + b\beta + c = d\beta^2 + e\beta + f \iff a = d,\ b = e \text{ and } c = f\,, \tag{P3}$$

$$a\gamma^2 + b\gamma + c = d\gamma^2 + e\gamma + f \iff a = d,\ b = e \text{ and } c = f\,, \tag{P4}$$

$$\frac{a\alpha^2+b\alpha+c}{d\alpha^2+e\alpha+f} = \frac{\begin{vmatrix} a & e & d \\ b & d+f & e \\ c & d & f \end{vmatrix}}{\begin{vmatrix} d+f & e & d \\ d+e & d+f & e \\ e & d & f \end{vmatrix}}\alpha^2 + \frac{\begin{vmatrix} d+f & a & d \\ d+e & b & e \\ e & c & f \end{vmatrix}}{\begin{vmatrix} d+f & e & d \\ d+e & d+f & e \\ e & d & f \end{vmatrix}}\alpha + \frac{\begin{vmatrix} d+f & e & a \\ d+e & d+f & b \\ e & d & c \end{vmatrix}}{\begin{vmatrix} d+f & e & d \\ d+e & d+f & e \\ e & d & f \end{vmatrix}}, \tag{P5}$$

for d, e, f not all zero; with similar expressions for β and γ.

Upon setting $a = 0 = b$ and $c = 1$, property P5 *gives*

$$\frac{1}{d\alpha^2+e\alpha+f} = \frac{e^2-d^2-fd}{\begin{vmatrix} d+f & e & d \\ d+e & d+f & e \\ e & d & f \end{vmatrix}}\alpha^2 + \frac{d^2-ef}{\begin{vmatrix} d+f & e & d \\ d+e & d+f & e \\ e & d & f \end{vmatrix}}\alpha + \frac{d^2+2fd+f^2-ed-e^2}{\begin{vmatrix} d+f & e & d \\ d+e & d+f & e \\ e & d & f \end{vmatrix}}. \tag{P6}$$

From P6*, by setting $d = 0$ and $f = 0$, in turn, we have*

$$\frac{1}{e\alpha+f} = \frac{e^2}{f^3-e^2f+e^3}\alpha^2 - \frac{ef}{f^3-e^2f+e^3}\alpha + \frac{f^2-e^2}{f^3-e^2f+e^3} \tag{P7}$$

and

$$\frac{1}{d\alpha^2+e\alpha} = \frac{e^2-d^2}{e^3+d^3-ed^2}\alpha^2 + \frac{d^2}{e^3+d^3-ed^2}\alpha + \frac{d^2-ed-e^2}{e^3+d^3-ed^2}\,. \tag{P8}$$

A note on notation

Let F be any expression of the form

$$F := f_2\alpha^2 + f_1\alpha + f_0\,, \tag{1.30}$$

where f_2, f_1 and f_0 are rational numbers. Every such F can be considered a vector with components f_2, f_1 and f_0 in a three-dimensional vector space with basis vectors $\{1, \alpha, \alpha^2\}$. We introduce the notation: $(F)_{\alpha^2} = f_2$, $(F)_{\alpha^1} = f_1$ and $(F)_{\alpha^0} = f_0$. Thus, the F in (1.30) can be written

$$F = (F)_{\alpha^2}\,\alpha^2 + (F)_{\alpha}\,\alpha + (F)_{\alpha^0}\,. \tag{1.31}$$

Similarly, if $G := g_2\beta^2 + g_1\beta + g_0$ and $H := h_2\gamma^2 + h_1\gamma + h_0$ where g_2, g_1, g_0 and h_2, h_1, h_0 are rational numbers, then,

$$G = (G)_{\beta^2}\,\beta^2 + (G)_{\beta}\,\beta + (G)_{\beta^0} \tag{1.32}$$

and

$$H = (H)_{\gamma^2}\,\gamma^2 + (H)_{\gamma}\,\gamma + (H)_{\gamma^0}\ ; \tag{1.33}$$

so that $(F)_{\alpha^j}$, $j \in \{0,1,2\}$, denotes the α^j component of F, ... etc. Thus, for example, from identities P1, P5 and P6, we can write

$$(\alpha^n)_{\alpha^2} = P_{n-4}, \quad (\alpha^n)_{\alpha} = P_{n-3}, \quad (\alpha^n)_{\alpha^0} = P_{n-5}\,, \tag{P9}$$

$$\left(\frac{a\alpha^2 + b\alpha + c}{d\alpha^2 + e\alpha + f}\right)_{\alpha^2} = \frac{\begin{vmatrix} a & e & d \\ b & d+f & e \\ c & d & f \end{vmatrix}}{\begin{vmatrix} d+f & e & d \\ d+e & d+f & e \\ e & d & f \end{vmatrix}} \tag{P10}$$

and

$$\left(\frac{1}{d\alpha^2 + e\alpha + f}\right)_{\alpha^2} = \frac{e^2 - d^2 - fd}{\begin{vmatrix} d+f & e & d \\ d+e & d+f & e \\ e & d & f \end{vmatrix}}\,. \tag{P11}$$

Theorem 1.5. *For all integers n, we have*

$$P_{-n} = P_{n-7}^2 - P_{n-6}P_{n-8} \tag{1.34}$$

and

$$2Q_{-n} = Q_n^2 - Q_{2n}\,. \tag{1.35}$$

Proof. We have

$$(\alpha^{-n})_{\alpha^2} = \left(\frac{1}{\alpha^n}\right)_{\alpha^2}\,,$$

which by P1 and P2 gives

$$P_{-n-4} = \frac{P_{n-3}^2 - P_{n-4}^2 - P_{n-4}P_{n-5}}{\begin{vmatrix} P_{n-4}+P_{n-5} & P_{n-3} & P_{n-4} \\ P_{n-3}+P_{n-4} & P_{n-4}+P_{n-5} & P_{n-3} \\ P_{n-3} & P_{n-4} & P_{n-5} \end{vmatrix}} = \frac{P_{n-3}^2 - P_{n-4}^2 - P_{n-4}P_{n-5}}{\begin{vmatrix} P_{n-2} & P_{n-3} & P_{n-4} \\ P_{n-1} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-4} & P_{n-5} \end{vmatrix}}\,,$$

in which the numerator factors into $P_{n-3}^2 - P_{n-2}P_{n-4}$ while the determinant in the denominator evaluates to 1 for all n; and hence identity (1.34).

Squaring (1.10) gives

$$\begin{aligned} Q_n^2 &= \alpha^{2n} + \beta^{2n} + \gamma^{2n} + 2\beta^n\gamma^n + 2\alpha^n\gamma^n + 2\alpha^n\beta^n \\ &= \underbrace{\alpha^{2n} + \beta^{2n} + \gamma^{2n}}_{Q_{2n}} + \underbrace{2\alpha^{-n} + 2\beta^{-n} + 2\gamma^{-n}}_{2Q_{-n}}\,, \end{aligned}$$

from which (1.35) follows. □

Theorem 1.6. *The following identities hold for integers p and n:*

$$Q_{-n}P_p - P_{p-n} = Q_nP_{p+n} - P_{p+2n} \tag{1.36}$$

and

$$P_pP_{-n-3} - P_{p+1}P_{-n-4} = P_{p+n+1}P_{n-4} - P_{p+n}P_{n-3}\,. \tag{1.37}$$

Proof. Set $x = \alpha$ and $y = \beta$ in the identity

$$x^{-n} + y^{-n} = (xy)^{-n}(x^n + y^n)\,,$$

multiply through by γ^{p+4} and make use of identity (1.11) to obtain

$$Q_{-n}\gamma^{p+4} - \gamma^{p-n+4} = Q_n\gamma^{p+n+4} - \gamma^{p+2n+4}\,,$$

from which identity (1.36) follows upon use of the γ version of property P9.

Set $x = \alpha$ and $y = \beta$ in the identity

$$\frac{x^{-n} - y^{-n}}{x - y} = -(xy)^{-n}\frac{x^n - y^n}{x - y}\,,$$

multiply through by γ^{p+4} and make use of identity (1.15) to obtain

$$\gamma^{p+4}P_{-n-3} - \gamma^{p+5}P_{-n-4} = \gamma^{p+n+5}P_{n-4} - \gamma^{p+n+4}P_{n-3}\,,$$

which then yields identity (1.37). □

Corollary 1.7. *Let $\lambda \in \{p : P_p = 0\}$, that is $\lambda \in \{-17, -8, -4, -3, -1\}$. Then the following identities hold for any integer n:*

$$P_{-n} = P_{2n+3\lambda} - Q_{n+\lambda}P_{n+2\lambda}\,, \tag{1.38}$$

$$P_{n+\lambda}Q_{-n} = P_{2n+\lambda}Q_n - P_{3n+\lambda} \tag{1.39}$$

$$Q_{-n} = Q_nP_n - Q_{n-1}P_{n-2} - P_{2n-2}\,, \tag{1.40}$$

$$P_{\lambda+1}P_{-n} = P_{\lambda+n-4}P_{n-7} - P_{\lambda+n-3}P_{n-8} \tag{1.41}$$

and

$$P_{\lambda-1}P_{-n} = P_{\lambda+n-3}P_{n-7} - P_{\lambda+n-4}P_{n-6}\,. \tag{1.42}$$

Lemma 1.8. *The following identities hold for integers r, s and t, $s \neq 0$:*

$$\left((\alpha^r + \beta^r)\gamma^t\right)_{\gamma^2} = Q_rP_{t-4} - P_{r+t-4} \tag{1.43}$$

and

$$\left(\frac{\alpha^r - \beta^r}{\alpha^s - \beta^s}\gamma^t\right)_{\gamma^2} = \frac{\begin{vmatrix} P_{t-3}P_{r-4} - P_{t-4}P_{r-3} & P_{s-4} & 0 \\ P_{t-2}P_{r-4} - P_{t-3}P_{r-3} & -P_{s-3} & P_{s-4} \\ P_{t-4}P_{r-4} - P_{t-5}P_{r-3} & 0 & -P_{s-3} \end{vmatrix}}{\begin{vmatrix} -P_{s-3} & P_{s-4} & 0 \\ P_{s-4} & -P_{s-3} & P_{s-4} \\ P_{s-4} & 0 & -P_{s-3} \end{vmatrix}}\,. \tag{1.44}$$

Proof. Using identity (1.11), we have

$$(\alpha^r + \beta^r)\gamma^t = (Q_r - \gamma^r)\gamma^t = Q_r\gamma^t - \gamma^{r+t}\,,$$

from which identity (1.43) follows when we use properties P1c and P10.

By (1.15),

$$\frac{\alpha^r - \beta^r}{\alpha^s - \beta^s}\gamma^t = \frac{\alpha^r - \beta^r}{\alpha - \beta}\,\frac{\alpha - \beta}{\alpha^s - \beta^s}\gamma^t = \frac{\gamma^{t+1}P_{r-4} - \gamma^tP_{r-3}}{\gamma P_{s-4} - P_{s-3}}\,, \tag{1.45}$$

from which identity (1.44) follows upon application of properties P1c and P10. □

Lemma 1.9. *Let a and b be rational numbers and f and g functions of α. The α^j components $(\cdot)_{\alpha^j}$ have the following composition rules:*

$$(af)_{\alpha^j} = a\,(f)_{\alpha^j}\ , \tag{1.46}$$

$$(af+bg)_{\alpha^j} = a\,(f)_{\alpha^j} + b\,(g)_{\alpha^j}\ , \tag{1.47}$$

$$(fg)_{\alpha^2} = (f)_{\alpha^0}(g)_{\alpha^2} + (f)_{\alpha^2}(g)_{\alpha^0} + (f)_{\alpha}(g)_{\alpha} + (f)_{\alpha^2}(g)_{\alpha^2}\ , \tag{R1}$$

$$(fg)_{\alpha} = (f)_{\alpha^0}(g)_{\alpha} + (f)_{\alpha}(g)_{\alpha^0} + (f)_{\alpha^2}(g)_{\alpha} + (f)_{\alpha}(g)_{\alpha^2} + (f)_{\alpha^2}(g)_{\alpha^2} \tag{R2}$$

and

$$(fg)_{\alpha^0} = (f)_{\alpha^0}(g)_{\alpha^0} + (f)_{\alpha}(g)_{\alpha^2} + (f)_{\alpha^2}(g)_{\alpha}\,. \tag{R3}$$

Theorem 1.10. *The following identity holds for integers m and n:*

$$P_{m+n} = P_m P_{n-5} + P_{m+1}P_{n-3} + P_{m+2}P_{n-4}\,.$$

Proof. Set $f=\alpha^m$ and $g=\alpha^n$ in R3, use P9 and note also that $\alpha^m\alpha^n=\alpha^{m+n}$. □

2 Summation identities

In this section, we will derive various summation identities involving the Padovan and Perrin numbers. Both summation identities not involving binomial coefficients as well as identities that involve binomial coefficients will be obtained.

2.1 Summation identities not involving binomial coefficients

In §2.1 – §2.1, summation identities involving Padovan and Perrin numbers but not containing binomial coefficients will be derived.

Sums of Padovan and Perrin numbers with subscripts in arithmetic progression

Setting $x=\alpha^p$ in the geometric sum identity

$$\sum_{j=0}^{n} x^j = \frac{x^{n+1}-1}{x-1} \tag{2.1}$$

and multiplying through by α^{q+4} gives

$$\sum_{j=0}^{n} \alpha^{pj+q+4} = \frac{\alpha^{pn+p+q+4}-\alpha^{q+4}}{\alpha^p-1}\,. \tag{2.2}$$

Thus, we have

$$\sum_{j=0}^{n}\left(\alpha^{pj+q+4}\right)_{\alpha^2} = \left(\frac{(P_{pn+p+q}-P_q)\alpha^2 + (P_{pn+p+q+1}-P_{q+1})\alpha + P_{pn+p+q-1}-P_{q-1}}{P_{p-4}\alpha^2+P_{p-3}\alpha+P_{p-5}-1}\right)_{\alpha^2}; \tag{2.3}$$

and hence, by P10,

$$\sum_{j=0}^{n} P_{pj+q} = \frac{\begin{vmatrix} P_{pn+p+q}-P_q & P_{p-3} & P_{p-4}\\ P_{pn+p+q+1}-P_{q+1} & P_{p-2}-1 & P_{p-3}\\ P_{pn+p+q-1}-P_{q-1} & P_{p-4} & P_{p-5}-1\end{vmatrix}}{\begin{vmatrix} P_{p-2}-1 & P_{p-3} & P_{p-4}\\ P_{p-1} & P_{p-2}-1 & P_{p-3}\\ P_{p-3} & P_{p-4} & P_{p-5}-1\end{vmatrix}}\,. \tag{2.4}$$

Using (2.1) and P1d, a similar calculation for the Perrin numbers yields

$$\sum_{j=0}^{n} Q_{pj+q} = \frac{\begin{vmatrix} Q_{pn+p+q} - Q_q & P_{p-3} & P_{p-4} \\ Q_{pn+p+q+1} - Q_{q+1} & P_{p-2} - 1 & P_{p-3} \\ Q_{pn+p+q-1} - Q_{q-1} & P_{p-4} & P_{p-5} - 1 \end{vmatrix}}{\begin{vmatrix} P_{p-2} - 1 & P_{p-3} & P_{p-4} \\ P_{p-1} & P_{p-2} - 1 & P_{p-3} \\ P_{p-3} & P_{p-4} & P_{p-5} - 1 \end{vmatrix}} . \tag{2.5}$$

In particular we have

$$\sum_{j=0}^{n} P_{j+q} = P_{n+q+5} - P_{q+4} \tag{2.6}$$

and

$$\sum_{j=0}^{n} Q_{j+q} = Q_{n+q+5} - Q_{q+4} . \tag{2.7}$$

Generating functions for Padovan and Perrin numbers with indices in arithmetic progression

Setting $x = y\alpha^p$ in the identity

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x} \tag{2.8}$$

and multiplying through by α^{q+4} gives

$$\sum_{j=0}^{\infty} \alpha^{pj+q+4} y^j = \frac{\alpha^{q+4}}{1-\alpha^p y} = \frac{P_q\alpha^2 + P_{q+1}\alpha + P_{q-1}}{-yP_{p-4}\alpha^2 - yP_{p-3}\alpha - yP_{p-5} + 1} . \tag{2.9}$$

Thus, by property P10, we have

$$\sum_{j=0}^{\infty} P_{pj+q} y^j = \frac{\begin{vmatrix} P_q & -yP_{p-3} & -yP_{p-4} \\ P_{q+1} & -yP_{p-2} + 1 & -yP_{p-3} \\ P_{q-1} & -yP_{p-4} & -yP_{p-5} + 1 \end{vmatrix}}{\begin{vmatrix} -yP_{p-2} + 1 & -yP_{p-3} & -yP_{p-4} \\ -yP_{p-1} & -yP_{p-2} + 1 & -yP_{p-3} \\ -yP_{p-3} & -yP_{p-4} & -yP_{p-5} + 1 \end{vmatrix}} . \tag{2.10}$$

Similarly,

$$\sum_{j=0}^{\infty} Q_{pj+q} y^j = \frac{\begin{vmatrix} Q_q & -yP_{p-3} & -yP_{p-4} \\ Q_{q+1} & -yP_{p-2} + 1 & -yP_{p-3} \\ Q_{q-1} & -yP_{p-4} & -yP_{p-5} + 1 \end{vmatrix}}{\begin{vmatrix} -yP_{p-2} + 1 & -yP_{p-3} & -yP_{p-4} \\ -yP_{p-1} & -yP_{p-2} + 1 & -yP_{p-3} \\ -yP_{p-3} & -yP_{p-4} & -yP_{p-5} + 1 \end{vmatrix}} . \tag{2.11}$$

Exponential generating functions for Padovan and Perrin numbers with indices in arithmetic progression

In addition to the ordinary generating functions of the Padovan and Perrin numbers, given in (2.10) and (2.11), it turns out that the exponential generating functions of these numbers are also readily obtained and have rather simple forms as presented in Theorem 2.1.

Theorem 2.1. *The following identities hold for integers p and q:*

$$\sum_{j=0}^{\infty} \frac{P_{pj+q}}{j!} y^j = \frac{-i}{\sqrt{23}} \begin{vmatrix} \alpha^{q+4} e^{\alpha^p y} & \alpha & 1 \\ \beta^{q+4} e^{\beta^p y} & \beta & 1 \\ \gamma^{q+4} e^{\gamma^p y} & \gamma & 1 \end{vmatrix} \tag{2.12}$$

and

$$\sum_{j=0}^{\infty} \frac{Q_{pj+q}}{j!} y^j = \frac{-i}{\sqrt{23}} \begin{vmatrix} (2\alpha^{q+2} + \alpha^{q-1}) e^{\alpha^p y} & \alpha & 1 \\ (2\beta^{q+2} + \beta^{q-1}) e^{\beta^p y} & \beta & 1 \\ (2\gamma^{q+2} + \gamma^{q-1}) e^{\gamma^p y} & \gamma & 1 \end{vmatrix} . \tag{2.13}$$

Proof. By Taylor series expansion and P1 we have

$$\begin{aligned} \alpha^{q+4} e^{\alpha^p y} &= \sum_{j=0}^{\infty} \frac{\alpha^{pj+q+4} y^j}{j!} \\ &= \alpha^2 \sum_{j=0}^{\infty} \frac{P_{pj+q} y^j}{j!} + \alpha \sum_{j=0}^{\infty} \frac{P_{pj+q+1} y^j}{j!} + \sum_{j=0}^{\infty} \frac{P_{pj+q-1} y^j}{j!} . \end{aligned}$$

Thus,

$$\alpha^{q+4} e^{\alpha^p y} = \alpha^2 A(p,q) + \alpha B(p,q) + C(p,q) , \tag{2.14}$$

where

$$A(p,q) = \sum_{j=0}^{\infty} \frac{P_{pj+q} y^j}{j!} \tag{2.15}$$

and

$$B(p,q) = \sum_{j=0}^{\infty} \frac{P_{pj+q+1} y^j}{j!}, \quad C(p,q) = \sum_{j=0}^{\infty} \frac{P_{pj+q-1} y^j}{j!} .$$

Similar calculations give

$$\beta^{q+4} e^{\beta^p y} = \beta^2 A(p,q) + \beta B(p,q) + C(p,q) \tag{2.16}$$

and

$$\gamma^{q+4} e^{\gamma^p y} = \gamma^2 A(p,q) + \gamma B(p,q) + C(p,q) . \tag{2.17}$$

Solving (2.14), (2.16) and (2.17) simultaneously for A, B and C, Crammer's rule gives

$$A(p,q) = \Delta_A / \Delta ,$$

where

$$\Delta = \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} = i\sqrt{23}$$

and

$$\Delta_A = \begin{vmatrix} \alpha^{q+4} e^{\alpha^p y} & \alpha & 1 \\ \beta^{q+4} e^{\beta^p y} & \beta & 1 \\ \gamma^{q+4} e^{\gamma^p y} & \gamma & 1 \end{vmatrix} .$$

The proof of (2.13) is similar. We start with

$$(2\alpha^{q+2} + \alpha^{q+2}) e^{\alpha^p y} = \sum_{j=0}^{\infty} (2\alpha^{pj+q+2} + \alpha^{pj+q-1}) \frac{y^j}{j!}$$

and make use of property P1d. □

Weighted Padovan and Perrin summation

Replacing x with x/y in identity (2.1) gives

$$(x-y)\sum_{j=0}^{n} y^{r-j}x^{j} = y^{r-n}x^{n+1} - y^{r+1}\,, \tag{2.18}$$

for integers r and n and arbitrary x and y.

Theorem 2.2. *The following identities hold for integers m, n and r, for each set of values of a, b, c, d and e given in the table:*

$$\begin{aligned} &bf^{n+1}\sum_{j=0}^{n} a^{n-j}P_{m+d+(m+c)r-4+(e-c)j} \\ &\quad = f^{n+1}P_{(m+c)r+(e-c)n+m+e-4} - a^{n+1}P_{(m+c)(r+1)-4} \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &bf^{n+1}\sum_{j=0}^{n} a^{n-j}Q_{m+d+(m+c)r-4+(e-c)j} \\ &\quad = f^{n+1}Q_{(m+c)r+(e-c)n+m+e-4} - a^{n+1}Q_{(m+c)(r+1)-4}\,. \end{aligned} \tag{2.20}$$

Proof. With identity (1.27) in mind, set $x = f\alpha^{m+e}$ and $y = a\alpha^{m+c}$ in identity (2.18). □

In particular, we have

$$\begin{aligned} &bf^{n+1}\sum_{j=0}^{n} a^{n-j}P_{m+d+(m+c)n-4+(e-c)j} \\ &\quad = f^{n+1}P_{(m+e)(n+1)-4} - a^{n+1}P_{(m+c)(n+1)-4} \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} &bf^{n+1}\sum_{j=0}^{n} a^{n-j}Q_{m+d+(m+c)n-4+(e-c)j} \\ &\quad = f^{n+1}Q_{(m+e)(n+1)-4} - a^{n+1}Q_{(m+c)(n+1)-4}\,. \end{aligned} \tag{2.22}$$

Here are explicit examples, with the indicated set of values of a, b, c, d, e and f as read from the table on page 637.

$$\text{Set 1: } \sum_{j=0}^{n} P_{m+(m+1)r-5-3j} = P_{(m+1)(r+1)-4} - P_{(m+1)r-3n+m-6}\,, \tag{2.23}$$

$$\sum_{j=0}^{n} Q_{m+(m+1)r-5-3j} = Q_{(m+1)(r+1)-4} - Q_{(m+1)r-3n+m-6}\,, \tag{2.24}$$

$$\text{Set 7: } \sum_{j=0}^{n} 2^{n-j}P_{m+(m-1)r-3-5j} = 2^{n+1}P_{(m-1)(r+1)-4} - P_{(m-1)r-5n+m-10}\,, \tag{2.25}$$

$$\sum_{j=0}^{n} 2^{n-j}Q_{m+(m-1)r-3-5j} = 2^{n+1}Q_{(m-1)(r+1)-4} - Q_{(m-1)r-5n+m-10}\,, \tag{2.26}$$

$$\text{Set 10: } \sum_{j=0}^{n} 2^{n-j}P_{m+(m+2)r-6+3j} = P_{(m+2)r+3n+m+1} - 2^{n+1}P_{(m+2)(r+1)-4}\,, \tag{2.27}$$

$$\sum_{j=0}^{n} 2^{n-j} Q_{m+(m+2)r-6+3j} = Q_{(m+2)r+3n+m+1} - 2^{n+1} Q_{(m+2)(r+1)-4}\,. \tag{2.28}$$

Further summation identities can be obtained from the following identities:

$$x\sum_{j=0}^{n} y^{r-j}(x+y)^j = y^{r-n}(x+y)^{n+1} - y^{r+1} \tag{2.29}$$

and

$$(x-y)\sum_{j=0}^{n} x^{r-j} y^j = x^{r+1} - x^{r-n} y^{n+1}\,. \tag{2.30}$$

Identity (2.29) is obtained by replacing x with $x+y$ in identity (2.18) while identity (2.30) is obtained by interchanging x and y in identity (2.18).

Sums of certain products of Padovan and Perrin numbers

Since

$$\sum_{j=0}^{n} y^{n-j} x^j = \sum_{j=0}^{n} y^j x^{n-j}\,,$$

identity (2.18) implies

$$\frac{1}{2}\sum_{j=0}^{n} (xy)^j (x^{n-2j} + y^{n-2j}) = \frac{x^{n+1} - y^{n+1}}{x-y}\,. \tag{2.31}$$

Theorem 2.3. *The following identities hold for integers p, q and n:*

$$\sum_{j=0}^{n} P_{q-pj} Q_{p(n-2j)} = \frac{\begin{vmatrix} P_{pn+3p+q} - P_{q-2pn} & P_{3p-3} & P_{3p-4} \\ P_{pn+3p+q+1} - P_{q-2pn+1} & P_{3p-2} - 1 & P_{3p-3} \\ P_{pn+3p+q-1} - P_{q-2pn-1} & P_{3p-4} & P_{3p-5} - 1 \end{vmatrix}}{\begin{vmatrix} P_{3p-2} - 1 & P_{3p-3} & P_{3p-4} \\ P_{3p-1} & P_{3p-2} - 1 & P_{3p-3} \\ P_{3p-3} & P_{3p-4} & P_{3p-5} - 1 \end{vmatrix}} + \frac{2\begin{vmatrix} P_{q+1}P_{pn+p-4} - P_q P_{pn+p-3} & P_{p-4} & 0 \\ P_{q+2}P_{pn+p-4} - P_{q+1}P_{pn+p-3} & -P_{p-3} & P_{p-4} \\ P_q P_{pn+p-4} - P_{q-1}P_{pn+p-3} & 0 & -P_{p-3} \end{vmatrix}}{\begin{vmatrix} -P_{p-3} & P_{p-4} & 0 \\ P_{p-4} & -P_{p-3} & P_{p-4} \\ P_{p-4} & 0 & -P_{p-3} \end{vmatrix}}\,. \tag{2.32}$$

Proof. Set $x = \alpha^p$, $y = \beta^p$ in identity (2.31) and make use of Lemma 1.8. □

A particular case is

$$\sum_{j=0}^{n} P_{q-j} Q_{n-2j} = P_{n+q+2} - P_{q-2n-1} - 2(P_{q+1}P_{n-3} - P_q P_{n-2})\,. \tag{2.33}$$

2.2 Binomial summation identities

Binomial summation identities involving the Padovan and Perrin numbers will be derived in §2.2 and §2.2. These will be facilitated by variations on the standard binomial theorem and the Waring identity and its dual.

Identities from the binomial formula

With identity (1.27) in mind; substitute $x = a\alpha^{m+c}$ and $y = b\alpha^{m+d}$ in the binomial formula

$$\sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j} = (x+y)^n \,, \tag{2.34}$$

multiply through by α^{p+4} and make use of properties P1 and P9 to obtain the following result:

Theorem 2.4. *The identity*

$$\sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} P_{(m+d)n+p+(c-d)j} = f^n P_{(m+e)n+p} \,,$$

holds for non-negative integer n, arbitrary integers m and p, and values of a, b, c, d and e as given in Table 1.

The corresponding Perrin version of the identity of Theorem 2.4 is

$$\sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} Q_{(m+d)n+p+(c-d)j} = f^n Q_{(m+e)n+p} \,. \tag{2.35}$$

Here are some explicit examples from Theorem 2.4 and sets of values from Table 1:

$$\text{Set 1: } \sum_{j=o}^{n} (-1)^j \binom{n}{j} P_{(m-1)n+p+2j} = (-1)^n P_{(m-2)n+p} \,, \tag{2.36}$$

$$\sum_{j=o}^{n} (-1)^j \binom{n}{j} Q_{(m-1)n+p+2j} = (-1)^n Q_{(m-2)n+p} \,, \tag{2.37}$$

$$\text{Set 4: } \sum_{j=o}^{n} \binom{n}{j} P_{(m-2)n+p+4j} = P_{(m+3)n+p} \,, \tag{2.38}$$

$$\sum_{j=o}^{n} \binom{n}{j} Q_{(m-2)n+p+4j} = Q_{(m+3)n+p} \,, \tag{2.39}$$

$$\text{Set 7: } \sum_{j=o}^{n} (-1)^j \binom{n}{j} 2^j P_{(m+1)n+p-2j} = (-1)^n P_{(m-6)n+p} \,, \tag{2.40}$$

$$\sum_{j=o}^{n} (-1)^j \binom{n}{j} 2^j Q_{(m+1)n+p-2j} = (-1)^n Q_{(m-6)n+p} \,, \tag{2.41}$$

$$\text{Set 10: } \sum_{j=o}^{n} 2^j \binom{n}{j} P_{(m-2)n+p+4j} = P_{(m+5)n+p} \,, \tag{2.42}$$

$$\sum_{j=o}^{n} 2^j \binom{n}{j} Q_{(m-2)n+p+4j} = Q_{(m+5)n+p} \,, \tag{2.43}$$

$$\text{Set 13: } \sum_{j=o}^{n} (-1)^j \binom{n}{j} P_{(m-7)n+p+14j} = (-1)^n 4^n P_{(m+2)n+p} \,, \tag{2.44}$$

$$\sum_{j=o}^{n} (-1)^j \binom{n}{j} Q_{(m-7)n+p+14j} = (-1)^n 4^n Q_{(m+2)n+p} \,. \tag{2.45}$$

Many more binomial identities can be derived by making appropriate substitutions in the following variations on the binomial formula:

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(x+y)^j y^{n-j}=(-1)^n x^n\,, \tag{2.46}$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}x^j(x+y)^{n-j}=y^n\,, \tag{2.47}$$

$$\sum_{j=0}^{n}\binom{n}{j}jx^{j-1}y^{n-j}=n(x+y)^{n-1}\,, \tag{2.48}$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j(x+y)^{j-1}y^{n-j}=(-1)^n nx^{n-1} \tag{2.49}$$

and

$$\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}x^{j-1}j(x+y)^{n-j}=ny^{n-1}\,. \tag{2.50}$$

Note that identities (2.46) and (2.47) are obtained from identity (2.34) by obvious transformations while identity (2.48) is obtained by differentiating the identity

$$\sum_{j=0}^{n}\binom{n}{j}x^j e^{jz}y^{n-j}=(xe^z+y)^n \tag{2.51}$$

with respect to z and then setting z to zero. More generally,

$$\sum_{j=0}^{n}\binom{n}{j}j^r x^j y^{n-j}=\left.\frac{d^r}{dz^r}(xe^z+y)^n\right|_{z=0}\,. \tag{2.52}$$

Identities (2.49) and (2.50) are variations on identity (2.48).

Identities from Waring identity

Waring's formula and its dual [1, Equations (22) and (1)] are

$$\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{n}{n-j}\binom{n-j}{j}(xy)^j(x+y)^{n-2j}=x^n+y^n \tag{2.53}$$

and

$$\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{n-j}{j}(xy)^j(x+y)^{n-2j}=\frac{x^{n+1}-y^{n+1}}{x-y}\,. \tag{2.54}$$

Identity (2.53) holds for positive integer n while identity (2.54) holds for any non-negative integer n.

Theorem 2.5. *The following identities hold for positive integer n and arbitrary integer p:*

$$\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{n}{n-j}\binom{n-j}{j}P_{p+n-3j}=(-1)^n(Q_nP_p-P_{n+p})\,, \tag{2.55}$$

$$\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{n-j}{j}P_{p+n-3j}=(-1)^{n-1}(P_{p+1}P_{n-3}-P_pP_{n-2})\,, \tag{2.56}$$

$$\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{n}{n-j}\binom{n-j}{j}P_{p-4n+8j}=P_{p+n}Q_{2n}-P_{p+3n}\,, \tag{2.57}$$

$$\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{n-j}{j}P_{p-4n+8j}=P_{p+n}P_{2n-2}-P_{p+n-1}P_{2n-1}\,, \tag{2.58}$$

$$\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{n}{n-j}\binom{n-j}{j}P_{p-3n+8j}=P_{2n+p}Q_{2n}-P_{p+4n} \tag{2.59}$$

and

$$\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{n-j}{j}P_{p-3n+8j}=P_{2n+p}P_{2n-2}-P_{2n+p-1}P_{2n-1}\,. \tag{2.60}$$

Proof. Identities (2.55), (2.57) and (2.59) are obtained by choosing $(x,y)=(\alpha,\beta)$, $(x,y)=(\alpha/\beta,\beta/\alpha)$ and $(x,y)=(1/\alpha^2,1/\beta^2)$ in identity (2.53), in turn, and making use of (1.4), (1.11), (1.22) and (1.23). Identities (2.56), (2.58) and (2.60) are obtained by choosing $(x,y)=(\alpha,\beta)$, $(x,y)=(\alpha/\beta,\beta/\alpha)$ and $(x,y)=(1/\alpha^2,1/\beta^2)$ in identity (2.54), in turn, and making use of (1.4), (1.11), (1.22), (1.23) and (1.15). □

Theorem 2.6. *The following identities hold for positive integer n and any integer p; where values of a, b, c, d, e and f are given in the attached table (Table 1 reproduced here for ease of reference):*

$$\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{n}{n-j}\binom{n-j}{j}a^jb^jf^{n-2j}P_{(m+e)n+p+(c-2e+d)j}=a^nP_{(m+c)n+p}+b^nP_{(m+d)n+p} \tag{2.61}$$

and

$$\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{n}{n-j}\binom{n-j}{j}a^jb^jf^{n-2j}Q_{(m+e)n+p+(c-2e+d)j}=a^nQ_{(m+c)n+p}+b^nQ_{(m+d)n+p}\,. \tag{2.62}$$

	a	b	c	d	e	f
Set 1:	1	−1	1	−1	−2	1
Set 2:	1	−1	1	−2	−1	1
Set 3:	1	1	−1	−2	1	1
Set 4:	1	1	2	−2	3	1
Set 5:	1	−1	3	2	−2	1
Set 6:	1	−1	3	−2	2	1
Set 7:	2	−1	−1	1	−6	1
Set 8:	2	−1	−1	−6	1	1

	a	b	c	d	e	f
Set 9:	1	1	1	−6	−1	2
Set 10:	2	1	2	−2	5	1
Set 11:	1	−1	5	−2	2	2
Set 12:	1	−2	5	2	−2	1
Set 13:	1	−1	7	−7	2	4
Set 14:	1	−4	7	2	−7	1
Set 15:	1	4	−7	2	7	1

Proof. Use $(x,y)=(a\alpha^{m+c},b\alpha^{m+d})$ in (2.53) while taking note of (1.27). □

Below we give explicit examples from identities (2.61) and (2.62), using the values of a, b, c, d, e and f as given in the indicated set, in each case, as seen from the attached table of Theorem 2.6.

$$\text{Set 1:}\ \sum_{j=0}^{\lfloor n/2\rfloor}\frac{n}{n-j}\binom{n-j}{j}P_{(m-2)n+p+4j}=P_{(m+1)n+p}+(-1)^nP_{(m-1)n+p}\,, \tag{2.63}$$

$$\text{Set 1:}\ \sum_{j=0}^{\lfloor n/2\rfloor}\frac{n}{n-j}\binom{n-j}{j}Q_{(m-2)n+p+4j}=Q_{(m+1)n+p}+(-1)^nQ_{(m-1)n+p}\,, \tag{2.64}$$

$$\text{Set 4: } \sum_{j=0}^{\lfloor n/2\rfloor} (-1)^j \frac{n}{n-j}\binom{n-j}{j} P_{(m+3)n+p-6j} = P_{(m+2)n+p} + P_{(m-2)n+p}\,, \tag{2.65}$$

$$\text{Set 4: } \sum_{j=0}^{\lfloor n/2\rfloor} (-1)^j \frac{n}{n-j}\binom{n-j}{j} Q_{(m+3)n+p-6j} = Q_{(m+2)n+p} + Q_{(m-2)n+p}\,, \tag{2.66}$$

$$\text{Set 7: } \sum_{j=0}^{\lfloor n/2\rfloor} \frac{n}{n-j}\binom{n-j}{j} 2^j P_{(m-6)n+p+12j} = 2^n P_{(m-1)n+p} + (-1)^n P_{(m+1)n+p}\,, \tag{2.67}$$

$$\text{Set 7: } \sum_{j=0}^{\lfloor n/2\rfloor} \frac{n}{n-j}\binom{n-j}{j} 2^j Q_{(m-6)n+p+12j} = 2^n Q_{(m-1)n+p} + (-1)^n Q_{(m+1)n+p}\,, \tag{2.68}$$

$$\text{Set 10: } \sum_{j=0}^{\lfloor n/2\rfloor} (-1)^j \frac{n}{n-j}\binom{n-j}{j} 2^j P_{(m+5)n+p-10j} = 2^n P_{(m+2)n+p} + P_{(m-2)n+p}\,, \tag{2.69}$$

$$\text{Set 10: } \sum_{j=0}^{\lfloor n/2\rfloor} (-1)^j \frac{n}{n-j}\binom{n-j}{j} 2^j Q_{(m+5)n+p-10j} = 2^n Q_{(m+2)n+p} + Q_{(m-2)n+p}\,, \tag{2.70}$$

$$\text{Set 13: } \sum_{j=0}^{\lfloor n/2\rfloor} \frac{n}{n-j}\binom{n-j}{j} 2^{2n-4j} P_{(m+2)n+p-4j} = P_{(m+7)n+p} + (-1)^n P_{(m-7)n+p} \tag{2.71}$$

and

$$\text{Set 13: } \sum_{j=0}^{\lfloor n/2\rfloor} \frac{n}{n-j}\binom{n-j}{j} 2^{2n-4j} Q_{(m+2)n+p-4j} = Q_{(m+7)n+p} + (-1)^n Q_{(m-7)n+p}\,. \tag{2.72}$$

2.3 Double binomial summation identities

To conclude our study of series involving the Padovan and Perrin numbers, we now present double binomial summation identities involving these numbers. Theorem 2.7 states identities from the standard double binomial identity while double binomial summation identities obtained from the Waring formula and its dual are presented in Theorem 2.8

Theorem 2.7. *The following identities hold for positive integer n and arbitrary integers m, p and q:*

$$\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k} P_{p-4}^k P_{p-3}^{j-k} P_{p-5}^{n-j} P_{mn+q+k+j} = P_{(m+p)n+q}\,, \tag{2.73}$$

$$\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k} P_{m-4}^k P_{m-3}^{j-k} P_{m-5}^{n-j} P_{pn+q+k+j} = P_{(m+p)n+q}\,, \tag{2.74}$$

$$\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k} P_{p-4}^k P_{p-3}^{j-k} P_{p-5}^{n-j} Q_{mn+q+k+j} = Q_{(m+p)n+q} \tag{2.75}$$

and

$$\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k} P_{m-4}^k P_{m-3}^{j-k} P_{m-5}^{n-j} Q_{pn+q+k+j} = Q_{(m+p)n+q}\,. \tag{2.76}$$

Theorem 2.8. *The following identities hold for positive integer n and arbitrary integers p and q:*

$$\sum_{j=0}^{\lfloor n/2\rfloor}\sum_{k=0}^{n-2j}(-1)^{j+k}\frac{n}{n-j}\binom{n-j}{j}\binom{n-2j}{k}Q_p^{n-2j-k}P_{q-pj+pk}=Q_{pn}P_q-P_{pn+q}\,, \tag{2.77}$$

$$\sum_{j=0}^{\lfloor n/2\rfloor}\sum_{k=0}^{n-2j}(-1)^{j+k}\binom{n-j}{j}\binom{n-2j}{k}Q_p^{n-2j-k}P_{q-pj+pk}$$
$$=\frac{\begin{vmatrix} P_{q+1}P_{pn+p-4}-P_qP_{pn+p-3} & P_{p-4} & 0\\ P_{q+2}P_{pn+p-4}-P_{q+1}P_{pn+p-3} & -P_{p-3} & P_{p-4}\\ P_qP_{pn+p-4}-P_{q-1}P_{pn+p-3} & 0 & -P_{p-3}\end{vmatrix}}{\begin{vmatrix} -P_{p-3} & P_{p-4} & 0\\ P_{p-4} & -P_{p-3} & P_{p-4}\\ P_{p-4} & 0 & -P_{p-3}\end{vmatrix}}\,. \tag{2.78}$$

$$\sum_{j=0}^{\lfloor n/2\rfloor}\sum_{k=0}^{n-2j}(-1)^{j+k}\frac{n}{n-j}\binom{n-j}{j}\binom{n-2j}{k}Q_{2p}^{k}P_{q+3pn-6pj-2pk}=(-1)^n(Q_{2pn}P_{pn+q}-P_{3pn+q})\,, \tag{2.79}$$

$$\sum_{j=0}^{\lfloor n/2\rfloor}\sum_{k=0}^{n-2j}(-1)^{j+k}\binom{n-j}{j}\binom{n-2j}{k}Q_{2p}^{k}P_{q+3pn-6pj-2pk}$$
$$=\frac{(-1)^n\begin{vmatrix} P_{pn+q}P_{2pn+2p-3}-P_{pn+q+1}P_{2pn+2p-4} & P_{2p-4} & 0\\ P_{pn+q+1}P_{2pn+2p-3}-P_{pn+q+2}P_{2pn+2p-4} & P_{2p-3} & P_{2p-4}\\ P_{pn+q-1}P_{2pn+2p-3}-P_{pn+q}P_{2pn+2p-4} & 0 & P_{2p-3}\end{vmatrix}}{\begin{vmatrix} P_{2p-3} & P_{2p-4} & 0\\ P_{2p-4} & P_{2p-3} & P_{2p-4}\\ P_{2p-4} & 0 & P_{2p-3}\end{vmatrix}}\,, \tag{2.80}$$

$$\sum_{j=0}^{\lfloor n/2\rfloor}\sum_{k=0}^{n-2j}(-1)^{j+k}\frac{n}{n-j}\binom{n-j}{j}\binom{n-2j}{k}Q_{2p}^{k}P_{q+4pn-8pj-2pk}=(-1)^n(Q_{2pn}P_{2pn+q}-P_{4pn+q}) \tag{2.81}$$

and

$$\sum_{j=0}^{\lfloor n/2\rfloor}\sum_{k=0}^{n-2j}(-1)^{j+k}\binom{n-j}{j}\binom{n-2j}{k}Q_{2p}^{k}P_{q+4pn-8pj-2pk}$$
$$=\frac{(-1)^n\begin{vmatrix} P_{2pn+q+1}P_{2pn+2p-4}-P_{2pn+q}P_{2pn+2p-3} & P_{2p-4} & 0\\ P_{2pn+q+2}P_{2pn+2p-4}-P_{2pn+q+1}P_{2pn+2p-3} & P_{2p-3} & P_{2p-4}\\ P_{2pn+q}P_{2pn+2p-4}-P_{pn+q-1}P_{2pn+2p-3} & 0 & P_{2p-3}\end{vmatrix}}{\begin{vmatrix} -P_{2p-3} & P_{2p-4} & 0\\ P_{2p-4} & -P_{2p-3} & P_{2p-4}\\ P_{2p-4} & 0 & -P_{2p-3}\end{vmatrix}}\,. \tag{2.82}$$

Proof. Identities (2.77), (2.79) and (2.81) are obtained by choosing $(x,y)=(\alpha^p,\beta^p)$, $(x,y)=((\alpha/\beta)^p,(\beta/\alpha)^p)$ and $(x,y)=(1/\alpha^{2p},1/\beta^{2p})$ in identity (2.53), in turn, and making use of Lemma 1.8. Identities (2.78), (2.80) and (2.82) are obtained by choosing $(x,y)=(\alpha^p,\beta^p)$, $(x,y)=((\alpha/\beta)^p,(\beta/\alpha)^p)$ and $(x,y)=(1/\alpha^{2p},1/\beta^{2p})$ in identity (2.54), in turn, and making use of Lemma 1.8. □

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Received: July 13, 2020.
Accepted: November 15, 2020.