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Homoclinic orbits in a degenerate differential system and their preservation after discretization by Euler's method

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Abstract In this paper we are interested in homoclinic orbits in a very degenerate differential system, where the linear part exhibits two zero eigenvalues, and in the discrete system obtained by Euler's method. We show that this differential system presents a homoclinic region surrounded by an infinite number of periodic orbits. Moreover, we show that the homoclinic orbits persist in the discretized system generated by Euler's iterative method.

1 Introduction

Knowing that after discretization by Euler's method the solution of a differential equation pulls outwards, contrary to the idea which consists in believing that homoclinic solutions (solutions tending to the equilibrium point when time tends to $+\infty$ and $-\infty$) do not persist in the associated discretized system, this work aims to show that in very degenerate systems, this is not necessarily the case. In the hyperbolic case of non-degenerate homoclinic orbits of an autonomous differential equation, works [3] and [15] give an answer to the natural question of persistence of homoclinic solutions in the discretized system. If a differential equation has to be solved only over a finite time interval, numerical methods aim to get a precise discrete approximation of the solution. However, if behaviour of solutions over infinite time intervals is of high interest, then the errors may grow, and it could be impossible to prove that the numerical solution is close to the exact solution. Otherwise, convergence of a method over finite intervals does not guarantee persistence of long-term characteristics of solutions in the numerical approximation, which may take on many possible phenomena when applied to certain dynamical systems.
Dynamical behaviours in discrete equations are more complicated than those corresponding continuous-time differential equations. Furthermore, the difficulty is more stressed because a solution obtained by Euler's method tends to go outwards of the exact solution. In the hyperbolic case, it is shown in [3] that, subject to certain conditions, the phase portrait of the differential system is correctly reproduced in the associated discretization by a one step method, on an arbitrary time interval.

In [9], the question of preservation of homoclinic orbits after discretization is studied, in the nonhyperbolic case. On the other hand, approximation of homoclinic orbits of differential equations is considered [6]; it has been shown that the homoclinic branch of the numeric method is $O(h)$ close to its continuous counterparts, for delay differential equation [14], where h is the step size of Euler scheme. The comparison between the dynamics of differential equations and their discretization received much attention, see for instance [12], [13]. Recently, several authors investigated numeric approximations to the solution of differential or fractional differential equations, by Euler scheme [1], [5], [7], [8]. Note that, in this work, the situation is much more degenerate.

Our main interest in the present paper is to study the homoclinic region (any orbit started from this region is homoclinic) for a planar differential system, in the nonhyperbolic case, and for the discrete system associated by Euler's method; we will describe this homoclinic region and show that the corresponding Euler discretized system has a homoclinic region converging to that of the continuous one when the step size of the discretization tends to zero.

Consider the vector field

$$\begin{cases} \dot{x} &= f_1(x,y) \\ \dot{y} &= g_1(x,y) \end{cases} \tag{1.1}$$

where $\dot{(\cdot)} = d/dt$ and f_1 and g_1 are both analytic functions from $\mathbb{R}^2$ to $\mathbb{R}$. We assume that the origin is an isolated equilibrium point for system (1.1), and we suppose that the matrix M associated to the linearized system of (1.1) has two real eigenvalues λ_1 and λ_2. If λ_1 and λ_2 are nonzero, the nature of the origin is given by Hartman-Grobman's theorem [11]. If only λ_1 is zero, the origin is a node, a saddle, or a saddle-node [4]. When $\lambda_1 = \lambda_2 = 0$, two cases arise. The first one is that when the matrix M is null. In this case, if the smallest degree of the non-linear terms of (1.1) is m, then the neighbourhood of the origin is splitted into $2(m+1)$ parabolic, hyperbolic or elliptic sectors. The number of elliptic sectors existing in system (1.1) depends on the index of the equilibrium point (cf. [10], p 151): let C be a Jordan curve containing $(0,0)$ and no other critical point of (1.1) in its interior, then the index of the equilibrium point $(0,0)$ with respect to (1.1) is given by

$$I_{(1.1)}(0,0) = I_{(1.1)}(C) = \frac{1}{2\pi}\oint_C \frac{f_1 dg_1 - g_1 df_1}{f_1^2 + g_1^2}.$$

The second case is which we are interested in, the matrix M can be reduced by linear transformations to $M' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Namely, system (1.1) can be reduced to

$$\begin{cases} \dot{x} &= y + f_2(x,y) \\ \dot{y} &= g_2(x,y) \end{cases} \tag{1.2}$$

where f_2 and g_2 are analytic in neighbourhood of origin with a total valuation equal at least two [2]. Setting $X = x$ and $Y = y + f_2(x,y)$ in system (1.2), we get

$$\begin{cases} \dot{X} &= Y \\ \dot{Y} &= G_2(X,Y) \end{cases} \tag{1.3}$$

Returning to the usual notations x and y, and isolating the terms of degree 0 and 1 in y, we transform system (1.3) to its normal form [2]:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= ax^r\,(1 + h\,(x)) + bx^p y\,(1 + g\,(x)) + y^2 f\,(x,y)\,, \end{cases} \tag{1.4}$$

where f, g and h are analytic functions, such that $h(0) = g(0) = f(0,0) = 0$. a and b are real parameters, r and p are integer parameters, satisfying $r = 2m+1$, $m \geq 1$, $a < 0$, $b \neq 0$, $p \geq 1$, p odd and either $p = m$ with $\lambda = b^2 + 4(m+1)a \geq 0$, or $p < m$.

We study in this work system (1.4) in case of $p < m$ with $p = 1$, $m = 2$ and $f \equiv g \equiv h \equiv 0$; the case $p = m$ was studied in [9]. Suppose that $b > 0$, the case $b < 0$ is similar. As $a < 0$ and $b > 0$, without loss of generality, all throughout of this paper we assume that $a = -1$ and $b = 1$. Thus we are reduced to study the system

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -x^5 + xy \end{cases} \tag{1.5}$$

Origin is the unique equilibrium point of system (1.5); it is nonhyperbolic. We will show that this system has a homoclinic region surrounded by an infinite number of periodic orbits. After discretization by Euler's method of system (1.5), we will show the main result of this work: orbits of the associated discretized system, emanating from the homoclinic region of system (1.5), remain homoclinic.

This paper is organized as follows. In section 2, we determine for system (1.5), a homoclinic orbit S'_u and we show that orbits of (1.5) starting from the region delimited by S'_u are homoclinic and all other orbits are periodic. In section 3, we show that there exists a subset of the homoclinic region of (1.5), such that orbits of the discrete system obtained from (1.5) by Euler scheme starting from it, are also homoclinic, and that this subset tends to that delimited by S'_u when h tends to zero. A numerical example illustrating the established result is provided. A conclusion is presented in section 4.

2 Homoclinic and periodic orbits in system (1.5)

At first, we determine the nature of the unique equilibrium point $(0,0)$ of system (1.5), which is a very degenerate nonhyperbolic point.
If $x \neq 0$, the component $\dot{y}$ vanishes on the curve defined by $y\,(x) = x^4$ (Fig. 1). Set $R_1 = \{(x,y) \in \mathbb{R}^2\,;\, y < x^4, y > 0, x > 0\}$, $R_2 = \{(x,y) \in \mathbb{R}^2\,;\, x > 0, y < 0\}$, $R_3 = \{(x,y) \in \mathbb{R}^2\,;\, x < 0, y < 0\}$ and $R_4 = \{(x,y) \in \mathbb{R}^2\,;\, y < x^4, x < 0, y > 0\}$.

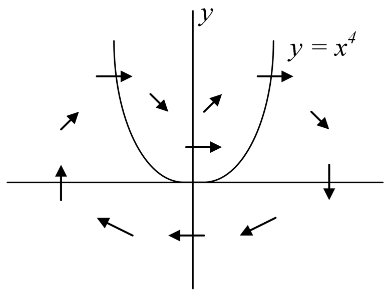

Figure 1. Vector field of system (1.5).

The application for system (1.5) of the blow-up

$$\begin{cases} x = uv \\ y = u^2 v \\ udt = d\tau \end{cases}$$

produces the system

$$\begin{cases} u' = -u\left(1 - v + u^2 v^4\right) \\ v' = v\left(2 - v + u^2 v^4\right) \end{cases} \tag{2.1}$$

with $(') = d/d\tau$. The origin and $(0,2)$ are the only equilibria points of (2.1), and they are saddles. Vectors $(0,1)$ and $(1,0)$ are tangent respectively to the stable separatrix and the unstable one of the saddle $(0,2)$. Besides, the vectors $(1,0)$ and $(0,1)$ are tangent (respectively) to the stable and unstable separatrices of the saddle $(0,0)$ (Fig. 2). The symmetry with respect to the v-axis allows to consider only the solutions of (2.1) starting at points with negative abscissas.
We denote by A the saddle point $(0,2)$ of system (2.1), S_u the unstable separatrix of $(0,2)$ and S'_u the orbit of system (1.5) corresponding to S_u in the (x,y)-plane.

We will determine the position of the unstable separatrix S'_u in the neighbourhood of the origin, by following trajectories of system (1.5) in the plane.
Equation $u' = 0$ implies $u = 0$ or

$$u = -\frac{\sqrt{v-1}}{v^2} \tag{2.2}$$

The curve given by (2.2) is represented in the (x,y)-plane for $-1/2 \le x \le 1/2$ (Fig. 2 and Fig. 3), by the union of two curves defined by

$$y = \frac{x^2}{2}\left(1 - \sqrt{1-4x^2}\right) \tag{2.3}$$

and

$$y = \frac{x^2}{2}\left(1 + \sqrt{1-4x^2}\right). \tag{2.4}$$

The curve

$$u = -\frac{\sqrt{v-2}}{v^2} \tag{2.5}$$

corresponding to $v' = 0$ is represented, for $-1/\sqrt{8} \le x \le 1/\sqrt{8}$ (Fig. 2 and Fig. 3), by the union of the two curves given by

$$y = \frac{x^2}{4}\left(1 - \sqrt{1-8x^2}\right) \tag{2.6}$$

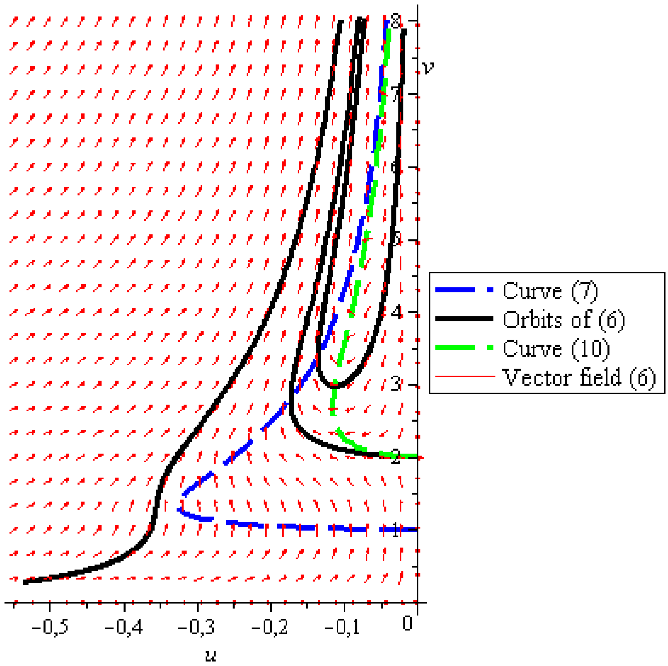

Figure 2. Part of the phase portrait of system (2.1) and the curves (2.2) and (2.5) ($u < 0$).

and

$$y = \frac{x^2}{4}\left(1 + \sqrt{1 - 8x^2}\right) \tag{2.7}$$

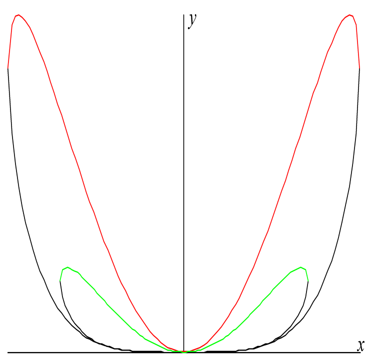

Figure 3. Curves (2.3), (2.4), (2.6) and (2.7).

Separatrix S_u passes between the two curves (2.2) and (2.5) above the line $v = 2$ (tangent to S_u at the saddle A), then crosses curve (2.2). If

$$v \leq 4 \tag{2.8}$$

it follows that

$$y \geq \frac{x^2}{4},$$

and thus,

$$y \geq \frac{x^2}{4}(1 - \sqrt{1 - 8x^2}) \tag{2.9}$$

Therefore, when points (u, v) of separatrix S_u satisfy (2.8), the corresponding part of separatrix S'_u is above the curve (2.6). If

$$v \geq 2, \tag{2.10}$$

then

$$y \leq \frac{x^2}{2}$$

Thus

$$y \leq \frac{x^2}{2}(1 + \sqrt{1 - 4x^2}) \tag{2.11}$$

This means that when points (u, v) of separatrix S_u satisfy (2.10), the corresponding part of S'_u is bellow the curve (2.4). When

$$-\frac{\sqrt{v-2}}{v^2} < u < -\frac{\sqrt{v-1}}{v^2}$$

we get

$$\frac{x^2}{2}(1 - \sqrt{1 - 4x^2}) < y < \frac{x^2}{2}(1 + \sqrt{1 - 4x^2})$$

with

$$y < \frac{x^2}{4}(1 - \sqrt{1 - 8x^2}), \tag{2.12}$$

or

$$y > \frac{x^2}{4}(1 + \sqrt{1 - 8x^2}) \tag{2.13}$$

If $v \leq 4$, according to (2.9), inequality (2.12) is excluded.
To summarize, orbit S'_u passes above the curve (2.7) and between the two curves (2.3) and (2.4).

Next proposition provides the global behaviour of separatrix S'_u.

Proposition 2.1. *Orbit S'_u is homoclinic, and any orbit of system (1.5) starting from the region delimited by S'_u is homoclinic.*

Proof. Let (u, v) be a point of unstable separatrix S_u. Then

$$\forall u < 0, v > 2. \tag{2.14}$$

It follows that in the (x, y)-plane,

$$\forall x < 0, y < \frac{x^2}{2} \tag{2.15}$$

As the parabola $y = x^2/2$ crosses the curve

$$y = x^4 \tag{2.16}$$

and S_u satisfies (2.14), it follows that by inverting time t, trajectory S'_u crosses the curve (2.16) and passes into the region R_4 (Fig. 1).
On the other hand, by inverting time, we consider trajectory γ of system

$$\begin{cases} \dot{x} &= -y \\ \dot{y} &= x^5 - xy \end{cases} \tag{2.17}$$

starting from a point (x_0, y_0) of region R_4 and assume that γ remains in R_4. Then

$$y(t) - y_0 = \int_0^t -x(s)\left(y(s) - x(s)^4\right) ds$$

$$\leq \int_0^t -x_0\left(y(s) - x(s)^4\right) ds$$

However, since

$$y(s) < y_0 \text{ and } x(s) < 0, \forall s > 0,$$

it follows that

$$y(t) \leq y_0 - x_0\left(y_0 - x_0^4\right) t.$$

As $y_0 > 0, x_0 < 0$ and $y_0 - x_0^4 < 0$, then

$$\exists\, t > 0,\, y(t) < 0$$

This contradicts the fact that γ does not leave region R_4. This means that γ reaches a point of region R_3 (Fig. 1).
Similarly, we show that orbits of system (2.17) starting from region R_3, passes into part R_2, then into region R_1 (Fig. 1).
It follows from the above that, trajectory S_u' goes from part R_3, then enters into part R_2. As solutions of system (1.5) are symmetric with respect to the line (yy'), we conclude that S_u' is homoclinic. □

When u tends to $-\infty$, as $(0,0)$ is a saddle for system (2.1) and according to the direction of vector fields (1.5) and (2.1), then v tends to 0 when τ tends to $-\infty$. We will describe in this case, the behaviour of the corresponding solutions of (1.5). Let γ be a trajectory of (2.1) such that u is in neighbourhood of $-\infty$ and let γ' be the corresponding trajectory of (1.5). We can write

$$u\left(1 - v + u^2v^4\right)\frac{dv}{du} = -v\left(2 - v + u^2v^4\right) \tag{2.18}$$

If v has an asymptotic expansion in the neighbourhood of infinity,

$$v = a_0 + \frac{a_1}{u} + \frac{a_2}{u^2} + \frac{1}{u^2}\varepsilon\left(\frac{1}{u}\right), \text{ with } \lim_{u\to\infty}\varepsilon\left(\frac{1}{u}\right) = 0 \tag{2.19}$$

substituting (2.19) in (2.18), we will have

$$a_0 = a_1 = 0$$

and $a_2 \in \mathbb{R}$. Thus,

$$v = \frac{a_2}{u^2} + \frac{1}{u^2}\varepsilon\left(\frac{1}{u}\right)$$

and

$$\lim_{u\to-\infty}\left(u^2v\right) = a_2.$$

Therefore, when u tends to $-\infty$, x tends to zero and y tends to a_2. If $a_2 \neq 0$, then orbit γ' is not homoclinic. If $a_2 = 0$, by induction, assuming that $a_{n-1} = 0$ for $n > 3$ and substituting v in (2.18), we get $a_n = 0$. Then,

$$v = \frac{1}{u^n}\varepsilon\left(\frac{1}{u}\right)$$

and

$$\lim_{u\to-\infty}\left(u^2v\right) = 0.$$

This fact means that when u is sufficiently large, x and y become close to zero. Thus two options arise: either γ' is homoclinic, or it goes to the half-plane $x > 0$, close to $(0,0)$.

In order to determine the homoclinic region, a sequence of changes of variables should be introduced. We will show that when $a_2 \neq 0$, the solution crosses the y-axis and is periodic. By setting the change of variables

$$\begin{cases} x = wy \\ y = y \end{cases} \tag{2.20}$$

system (1.5) becomes

$$\begin{cases} \dot{w} &= 1 - yw^2 + y^4w^6 \\ \dot{y} &= y^2w - y^5w^5 \end{cases} \tag{2.21}$$

System (2.21) does not attain equilibrium points, and the w-axis is invariant under (2.21) ($y = 0$ is a solution of (2.21)).
Equation $\dot{y} = 0$ gives

$$y = 0,\, y = w^{-4/3} \text{ or } w = 0$$

Equation $\dot{w} = 0$ can be written as

$$1 - yw^2 + y^4w^6 = 0 \tag{2.22}$$

The following lemma gives the solutions of equation (2.22).

Lemma 2.2. *Since w is fixed, equation (2.22) admits: a unique solution $\bar{y}$ if $w = \bar{w} := 2^4 3^{-3/2}$; two solutions if $w > \bar{w}$ and no solution when $0 < w < \bar{w}$.*

Proof. The line $w = 0$ is not a solution of (2.22). By symmetry, we consider $w > 0$. Observe that no point of the half-plane $y \leq 0$ is a solution of equation (2.22). Then, assume $y > 0$ and put

$$\varphi_w(y) := 1 - yw^2 + y^4w^6 = 0.$$

Function φ'_w vanishes at $\bar{y} = \sqrt[3]{1/4w^4}$. Denote $\varphi_{min}(w)$ the value of φ_w at $\bar{y}$. Then

$$\varphi_{min}(w) = \varphi_w(\bar{y}) = 1 - \frac{3}{4\sqrt[3]{4}} w^{2/3}$$

and

$$\varphi'_{min}(w) = -\frac{1}{2\sqrt[3]{4}w^{1/3}} < 0$$

Function φ_{min} vanishes at $\bar{w} = \left(4\sqrt[3]{4}/3\right)^{3/2}$. When w varies from 0 to $+\infty$, function φ_{min} decreases from 1 to $-\infty$, taking 0 at $\bar{w}$. This means that when $w < \bar{w}$, we have $\varphi_{min}(w) > 0$ and equation (2.22) has no solution.
When $w > \bar{w}$, we have $\varphi_{min}(w) < 0$. Namely, since w is fixed, equation (2.22) admits at least one solution (Fig. 4).
If w is fixed, when y varies from 0 to $+\infty$, then function φ_w decreases from 1 to $\varphi_w(\bar{y}) =$

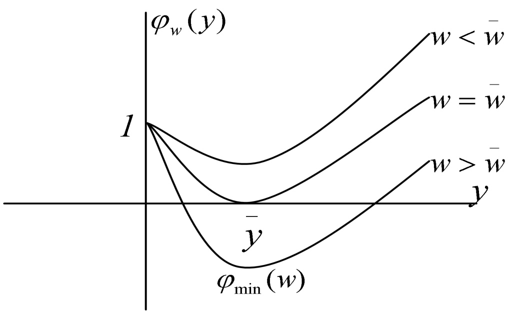

Figure 4. Graph of φ_w according to values of w.

$\varphi_{min}(w)$, and increases from $\varphi_{min}(w)$ to $+\infty$. Thus, when $w > \bar{w}$, equation $\varphi_w(y) = 0$ has two solutions y_1 and y_2. When $w = \bar{w}$, equation (2.22) has a unique solution $\bar{y}$ because $\varphi_{min}(\bar{w}) = 0$ (Fig. 4). □

When

$$y \geq w^{-4/3}$$

then,

$$1 - yw^2 + y^4w^6 \geq 1.$$

Therefore, the curve corresponding to the solution of (2.22) is below the curve $y = \sqrt[3]{1/w^4}$. The vector field (2.21) is illustrated in Figure 5.

The following proposition gives the existence of an infinite number of periodic orbits in system (1.5).

Proposition 2.3. *Any solution of system (1.5) starting above S'_u, is periodic.*

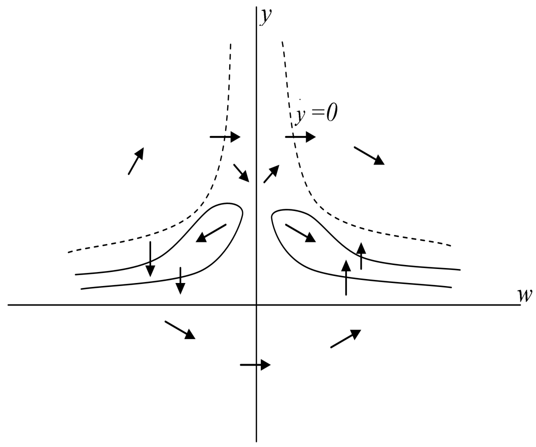

Figure 5. Vector field (2.21).

Proof. When $y > 0$, inequality $y > \sqrt[3]{1/w^4}$ is equivalent to $y < x^4$. The w-axis in the (w, y)-plane, is represented by the equilibrium point $(0, 0)$ in the (x, y)-plane.
Solutions of (2.21) satisfying

$$\exists t_0 > 0,\ \forall t > t_0,\ w(t) > \bar{w}$$

cross the y-axis at points with strictly positive ordinates. Therefore, the corresponding solutions of (1.5) cross the y-axis (Fig. 6); because of symmetry, they are periodic (we show that these solutions move from one quadrant to a neighbouring quadrant in a finite time) (Fig. 7). As

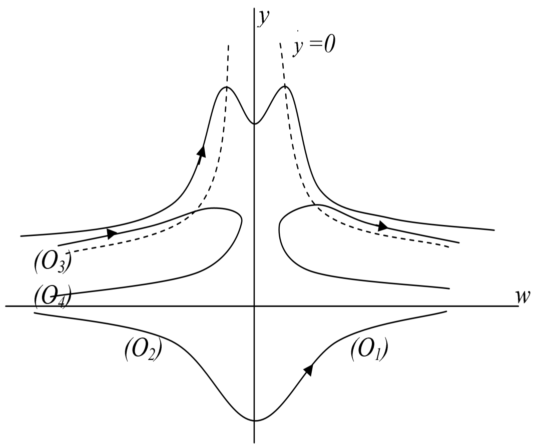

Figure 6. Orbits of system (2.21).

$w = 1/u$, solutions of (2.1) such that u tends to infinity satisfy, in (w, y)-plane, that w tends to zero. This means that the corresponding solutions of (1.5) are periodic. Thus, trajectory S'_u is the one that separates the homoclinic region from the periodic region (any orbit started from this region is periodic) of system (1.5). □

3 Homoclinic orbits in the discrete system associated to (1.5)

A discretization of system (1.5) by Euler's method leads to the discrete system

$$\begin{cases} x_{n+1} = x_n + hy_n \\ y_{n+1} = y_n + h(-x_n^5 + x_n y_n) \end{cases} \tag{3.1}$$

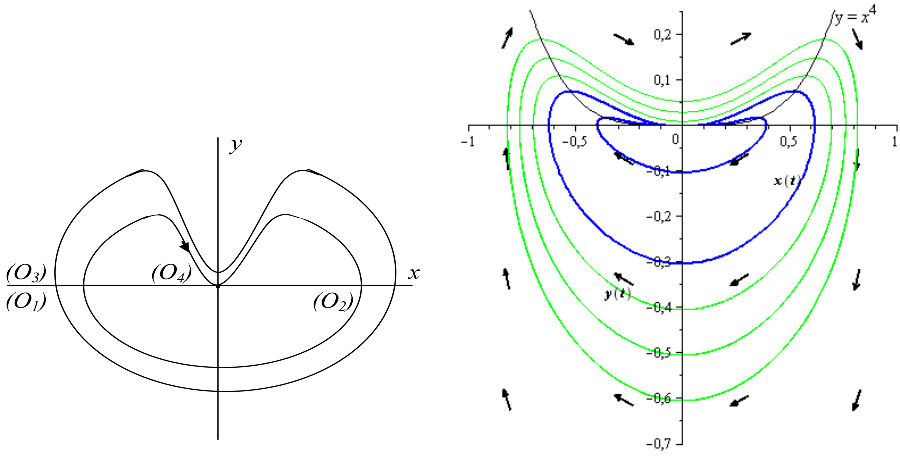

Figure 7. Phase portrait of system (1.5).

where h is the discretization step size.
Denote for a given r in $\mathbb{R}$, the ball $\mathcal{B}_r = \{(x, y) \in \mathbb{R}^2 \; ; x^2 + y^2 \leq r^2\}$ and $l = y_0/x_0^2$. Let (x_0, y_0) be a point satisfying $x_0 < 0$ and $y_0 > 0$. On account on what was shown above, for x_0 and y_0 small enough, we have:
- if $l < 1/2$, the solution of system (1.5) starting from (x_0, y_0) is homoclinic,
- if $l > 1/2$, solution of system (1.5) starting from (x_0, y_0) is periodic,
- and if $l = 1/2$, the two previous situations are possible.

Next lemmas are crucial to show our main result:

Lemma 3.1. *There exists a $h_0 > 0$ and a $r_0 > 0$ such that for any $h \in (0, h_0]$, any point (x_0, y_0) in the ball $\mathcal{B}_{r_0}$ has a unique predecessor in $\mathcal{B}_{r_0+1}$ by system (3.1).*

Proof. We will show that there exists a $h_0 > 0$ and a $r_0 > 0$ such that for any $h \in (0, h_0]$ and any point (z, t) with $z^2 + t^2 \leq r_0^2$, there exists a unique point (x, y) satisfying $x^2 + y^2 \leq (r_0 + 1)^2$ and $F_h(x, y) = (z, t)$, where $F_h(x, y) = (x + hy, y + h(-x^5 + xy))$. Put $f(x, y) = -x^5 + xy$. By the definition of F_h, we get $x = z - hy$ and $t = y + hf(z - hy, y)$. For a fixed (z, t), denote $g(y) = t - y - hf(z - hy, y)$. Then,

$$\exists h_0 > 0, \exists r_0 > 0, \forall h \in (0, h_0], \forall (z, t) \in \mathcal{B}_{r_0}, g(t + 1) < 0 \text{ and } g(t - 1) > 0.$$

For any positive y in $]t - 1, t + 1[$, $g'(y) < 0$. Since g is a continuous function, equation $g(y) = 0$ admits a unique solution y in $]t - 1, t + 1[$. Furthermore, we get from the expression of x that $x \in]z - 1, z + 1[$. □

Set

$$\bar{y}_{-n} = \frac{2 + hx_{-n} + 2\sqrt{1 + hx_{-n} - h^3 x_{-n}^5}}{h^2} > 0$$

and

$$\bar{\bar{y}}_{-n} = \frac{2 + hx_{-n} - 2\sqrt{1 + hx_{-n} - h^3 x_{-n}^5}}{h^2} > 0$$

We have the following lemma:

Lemma 3.2. *For any n in $\mathbb{N}$, inequality $y_{-n+1} > x_{-n+1}^2/4$ is satisfied if and only if $\bar{\bar{y}}_{-n} < y_{-n} < \bar{y}_{-n}$.*

Proof. We have

$$y_{-n+1} - \frac{1}{4}x_{-n+1}^2 = \frac{1}{4}y_{-n}^2 + (1 + \frac{1}{2}hx_{-n})y_{-n} - hx_{-n}^5 - \frac{1}{4}x_{-n}^2. \tag{3.2}$$

The discriminant Δ associated to the polynomial of degree two in y_{-n} is given by

$$\Delta = (1+\frac{1}{2}hx_{-n})^2 - h^2(\frac{1}{4}x_{-n}^2 + x_{-n}^5) > 0.$$

So, $y_{-n+1} - x_{-n+1}^2/4 > 0$ if and only if $\bar{\bar{y}}_{-n} < y_{-n} < \bar{y}_{-n}$. □

We deduce from 3.2 that there exists a $h_0 > 0$ such that for any $h \in (0, h_0]$, if $y_{-n+1} < x_{-n+1}^2/4$ then $y_{-n} < \bar{\bar{y}}_{-n}$, and therefore $x_{-n} > 0$.

Lemma 3.3. *Let (x_0, y_0) be a point of the plane satisfying one of the two following conditions:*
1)

$$0 < x_0 < \bar{x} = \frac{1}{\frac{h}{16} + \sqrt{\frac{h^2}{16^2} + 8}} \text{ and } 0 < y_0 \le \frac{x_0^2}{4},$$

2)

$$0 < x_0 < \frac{1}{2\sqrt{2}} \text{ and } 0 < y_0 < \frac{x_0^2}{4} - \sqrt{h}x_0.$$

Then, the orbit $(x_{-n}, y_{-n})_{n\in\mathbb{N}*}$ of system (3.1) starting at (x_0, y_0), does not leave the region $\{(x,y) \in \mathbb{R}^2\ ;\ 0 < y < x^2/4\}$.

Proof. Regarding the first condition, we have $\bar{\bar{y}}_0 \le x_0^2/4$ when $0 < x_0 < \bar{x}$. By induction, $\bar{\bar{y}}_{-n} \le x_{-n}^2/4$ when $0 < x_0 < \bar{x}$. On the other hand,

$$y_0 - \frac{1}{4}x_0^2 = y_{-1} + h(-x_{-1}^5 + x_{-1}y_{-1}) - \frac{1}{4}(x_{-1} + hy_{-1})^2$$

It follows that

$$(y_{-1} - \frac{1}{4}x_{-1}^2)(1 + \frac{h}{2}x_{-1}) = y_0 - \frac{1}{4}x_0^2 + \frac{1}{4}h^2y_{-1}^2 - hx_{-1}^3(\frac{1}{8} - x_{-1})$$

We have $0 < y_0 < x_0^2/4$. According to 3.2, $x_{-1} > 0$. Thus, as $x_{-1} < 1/(2\sqrt{2})$, we get $hx_{-1}^3((1/8) - x_{-1}) > 0$ and

$$y_{-1} - \frac{1}{4}x_{-1}^2 < (y_{-1} - \frac{1}{4}x_{-1}^2)(1 + \frac{h}{2}x_{-1}) < y_0 - \frac{1}{4}x_0^2 + \frac{1}{4}h^2y_{-1}^2$$

Thereby,

$$y_{-1} - \frac{1}{4}x_{-1}^2 < y_0 - \frac{1}{4}x_0^2 + \frac{1}{4}h\sqrt{h}y_{-1}.$$

We get

$$y_{-n} - \frac{1}{4}x_{-n}^2 < y_0 - \frac{1}{4}x_0^2 + \frac{1}{4}h\sqrt{h}(y_{-1} + y_{-2} + ... + y_{-n}).$$

Thus,

$$y_{-n} - \frac{1}{4}x_{-n}^2 < y_0 - \frac{1}{4}x_0^2 + \frac{1}{4}\sqrt{h}(x_0 - x_{-n}).$$

Therefore,

$$y_{-n} - \frac{1}{4}x_{-n}^2 < y_0 - (\frac{1}{4}x_0^2 - \frac{1}{4}\sqrt{h}x_0) < 0.$$

□

Lemma 3.4. *For any $n \in \mathbb{N}$ such that $(-1/(2\sqrt{2})) < x_n < 0$ and $y_n < x_n^2/4$, we have $(-1/(2\sqrt{2})) < x_{n+1} < 0$ and $y_{n+1} < x_{n+1}^2/4$.*

Proof. Let be $x_n < 0$ and $y_n < x_n^2/4$. According to the proof of 3.2, $y_{n+1} - x_{n+1}^2/4 < 0$ if and only if $y_n < \bar{\bar{y}}_n$. However $\bar{\bar{y}}_n > x_n^2/4$ when $(-1/(2\sqrt{2})) < x_n < 0$, and therefore, the desired result follows. □

Let now (x_0, y_0) be a point in the homoclinic region of (1.5) such that $x_0 < 0$ and $y_0 < 0$. We locally parametrize the orbit of (1.5) starting from (x_0, y_0) by

$$y = \varphi_a(x), \text{ where } a = \lim_{x \to 0^-} \frac{y}{x^2} \tag{3.3}$$

We show in the following theorem that, in some region of the plane, solutions of (3.1) are homoclinic.

Theorem 3.5. *There exists a $h_0 > 0$, $b > 0$ and a $c < 0$ such that for any $h \in (0, h_0]$, if $x_0^2 + y_0^2 \geq b^2$ and $a - (1/4) < c$, where a is given by (3.3), then solution of system (3.1) starting from point (x_0, y_0) is homoclinic.*

Proof. There exists a $r_1 > 0$ and a $r_2 > 0$, for any $n \in \mathbb{N}$ such that $nh < r_1$,

$$\left\| (x_n, y_n) - \mathcal{O}^+ (x_0, y_0) \right\| < r_2$$

where $\mathcal{O}^+(x_0, y_0)$ is the orbit of system (1.5) (Fig. 8) defined by

$$\mathcal{O}^+ (x_0, y_0) = \{(x(t), y(t)) \text{ solution of (1.5)}; t \geq 0 \text{ and } (x(0), y(0)) = (x_0, y_0)\}$$

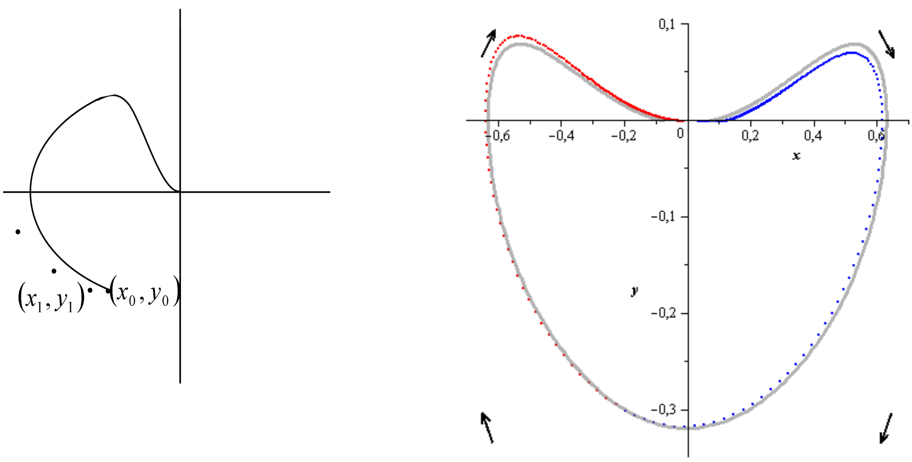

Figure 8. Orbit $(x_n, y_n)_{n \in \mathbb{Z}}$ of system (3.1) starting at $(x_0, y_0) = (-0.2, -0.3)$ (red points correspond to the part $(x_n, y_n)_{n \in \mathbb{N}}$, blue points correspond to the part $(x_{-n}, y_{-n})_{n \in \mathbb{N}}$) and orbit of system (1.5) starting at the same point (gray line).

Thus, there exists a $r_3 > 0$, for any $n \in \mathbb{N}$ such that $nh < r_1$,

$$\left| \frac{y_n}{x_n^2} - \frac{\varphi_a (x_n)}{x_n^2} \right| < r_3$$

The inflection points (x, y) of orbits of (1.5) are given by equation

$$f^2(x, y) \frac{\partial g}{\partial x}(x, y) + f(x, y) g(x, y) \left(\frac{\partial g}{\partial y}(x, y) - \frac{\partial f}{\partial x}(x, y) \right) - g^2(x, y) \frac{\partial f}{\partial y}(x, y) = 0$$

Therefore,

$$y^3 - 5y^2 x^4 + y x^6 - x^{10} = 0$$

In the upper half-plane, denote by *the convex zone*, the part whose abscissa's points are greater than the abscissa of the inflection point of $\mathcal{O}^+ (x_0, y_0)$, and denote by *the concave zone* that whose abscissa's points are less than the abscissa of the inflection point of $\mathcal{O}^+ (x_0, y_0)$. Let n be large enough such that $nh < r_1$, and (x_n, y_n) is in the convex zone. Let a_n be such that the trajectory of system (1.5) which is locally parametrized by $y = \varphi_{a_n}(x)$, passes through (x_n, y_n). According to 3.4, there exists a $c_1 > 0$, for any $n \in \mathbb{N}$ such that $nh < r_1$,

$$0 < a_n - a < c_1,$$

with $a < 1/4$. However, there exists a $h_0 > 0$, $b > 0$ and a $c < 0$ such that for any $h \in (0, h_0]$, if $x_0^2 + y_0^2 \geq b^2$ and $a - (1/4) < c$:
- If (x_{n+1}, y_{n+1}) is in the convex zone, then $a_{n+1} < a_n < 1/4$. It follows that

$$\forall m > n,\, a_m < a_n < \frac{1}{4}. \tag{3.4}$$

- If (x_{n+1}, y_{n+1}) is in the concave zone, then $a_n < a_{n+1}$, and two cases are possible: there exists a $r > 0$, for any $n \in \mathbb{N}$ such that $nh < r_1$,
- If (x_{n+1}, y_{n+1}) is in $\mathcal{B}_r$, then inequality (3.4) follows by 3.4.
- If (x_{n+1}, y_{n+1}) is not in $\mathcal{B}_r$, then solution $(x_n, y_n)_{n\in\mathbb{N}}$ reaches a point (x_p, y_p) for $p > n$ such that $-1/(2\sqrt{2}) < x_p < 0$ and $y_p < 1/4x_p^2$. Thus, according to 3.4,

$$\forall q > p,\, a_q < \frac{1}{4}.$$

Hence, the solution of (3.1) reaching (x_0, y_0), tends to $(0,0)$ when n tends to $+\infty$. On the other hand, in the zone given by $0 < x < -1/(2\sqrt{2})$ and $x^4 < y < x^2/4$, sequences $(x_{-n})_{n\in\mathbb{N}}$ and $(y_{-n})_{n\in\mathbb{N}}$ are decreasing and bounded below. Therefore, they are convergent; they converge to $(0,0)$. □

Example. Taking for the differential system (1.5), the initial conditions $(0.1, -0.27)$, $(0.2, -0.2)$, $(-0.1, -0.1)$, $(0.1, -0.31)$, $(-0.1, -0.15)$ and $(0.05, -0.08)$, such that the orbits starting at these points are homoclinic, we obtain that the orbits of the discrete system (3.1) starting at the same points are also homoclinic; we take for (3.1), the step size $h = 0.1$ and the number of iterations $n = 10^5$ for each orbit. The corresponding phase portraits are illustrated in Fig. 9.

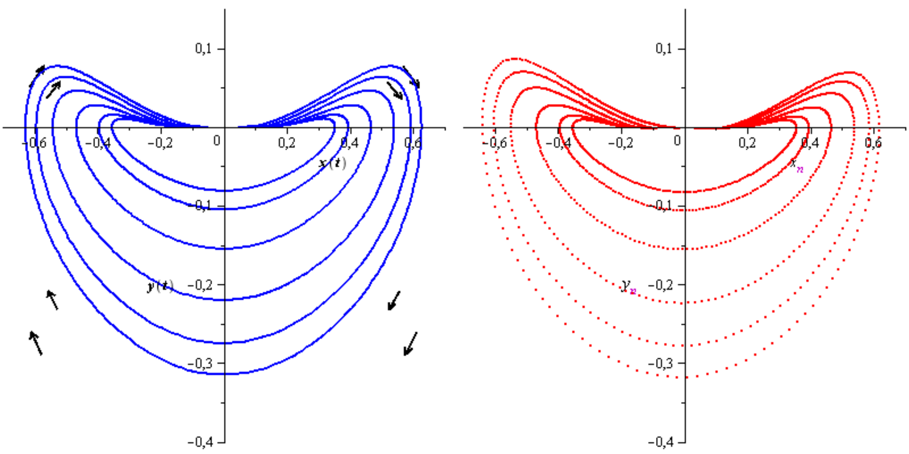

Figure 9. Orbits of systems (1.5) (on the left) and (3.1) (on the right) starting at $(0.1, -0.27)$, $(0.2, -0.2)$, $(-0.1, -0.1)$, $(0.1, -0.31)$, $(-0.1, -0.15)$ and $(0.05, -0.08)$, with the step size $h = 0.1$ and number of iterations 2.10^5.

4 Conclusion

The present paper is concerned with the question of the preservation of the homoclinic orbits after discretization by Euler scheme, in a very degenerate case. For the differential system studied in this work, we showed that it admits a region of homoclinic orbits surrounded by an infinite number of periodic orbits; we described this region and we showed that the associated Euler discretized system has a homoclinic region converging to that of the continuous system when the step size of the discretization tends to zero.

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