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Iterative construction of common fixed points of a pair of uniformly L-Lipschitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense

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Abstract The purpose of this paper is to propose an implicit iteration scheme for approximating the common fixed points of two uniformly L-Lipschitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense. We provide necessary and sufficient conditions for the strong convergence of our proposed iteration scheme to the common fixed points of two uniformly L-Lipschitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense in Banach spaces. Our new iteration scheme contains several other iteration schemes which have been used by several authors to approximate the fixed points of different classes of mappings. Hence, our results extend, improve, generalize and unify several well known results in the existing literature.

1 Introduction

Let X be a real Banach space with dual X^* and K be a nonempty closed convex subset of X. We denote by J the *normalized duality* mapping from X into 2^{X^*} defined by

$$J(\psi) = \{f^* \in X^* : \langle \psi, f^* \rangle = \|\psi\|^2 = \|f^*\|^2\}, \ \forall\, \psi \in X, \tag{1.1}$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. The single-valued-normalized duality mapping is denoted by j and $F(G)$ denotes the set of fixed points of mapping $G : K \to K$, i.e., $F(G) = \{\psi \in X : G\psi = \psi\}$.

In the sequel, we give the following definitions which will be useful in this study.

Definition 1.1. A mapping $G : K \to K$ is said to be:

(1) *contraction* if there exists a constant $\kappa \in (0, 1)$ such that

$$\|G\psi - G\zeta\| \leq \kappa\|\psi - \zeta\|, \ \forall\ \psi, \zeta \in K; \tag{1.2}$$

(2) *nonexpansive* if

$$\|G\psi - G\zeta\| \leq \|\psi - \zeta\|, \ \forall\ \psi, \zeta \in K; \tag{1.3}$$

(3) *strongly pseudocontractive* (Kim et al. [31]) if for all $\psi, \zeta \in K$, there exists a constant $k \in (0, 1)$ and $j(\psi - \zeta) \in J(\psi - \zeta)$ satisfying

$$\langle G\psi - G\zeta, j(\psi - \zeta)\rangle \leq k\|\psi - \zeta\|^2; \tag{1.4}$$

(4) *ϕ-strongly pseudocontractive* (Kim et al. [31]) if for all $\psi, \zeta \in K$, there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $j(\psi - \zeta) \in J(\psi - \zeta)$ satisfying

$$\langle G\psi - G\zeta, j(\psi - \zeta)\rangle \leq \|\psi - \zeta\|^2 - \phi(\|\psi - \zeta\|)\|\psi - \zeta\|; \tag{1.5}$$

In [41], it is proved that the class of strongly pseudocontractive mappings is a proper subclass of ϕ-strongly pseudocontractive mappings by taking $\Phi(t) = ts$ for all $t \in [0, \infty)$, where $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing function with $\phi(0) = 0$. However, the converse fails. These classes of mappings has been studied recently by several authors (see for example, Chidume et al. [11], Okeke [39], Okeke et al. [40] and the references therein).

(5) *generalized* Φ*-pseudocontractive* (Albert et al. [3]) if for all $\psi, \zeta \in K$, there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ and $j(\psi-\zeta) \in J(\psi-\zeta)$ satisfying

$$\langle G\psi - G\zeta, j(\psi - \zeta)\rangle \leq \|\psi - \zeta\|^2 - \Phi(\|\psi - \zeta\|); \tag{1.6}$$

The class of generalized Φ-pseudocontractive mappings is also called uniformly pseudocontractive mappings (see [10]). Clearly, the class of generalized Φ-pseudocontractive mappings properly contains the class of ϕ-pseudocontractive mappings;

(6) *generalized* Φ*-hemicontractive* if $F(G) = \{\psi \in K : G\psi = \psi\} \neq \emptyset$, and there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$, such that for all $\psi \in K$, $p \in F(G)$, there exists $j(\psi - p) \in J(\psi - p)$ such that the following inequality holds:

$$\langle G\psi - p, j(\psi - p)\rangle \leq \|\psi - p\|^2 - \Phi(\|\psi - p\|); \tag{1.7}$$

Clearly, the class of generalized Φ-hemicontractive mappings includes the class of generalized Φ-pseudocontractive mappings in which the fixed points set $F(G)$ is nonempty;

(7) *asymptotically generalized* Φ*-pseudocontractive* (Kim et al. [31]) with sequence $\{h_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} h_n = 1$, if for each $\psi, \zeta \in K$, there exist a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ satisfying

$$\langle G^n\psi - G^n\zeta, j(\psi - \zeta)\rangle \leq h_n\|\psi - \zeta\|^2 - \Phi(\|\psi - \zeta\|),\ \forall\ n \geq 1. \tag{1.8}$$

The class of asymptotically generalized Φ-pseudocontractive mappings is a generalization of the class of strongly pseudocontractive maps and the class of ϕ-strongly peudocontractive maps. The class of asymptotically generalized Φ-pseudocontractive mappings was introduced by Kim et al. [31] in 2009;

(8) *asymptotically generalized* Φ*–hemicontractive* with sequence $\{h_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} h_n = 1$ if there exist a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$, such that for each $\psi \in K$, $p \in F(G)$, there exists $j(\psi - p) \in J(\psi - p)$ such that the following inequality holds:

$$\langle G^n - p, j(\psi - p)\rangle \leq h_n\|\psi_n - p\|^2 - \Phi(\|\psi - p\|),\ \ \forall\ n \geq 1. \tag{1.9}$$

Clearly, every asymptotically generalized Φ–pseudocontractive mapping with a nonempty fixed point set is an asymptotically generalized Φ–hemicontractive mapping;

(9) *asymptotically generalized* Φ*-hemicontractive in the intermediate sense* (see Okeke et al. [40]) with sequence $\{h_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} h_n = 1$, if $F(G) \neq \emptyset$ and for each $n \in \mathbb{N}$, $\psi \in K$ and $p \in F(G)$, there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ and $j(\psi - p) \in J(\psi - p)$ satisfying

$$\limsup_{n\to\infty} \sup_{(\psi,p)\in K\times F(G)} (\langle G^n\psi - p, j(\psi - p)\rangle - h_n\|\psi - p\|^2 + \Phi(\|\psi - p\|)) \leq 0. \tag{1.10}$$

Set

$$\tau_n = \max\left\{0, \sup_{(\psi,p)\in K\times F(G)} (\langle G^n\psi - p, j(\psi - p)\rangle - h_n\|\psi - p\|^2 + \Phi(\|\psi - p\|))\right\}.$$

It follows that $\tau_n \geq 0$, $\tau_n \to 0$ as $n \to \infty$. Hence, (1.10) yields the following inequality:

$$\langle G^n\psi - p, j(\psi - p)\rangle \leq h_n\|\psi - p\|^2 + \tau_n - \Phi(\|\psi - p\|). \tag{1.11}$$

Clearly, the class of asymptotically generalized Φ-hemicontractive mappings is a proper subclass of the class of asymptotically generalized Φ-hemicontractive mappings in the intermediate sense. If we set $\tau_n = 0$ in (1.11), then the class of asymptotically generalized Φ-hemicontractive mappings in the intermediate sense reduces to the class of asymptotically generalized Φ-hemicontractive mappings. Hence, the class of asymptotically generalized Φ-hemicontractive mappings in the intermediate properly includes the class of asymptotically generalized Φ-hemicontractive maps.

For recent results on the approximation of fixed points of mappings which are asymptotically generalized Φ-hemicontractive mappings (see for example, [7, 10, 11, 21, 22, 30, 45, 57, 61] and the references there in) and for recent results on the approximation of fixed points of mappings which are asymptotically generalized Φ-hemicontractive mappings in the intermediate sense, (see for example, Chidume et al. [14], Okeke et al. [40], Olaleru and Okeke [38], Kaczor et al. [25], Qin et al. [43], and the references contained in them).

On the other hand, fixed point theory as an aspect of operator theory is an interesting discipline within functional analysis with applications in game theory, economics, etc. Fixed point theory is a combination of geometry, functional analysis and topology. It is one of the most powerful tools in modern mathematics in general and functional analysis in particular. Fixed point theorems are concerned with the existence, uniqueness and properties of fixed points of given operators.

In 1886, Poincare [42] was the first to work on fixed point theory. Later, Browder [6] proved fixed point theorem for the solution of the equation $G\psi = \psi$. He also proved fixed point theorem for a square, a sphere and their n-dimension counterparts. In 1922, Banach [5] proved that a contraction mapping in a complete metric space has a unique fixed point. Fixed point theory is an interdisciplinary topic which can be applied in various aspect of mathematics and sciences like game theory, mathematical economics, mathematical physics, optimization theory, approximation theory and variational inequalities.

A very interesting and useful result on fixed theory is the contraction mapping principle credited to Banach [5]. The principle is a fixed point theorem which guarantees that every contraction mapping of a complete metric space into itself has a unique fixed point and it also provided a constructive method which is used to approximate the unique fixed point. The principle is as follows:

Theorem 1.2 (see [5]). *Let (X, ρ) be a metric space and $G : X \to X$ be a contraction, then G has a unique fixed point in X. i.e., there exists a unique $p \in X$ such that $Gp = p$. Moreover, for arbitrary $x_0 \in X$, the sequence $\{\psi_n\}_{n\geq 0}$ defined by*

$$\psi_{n+1} = G\psi_n, \ \ n \geq 0, \tag{1.12}$$

converges to the unique fixed point of the map G.

The iteration scheme (1.12) is known as Picard iteration formula. The Banach contraction principle is important because it is a source of the existence and uniqueness theorems in different branches of science. It brings to the limelight the powerful analytic method and usefulness of fixed point theorem in analysis. It gives the existence, uniqueness and the sequence of the successive approximations converging to a solution of the problem (see Igbokwe [23]).

A nonexpansive mapping may not have a fixed point in a complete metric space. For example, if $G : \mathbb{R} \to \mathbb{R}$ is given by the translation mapping $G\psi = \psi + k$, where k is any real number, then G has no fixed point. The identity mapping $I : \mathbb{R} \to \mathbb{R}$ given by $I\psi = \psi$ is also nonexpanive mapping and it has each point of $\mathbb{R}$ as a fixed point. These examples shows that a nonexpansive map, unlike contraction map need not have a fixed point and even if it has, it may not be unique and the so called Picard sequence of the approximation (1.12) may fail to converge to such a fixed point.

For example, let $X = \mathbb{R}$, the set of real numbers with the usual norm. Define the mapping $G : X \to X$ by $G\psi = -\psi$. Then G is a nonexpansive mapping with the origin zero as the only fixed point. For arbitrary $\psi_0 \in \mathbb{R}$, $\psi_0 \neq 0$, the Picard iteration process (1.12) fails to converge to zero. To see this, observe that for $\psi_0 \in \mathbb{R}$, $\psi_1 = G\psi_0 = -\psi_0$; $\psi_2 = G\psi_1 = -\psi_1$; $\psi_3 = G\psi_2 = -\psi_2 = -\psi_0$; ... Hence, $\{\psi_n\}_{n\geq 0} = \{(-1)^n\}_{n\geq 0}$ is an alternating sequence of oscillatory sequence of real number which is known to be divergent.

Also, if we consider an anti-clockwise rotation of the unit disc of $\mathbb{R}^2$ about the origin through an angle θ. Then G can be shown to be nonexpansive with the origin $(0,0)$ as the only fixed point. The Picard iteration process (1.12) again fails to converge to any point $\psi_0 \neq 0$.

In 1955, Kransnosel'skii [28] introduced an iterative scheme for approximating fixed point of nonexpansive mapping. A generalization of the Krasnosel'skii iterative method was given by Schaefer [50] as follows:

$$\begin{cases} \psi_0 \in K, \\ \psi_{n+1} = (1-\lambda)\psi_n + \lambda G\psi_n, \end{cases} \qquad \forall n \geq 1, \tag{1.13}$$

where λ is a constant in $(0,1)$.

On the other hand, the most general iteration process which has been widely studied for the approximation of fixed points of nonexpansive maps, called the Mann iteration process [34] is defined as follows:

$$\begin{cases} \psi_0 \in K, \\ \psi_{n+1} = (1-s_n)\psi_n + s_n G\psi_n, \end{cases} \qquad \forall n \geq 1, \tag{1.14}$$

where $\{s_n\}$ is a sequence in $(0,1)$ satisfying some appropriate conditions.

Nevertheless, the success of Mann iterative scheme has not carried over to the more general class of pseudocontractive mappings. It is well known that Mann's iteration scheme fails to converge to fixed point of Lipschitz pseudocontractive mappings in a compact convex subset of a Hilbert space. Chidume and Mutangadura [17] provided an example of a Lipschitz pseudocontractive mapping with a unique fixed point for which the Mann iteration process does not converge in a compact subset of a Hilbert space.

In 1974, Ishikawa [24] introduced an iteration process which in some sense, is more general than that of Mann and proved its convergence to a fixed point of Lipschitz pseudocontractive mapping in a compact convex subset of a Hilbert space. Specifically, he proved the following result:

Theorem 1.3. *If K is a compact convex subset of a Hilbert space, $G : K \to K$ is a Lipshitzian pseudocontractive map, the sequence $\{\psi_n\}$ converges strongly to a fixed point of G, where ψ_n is defined iteratively by*

$$\begin{cases} \psi_0 \in K, \\ \psi_{n+1} = (1-s_n)\psi_n + s_n G\zeta_n, \\ \zeta_n = (1-s'_n)\psi_n + s'_n G\psi_n, \end{cases} \qquad \forall n \geq 1, \tag{1.15}$$

and $\{s_n\}, \{s'_n\}$ are sequences of positive numbers satisfying: (i) $0 \leq s_n \leq s'_n < 1$, (ii) $\lim\limits_{n\to\infty} s'_n = 0$ (iii)$\sum_{n\geq 0} s_n s'_n = \infty$.

The iteration process (1.15) is known as the Ishikawa iteration process and reduces to the Mann iterative scheme (1.14) if $s'_n = 0$.

The limitations of Picard and Mann iterative scheme has gotten many researchers busy with constructing and studying new explicit and implicit iterative methods for approximating the fixed points of the classes of nonexpansive mappings, pseudocontractive mappings and other more general classes of mappings.

Some notable iterative schemes in the existing literature includes: Krasnoselskii [28], Mann [34], Ishikawa [24], Agarwal [2], Noor [35], Abbas [1] and so on.

Motivated by the above results, we introduce the following implicit iteration process for approximating the common fixed points of two uniformly L–Lipschitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense:

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1-s_n-q_n)\psi_{n-1} + s_n G^{k(n)}_{i(n)}\zeta_n + q_n H^{k(n)}_{i(n)}\psi_n, \\ \zeta_n = (1-s'_n-q'_n-t'_n)\psi_{n-1} + s'_n\psi_n + q'_n H^{k(n)}_{i(n)}\psi_{n-1} + t'_n G^{k(n)}_{i(n)}\psi_n, \end{cases} \tag{1.16}$$

$\forall n \geq 1$, where $\{s_n\}, \{q_n\}, \{s'_n\}, \{q'_n\}$ and $\{t'_n\}$ are real sequences in [0,1] satisfying $s_n + q_n \leq 1$ and $s'_n + q'_n + t'_n \leq 1$, $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

Interestingly, our new iteration process contains many iterative schemes which has been employed by several well known researchers in approximating fixed points of different classes of operators as we will see in the following special cases of our new iteration process:

- If we take $q_n = s'_n = q'_n = 0$ in (1.16) then we obtain

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1 - s_n)\psi_{n-1} + s_n G^{k(n)}_{i(n)} \zeta_n, \qquad \forall n \geq 1, \\ \zeta_n = (1 - t'_n)\psi_{n-1} + t'_n G^{k(n)}_{i(n)} \psi_n \end{cases} \tag{1.17}$$

  where $\{s_n\}$ and $\{t'_n\}$ are sequences in [0,1] and $n = (k-1)N + i$, $i = i(n) \in \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

  The composite implicit iterative process (1.17) was introduced in 2007 by Thahur [56] for a finite family of asymptotically nonexpansive mappings.

- If we take $s'_n = q'_n = t'_n = 0$ in (1.16) then we obtain

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1 - s_n - q_n)\psi_{n-1} + s_n G^{k(n)}_{i(n)} \psi_{n-1} + q_n H^{k(n)}_{i(n)} \psi_n, \end{cases} \quad \forall n \geq 1 \tag{1.18}$$

  where $\{s_n\}$ and $\{q_n\}$ are real sequences in [0,1] satisfying $s_n + q_n \leq 1$, $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

  The iterative algorithm (1.18) was introduced and studied by Khan et al. [26] for finite families of uniformly L–Lipschitzian generalized Φ-hemicontractive mappings in Banach spaces.

- If we take $s'_n = q'_n = t'_n = 0$, $G^n = G$ and $H^n = H$ in (1.16) then we obtain

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1 - s_n - q_n)\psi_{n-1} + s_n G_n \psi_{n-1} + q_n H_n \psi_n, \end{cases} \quad \forall n \geq 1 \tag{1.19}$$

  where $G_n = G_{n(\,mod N)}$, $H_n = H_{n(\,mod\, N)}$ and $\{s_n\}$, $\{q_n\}$ are two real sequences in [0,1] satisfying $s_n + q_n \leq 1$.

  Again, in [27], Khan et al. introduced and studied the iterative algorithm (1.19) for finite families of two uniformly L–Lipschitzian generalized Φ-hemicontractive mappings in Banach Spaces.

- If we take $s'_n = q'_n = t'_n = s_n = 0$ in (1.16) then we obtain

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1 - q_n)\psi_{n-1} + q_n H^{k(n)}_{i(n)} \psi_n, \end{cases} \quad \forall n \geq 1, \tag{1.20}$$

  where $\{q_n\}$ is a sequence in [0,1], $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

  In 2003, Sun [52] introduced the iterative scheme (1.20) for the approximation of fixed points of asymptotically quasi-nonexpansive mappings.

- If we take $s'_n = q'_n = t'_n = s_n = 0$, $H^n = H$ in (1.16) then we obtain

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1 - q_n)\psi_{n-1} + q_n H_n \psi_n, \end{cases} \quad \forall n \geq 1, \tag{1.21}$$

where $H_n = H_{n(\,mod\,N)}$ and $\{q_n\}$ a sequence in [0,1].

In 2001, Xu and Ori [59] first introduced and studied the implicit iteration process (1.21) for finite family of nonexpansive self-mapping in Hilbert spaces.

- If we take $s'_n = q'_n = t'_n = s_n = 0$, $H^n = H$, $N = 1$ in (1.16) then we obtain

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1-q_n)\psi_{n-1} + q_n H\psi_n, \end{cases} \qquad \forall n \geq 1, \tag{1.22}$$

where $\{q_n\}$ is a sequence in [0,1]. In [45], Rafiq and Imdad characterized conditions for the convergence of the implicit Mann iterative scheme (1.22) to the unique fixed point of generalized Φ-hemicontractive mappings defined on a nonempty convex subset of an arbitrary Banach space.

- If we take $s'_n = q'_n = t'_n = q_n = 0$ in (1.16) then we obtain

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1-s_n)\psi_{n-1} + s_n G^{k(n)}_{i(n)}\psi_{n-1}, \end{cases} \qquad \forall n \geq 1, \tag{1.23}$$

where $\{s_n\}$ is a sequence in [0,1], $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

The above modified averaging iteration process (1.23) was considered in 2014 by Saluja [46] for the approximation of common fixed point of a finite family of strictly asymptotically pseudocontractive mappings in the intermediate sense in Hilbert spaces.

- If we take $s'_n = q'_n = t'_n = q_n = 0$, $N = 1$ in (1.16) then we obtain

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1-s_n)\psi_{n-1} + s_n G^n\psi_{n-1}, \end{cases} \qquad \forall n \geq 1, \tag{1.24}$$

where $\{s_n\}$ is a sequences in [0,1]. The Mann-type iterative scheme (1.24) was introduced in 1991 by Schu [47] for an asymptotically nonexpansive in Hilbert spaces. Several authors have studied the approximation of (1.24) for the classes of asymptotically nonexpansive and asymptotically generalized Φ-hemicontractive mappings (see for example Chang [7], Ofoedu [37], Schu [48], Cho et al. [16], Zeng [63]-[64], Kim [31], Chidume [12]-[13], Gu [21]).

- If we take $s'_n = q'_n = t'_n = q_n = 0$, $G^n = G$, $N = 1$ in (1.16) then we obtain

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1-s_n)\psi_{n-1} + s_n G\psi_{n-1}, \end{cases} \qquad \forall n \geq 1, \tag{1.25}$$

where $\{s_n\}$ is a sequences in [0,1]. The iteration process (1.25) is the famous Mann iteration scheme which was introduced by Mann [34] in 1953.

From the demonstration above, we can clearly see that our new iteration process (1.16) properly contains the iteration processes (1.17)–(1.25) and several others in the existing literature.

Now, we show that our new implicit iteration process (1.16) can be employed for approximating common fixed points of two finite families of uniformly L-Lipschitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense.

Let G_i be a L^i_g-Lipschitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense with sequence $\{a^i_n\} \in [1, \infty)$ such that $a^i_n \to 1$ as $n \to \infty$ and H_i be a L^i_h-Lipschitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense with sequence $\{b^i_n\} \in [1, \infty)$ such that $b^i_n \to 1$ as $n \to \infty$ for each $1 \leq i \leq N$. Let $\{\psi_n\}$ be defined by (1.16). Define a mapping $\Omega_n : K \to K$ by

$$\begin{aligned} \Omega_n(\psi) \;\; &= \;\; (1 - s_n - q_n)\psi_{n-1} + s_n G^{k(n)}_{i(n)}[(1 - s'_n - q'_n - t'_n)\psi_{n-1} + s'_n\psi \\ &\quad + q'_n H^{k(n)}_{i(n)}\psi_{n-1} + t'_n G^{k(n)}_{i(n)}\psi] + q_n H^{k(n)}_{i(n)}\psi, \end{aligned}$$

for all $\psi \in K$ and $\forall\, n \geq 1$.

Now for any $\psi, \zeta \in K$ and $\forall n \geq 1$, we have

$$\begin{aligned}
\|\Omega_n(\psi) - \Omega_n(\zeta)\| &= s_n \|G^{k(n)}_{i(n)}[(1 - s'_n - q'_n - t'_n)\psi_{n-1} + s'_n\psi \\
&\quad + q'_n H^{k(n)}_{i(n)}\psi_{n-1} + t'_n G^{k(n)}_{i(n)}\psi] + q_n H^{k(n)}_{i(n)}\psi \\
&\quad - G^{k(n)}_{i(n)}[(1 - s'_n - q'_n - t'_n)\psi_{n-1} + s'_n\zeta \\
&\quad + q'_n H^{k(n)}_{i(n)}\psi_{n-1} + t'_n G^{k(n)}_{i(n)}\zeta] + q_n H^{k(n)}_{i(n)}\zeta\| \\
&\leq s_n L\|s'_n(\psi - \zeta) + t'_n(G^{k(n)}_{i(n)}\psi - G^{k(n)}_{i(n)}\zeta) \\
&\quad + q_n(H^{k(n)}_{i(n)}\psi - H^{k(n)}_{i(n)}\zeta)\| \\
&\leq s_n L(s'_n\|\psi - \zeta\| + t'_n L\|\psi - \zeta\| + q_n L\|\psi - \zeta\|) \\
&= s_n L(s'_n + t'_n L + q_n L)\|\psi - \zeta\|,
\end{aligned}$$

where $L = \max\{L^1_g, ..., L^N_g, L^1_h, ..., L^N_h\}$.

If $s_n L(s'_n + t'_n L + q_n L) < 1$, then Ω_n is a contraction mapping. By Banach contraction principle, Ω_n, $n \geq 1$ has a unique fixed point. Thus, the implicit iteration process (1.16) is well defined.

The purpose of this paper is to prove strong convergence of our new implicit iteration process (1.16) to the common fixed points of two uniformly L-Lipschitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense. Our results can be seen as a generalization, extension and complementation of several well known results in the literature owing to the fact that our new scheme properly includes the iterative schemes (1.17)–(1.25) which has been considered by several authors.

2 Preliminaries

In the sequel, we will need the following Lemmas.

Lemma 2.1 ((see [7])). *Let $J : X \to 2^{X^*}$ be the normalized duality mapping. Then for any $\psi, \zeta \in E$, one has*

$$\|\psi + \zeta\|^2 \leq \|\psi\|^2 + 2\langle\zeta, j(\psi + \zeta)\rangle, \;\; \forall j(\psi + \zeta) \in J(\psi + \zeta). \tag{2.1}$$

Lemma 2.2 (see [59]). *Let $\{\vartheta_n\}$ and $\{\Lambda_n\}$, $\{\sigma_n\}$ be sequences of nonnegative real numbers satisfying the following inequality:*

$$\vartheta_n \leq (1 + \Lambda_n)\vartheta_n + \sigma_n, \;\; n \geq 1. \tag{2.2}$$

If $\sum_{n=1}^{\infty} \Lambda_n < \infty$, $\sum_{n=1}^{\infty} \sigma_n < \infty$ then $\lim_{n\to\infty} \vartheta_n$ exists. Additionally, if $\{\vartheta_n\}$ has a subsequence $\{\vartheta_{n_i}\}$ such that $\vartheta_{n_i} \to 0$, then $\lim_{n\to\infty} \vartheta_n = 0$.

Lemma 2.3 (see [61]). *Let $\Phi : [0, \infty] \to [0, \infty)$ be a strictly increasing function with $\Phi(0) = 0$ and let $\{\gamma_n\}, \{\eta_n\}, \{\varpi_n\}$ be nonnegative real sequences such that $0 \leq \eta_n \leq 1$, $\sum\limits_{n=1}^{\infty} \eta_n = \infty$. If there exists a positive integer n_0 such that*

$$\gamma^2_{n+1} \leq \gamma^2_n - \eta_n\Phi(\gamma_{n+1}) + \varpi_n, \tag{2.3}$$

for all $n \geq n_0$, with $\varpi_n \geq 0$, $\forall n \geq 1$, $\varpi_n = o(\eta_n)$, then

$$\lim_{n\to\infty} \gamma_n = 0. \tag{2.4}$$

3 Main Results

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $G_i : K \to K$ be a uniformly L_g^i–Lipschitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense with sequence $\{a_n^i\} \subset [1, \infty)$, where $a_n^i \to 1$ as $n \to \infty$ and $H_i : K \to K$ be a uniformly L_h^i–Lipscitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense with sequence $\{b_n^i\} \subset [1, \infty)$, where $b_n^i \to 1$ as $n \to \infty$, for each $i \in I$. Let $h_n = \max\{\mu_n, \nu_n\}$, where $\mu_n = \max\{a_n^i : i \in I\}$ and $\nu_n = \max\{b_n^i : i \in I\}$. Assume that $\boldsymbol{F} = (\bigcap_{i=1}^N F(G_i)) \bigcap (\bigcap_{i=1}^N F(H_i)) \neq \emptyset$. Let $\{s_n\}$, $\{q_n\}$, $\{s_n'\}$, $\{q_n'\}$, $\{t_n'\}$ be sequences in [0,1] such that $s_n + q_n \leq 1$ and $s_n' + q_n' + t_n' \leq 1$ for each $n \geq 1$. Let $\{\psi_n\}$ be a sequence generated in (1.16). Put*

$$\ell_n^i = \max\left\{0, \sup_{(\psi,p)\in K\times F(G)} (\langle G_i^n\psi - p, j(\psi - p)\rangle - a_n^i\|\psi - p\|^2 + \Phi_i(\|\psi - p\|))\right\},$$

$$\Gamma_n^i = \max\left\{0, \sup_{(\psi,p)\in K\times F(H)} (\langle H_i^n\psi - p, j(\psi - p)\rangle - b_n^i\|\psi - p\|^2 + \Phi_i(\|\psi - p\|))\right\}.$$

Let $\tau_n = \max\{\ell_n, \Gamma_n\}$, where $\ell_n = \max\{\ell_n^i : i \in I\}$, $\Gamma_n = \max\{\Gamma_n^i : i \in I\}$. Let $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in I\}$, for each $\wp \geq 0$. Assume that the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty}(s_n + q_n) = \infty$;

(ii) $\sum_{n=0}^{\infty}(s_n + q_n)^2 < \infty$;

(iii) $\sum_{n=0}^{\infty}(s_n + q_n)(h_n - 1) < \infty$, $\sum_{n=0}^{\infty}\tau_n(s_n + q_n) < \infty$;

(iv) $\sum_{n=0}^{\infty} s_n s_n' < \infty$, $\sum_{n=0}^{\infty} s_n q_n' < \infty$, $\sum_{n=0}^{\infty} s_n t_n' < \infty$;

(v) $s_n L(s_n' + t_n' L + q_n L) < 1$, $\forall n \geq 1$.

Then the sequence $\{\psi_n\}$ converges strongly to a point in $\boldsymbol{F}$.

Proof. Fixing $p \in \mathbf{F}$, then from (1.16) we see that

$$\begin{aligned}
\|\zeta_n - p\| &= \|(1 - s_n' - q_n' - t_n')\psi_{n-1} + s_n'\psi_n \\
&\quad + q_n' H_{i(n)}^{k(n)}\psi_{n-1} + t_n' G_{i(n)}^{k(n)}\psi_n - p\| \\
&= \|(1 - s_n' - q_n' - t_n')(\psi_{n-1} - p) + s_n'(\psi_n - p) \\
&\quad + q_n'(H_{i(n)}^{k(n)}\psi_{n-1} - p) + t_n'(G_{i(n)}^{k(n)}\psi_n - p)\| \\
&\leq (1 - s_n' - q_n' - t_n')\|\psi_{n-1} - p\| + s_n'\|\psi_n - p\| \\
&\quad + q_n'\|H_{i(n)}^{k(n)}\psi_{n-1} - p\| + t_n'\|G_{i(n)}^{k(n)}\psi_n - p\| \\
&\leq \|\psi_{n-1} - p\| + s_n'\|\psi_n - p\| \\
&\quad + q_n' L\|\psi_{n-1} - p\| + t_n' L\|\psi_n - p\| \\
&= (1 + q_n' L)\|\psi_{n-1} - p\| + (s_n' + t_n' L)\|\psi_n - p\| \\
&\leq (1 + L)\|\psi_{n-1} - p\| + (s_n' + t_n' L)\|\psi_n - p\|. \qquad (3.1)
\end{aligned}$$

Notice from (1.16) that

$$
\begin{aligned}
\|\zeta_n-\psi_n\| &= \|\zeta_n-\psi_{n-1}+\psi_{n-1}-\psi_n\| \\
&\leq \|\zeta_n-\psi_{n-1}\|+\|\psi_{n-1}-\psi_n\| \\
&= \|[(1-s'_n-q'_n-t'_n)\psi_{n-1}+s'_n\psi_n \\
&\quad +q'_nH^{k(n)}_{i(n)}\psi_{n-1}+t'_nG^{k(n)}_{i(n)}\psi_n]-\psi_{n-1}\| \\
&\quad +\|\psi_{n-1}-[(1-s_n-q_n)\psi_{n-1}+s_nG^{k(n)}_{i(n)}\zeta_n+q_nH^{k(n)}_{i(n)}\psi_n]\| \\
&= \|s'_n(\psi_n-\psi_{n-1})+q'_n(H^{k(n)}_{i(n)}\psi_{n-1}-\psi_{n-1})+t'_n(G^{k(n)}_{i(n)}\psi_n-\psi_{n-1})\| \\
&\quad +\|s_n(\psi_{n-1}-G^{k(n)}_{i(n)}\zeta_n)+q_n(\psi_{n-1}-H^{k(n)}_{i(n)}\psi_n)\| \\
&\leq s'_n\|\psi_n-p\|+s'_n\|\psi_{n-1}-p\|+q'_n\|H^{k(n)}_{i(n)}\psi_{n-1}-p\| \\
&\quad +q'_n\|\psi_{n-1}-p\|+t'_n\|G^{k(n)}_{i(n)}\psi_n-p\|+t'_n\|\psi_{n-1}-p\| \\
&\quad +s_n\|\psi_{n-1}-p\|+s_n\|G^{k(n)}_{i(n)}\zeta_n-p\| \\
&\quad +q_n\|\psi_{n-1}-p\|+q_n\|H^{k(n)}_{i(n)}\psi_n-p\| \\
&\leq s'_n\|\psi_n-p\|+s'_n\|\psi_{n-1}-p\|+q'_nL\|\psi_{n-1}-p\| \\
&\quad +q'_n\|\psi_{n-1}-p\|+t'_nL\|\psi_n-p\|+t'_n\|\psi_{n-1}-p\| \\
&\quad +s_n\|\psi_{n-1}-p\|+s_nL\|\zeta_n-p\| \\
&\quad +q_n\|\psi_{n-1}-p\|+q_nL\|\psi_n-p\|.
\end{aligned} \tag{3.2}
$$

Putting (3.1) into (3.2) we obtain

$$
\begin{aligned}
\|\zeta_n-\psi_n\| &\leq s'_n\|\psi_n-p\|+s'_n\|\psi_{n-1}-p\|+q'_nL\|\psi_{n-1}-p\| \\
&\quad +q'_n\|\psi_{n-1}-p\|+t'_nL\|\psi_n-p\|+t'_n\|\psi_{n-1}-p\| \\
&\quad +s_n\|\psi_{n-1}-p\|+s_nL[(1+L)\|\psi_{n-1}-p\| \\
&\quad +(s'_n+t'_nL)\|\psi_n-p\|] \\
&\quad +q_n\|\psi_{n-1}-p\|+q_nL\|\psi_n-p\| \\
&= [s'_n+q'_n+t'_n+q'_nL+s_n+s_nL(1+L)+q_n]\|\psi_{n-1}-p\| \\
&\quad +[s'_n+t'_nL+s_nL(s'_n+t'_nL)+q_nL]\|\psi_n-p\| \\
&= [s'_n+q'_n(1+L)+t'_n+s_n+s_nL(1+L)+q_n]\|\psi_{n-1}-p\| \\
&\quad +[s'_n(1+s_nL)+t'_nL(1+s_nL)+q_nL]\|\psi_n-p\| \\
&\leq [s'_n+q'_n(1+L)+t'_n+s_n+s_nL(1+L)+q_n]\|\psi_{n-1}-p\| \\
&\quad +[s'_n(1+L)+t'_nL(1+L)+q_nL]\|\psi_n-p\|.
\end{aligned} \tag{3.3}
$$

Using (1.16) and Lemma 2.1, we have

$$
\begin{aligned}
\|\psi_n-p\|^2 &= \|(1-s_n-q_n)\psi_{n-1}+s_nG^{k(n)}_{i(n)}\zeta_n+q_nH^{k(n)}_{i(n)}\psi_n-p\|^2 \\
&= \|(1-s_n-q_n)(\psi_{n-1}-p)+s_n(G^{k(n)}_{i(n)}\zeta_n-p) \\
&\quad +q_n(H^{k(n)}_{i(n)}\psi_n-p)\|^2 \\
&\leq (1-s_n-q_n)^2\|\psi_{n-1}-p\|^2+2\langle s_n(G^{k(n)}_{i(n)}\zeta_n-p) \\
&\quad +q_n(H^{k(n)}_{i(n)}\psi_n-p),j(\psi_n-p)\rangle \\
&= (1-s_n-q_n)^2\|\psi_{n-1}-p\|^2+2s_n\langle G^{k(n)}_{i(n)}\zeta_n-p,j(\psi_n-p)\rangle \\
&\quad +2q_n\langle H^{k(n)}_{i(n)}\psi_n-p,j(\psi_n-p)\rangle
\end{aligned}
$$

$$\begin{aligned}
&= (1-s_n-q_n)^2\|\psi_{n-1}-p\|^2 + 2s_n\langle G_{i(n)}^{k(n)}\zeta_n - G_{i(n)}^{k(n)}\psi_n \\
&\quad + G_{i(n)}^{k(n)}\psi_n - p, j(\psi_n - p)\rangle + 2q_n\langle H_{i(n)}^{k(n)}\psi_n - p, j(\psi_n-p)\rangle \\
&= (1-s_n-q_n)^2\|\psi_{n-1}-p\|^2 + 2s_n\langle G_{i(n)}^{k(n)}\zeta_n - G_{i(n)}^{k(n)}\psi_n, j(\psi_n-p)\rangle \\
&\quad + 2s_n\langle G_{i(n)}^{k(n)}\psi_n - p, j(\psi_n-p)\rangle + 2q_n\langle H_{i(n)}^{k(n)}\psi_n - p, j(\psi_n-p)\rangle \\
&\le (1-s_n-q_n)^2\|\psi_{n-1}-p\|^2 + 2s_n\|G_{i(n)}^{k(n)}\zeta_n - G_{i(n)}^{k(n)}\psi_n\|\|\psi_n-p\| \\
&\quad + 2s_n\langle G_{i(n)}^{k(n)}\psi_n - p, j(\psi_n-p)\rangle \\
&\quad + 2q_n\langle H_{i(n)}^{k(n)}\psi_n - p, j(\psi_n-p)\rangle \\
&\le (1-s_n-q_n)^2\|\psi_{n-1}-p\|^2 + 2s_nL\|\zeta_n-\psi_n\|\|\psi_n-p\| \\
&\quad + 2s_n\langle G_{i(n)}^{k(n)}\psi_n - p, j(\psi_n-p)\rangle \\
&\quad + 2q_n\langle H_{i(n)}^{k(n)}\psi_n - p, j(\psi_n-p)\rangle. \qquad (3.4)
\end{aligned}$$

Substituting (3.3) into (3.4), we obtain

$$\begin{aligned}
\|\psi_n-p\|^2 &\le (1-s_n-q_n)^2\|\psi_{n-1}-p\|^2 + 2s_nL\{[s_n' + q_n'(1+L) \\
&\quad + t_n' + s_n + s_nL(1+L)) + q_n]\|\psi_{n-1}-p\| \\
&\quad + [s_n'(1+L) + t_n'L(1+L) + q_nL]\|\psi_n-p\|\}\|\psi_n-p\| \\
&\quad + 2s_n\langle G_{i(n)}^{k(n)}w_n - p\, j(\psi_n-p)\rangle \\
&\quad + 2q_n\langle H_{i(n)}^{k(n)}\psi_n - p\, j(\psi_n-p)\rangle. \\
&= (1-s_n-q_n)^2\|\psi_{n-1}-\alpha z\|^2 + 2s_nL[s_n' + q_n'(1+L) \\
&\quad + t_n' + s_n + s_nL(1+L) + q_n]\|\psi_{n-1}-p\|\|\psi_n-p\| \\
&\quad + 2s_nL[s_n'(1+L) + t_n'L(1+L) + q_nL]\|\psi_n-p\|^2 \\
&\quad + 2s_n\langle G_{i(n)}^{k(n)}\psi_n - p, j(\psi_n-p)\rangle \\
&\quad + 2q_n\langle H_{i(n)}^{k(n)}\psi_n - p, j(\psi_n-p)\rangle. \qquad (3.5)
\end{aligned}$$

Now, using the fact that

$$\|\psi_{n-1}-p\|\|\psi_n-p\| \le \frac{1}{2}(\|\psi_{n-1}-p\|^2 + \|\psi_n-p\|^2), \qquad (3.6)$$

then from (3.5) and (3.6) we obtain

$$\begin{aligned}
\|\psi_n-p\|^2 &\le (1-s_n-q_n)^2\|\psi_{n-1}-p\|^2 + 2s_nL[s_n' + q_n'(1+L) \\
&\quad + t_n' + s_n + s_nL(1+L) + q_n] \times \frac{1}{2}(\|\psi_{n-1}-p\|^2 + \|\psi_n-p\|^2) \\
&\quad + 2s_nL[s_n'(1+L) + t_n'L(1+L) + q_nL]\|\psi_n-p\|^2 \\
&\quad + 2s_n\langle G_{i(n)}^{k(n)}\psi_n - p, j(\psi_n-p)\rangle \\
&\quad + 2q_n\langle H_{i(n)}^{k(n)}\psi_n - p, j(\psi_n-p)\rangle
\end{aligned}$$

$$\begin{aligned}
&= \{(1-s_n-q_n)^2 + s_nL[s'_n + q'_n(1+L) \\
&\quad +t'_n + s_n + s_nL(1+L) + q_n]\}\|\psi_{n-1}-p\|^2 \\
&\quad +\{s_nL[s'_n + q'_n(1+L) + t'_n + s_n + s_nL(1+L) + q_n] \\
&\quad +2s_nL[s'_n(1+L) + t'_nL(1+L) + q_nL]\}\|\psi_n - p\|^2 \\
&\quad +2s_n\langle G^{k(n)}_{i(n)}\psi_n - p, j(\psi_n - p)\rangle \\
&\quad +2q_n\langle H^{k(n)}_{i(n)}\psi_n - p, j(\psi_n - p)\rangle. \qquad (3.7)
\end{aligned}$$

S_i and H_i are two finite families of asymptotically generalized Φ-hemicontractive mappings in the intermediate sense, we have from (3.7) that

$$\begin{aligned}
\|\psi_n - p\|^2 &\leq \{(1-s_n-q_n)^2 + s_nL[s'_n + q'_n(1+L) \\
&\quad +t'_n + s_n + s_nL(1+L) + q_n]\}\|\psi_{n-1}-p\|^2 \\
&\quad +\{s_nL[s'_n + q'_n(1+L) + t'_n + s_n + s_nL(1+L) + q_n] \\
&\quad +2s_nL[s'_n(1+L) + t'_nL(1+L) + q_nL]\}\|\psi_n - p\|^2 \\
&\quad +2s_n(h_n\|\psi_n - p\|^2 + \tau_n - \Phi(\|\psi_n - p\|)) \\
&\quad +2q_n(h_n\|\psi_n - p\|^2 + \tau_n - \Phi(\|\psi_n - p\|)) \\
&= \{(1-s_n-q_n)^2 + s_nL[s'_n + q'_n(1+L) \\
&\quad +t'_n + s_n + s_nL(1+L) + q_n]\}\|\psi_{n-1}-p\|^2 \\
&\quad +\{s_nL[s'_n + q'_n(1+L) + t'_n + s_n + s_nL(1+L) + q_n] \\
&\quad +2s_nL[s'_n(1+L) + t'_nL(1+L) + q_nL]\}\|\psi_n - p\|^2 \\
&\quad +2h_n(s_n+q_n)\|\psi_n - p\|^2 + 2\tau_n(s_n+q_n) - 2(s_n+q_n)\Phi(\|\psi_n - p\|) \\
&= \{(1-s_n-q_n)^2 + s_nL[s'_n + q'_n(1+L) \\
&\quad +t'_n + s_n + s_nL(1+L) + q_n]\}\|\psi_{n-1}-p\|^2 \\
&\quad +\{s_nL[s'_n + q'_n(1+L) + t'_n + s_n + s_nL(1+L) + q_n] \\
&\quad +2s_nL[s'_n(1+L) + t'_nL(1+L) + q_nL] + 2h_n(s_n+q_n)\}\|\psi_n - p\|^2 \\
&\quad +2\tau_n(s_n+q_n) - 2(s_n+q_n)\Phi(\|\psi_n - p\|) \\
&= m_n\|\psi_{n-1}-p\|^2 + \omega_n\|\psi_n - p\|^2 + 2\tau_n(s_n+q_n) \\
&\quad -2(s_n+q_n)\Phi(\|\psi_n - p\|), \qquad (3.8)
\end{aligned}$$

where

$$\begin{aligned}
m_n &= (1-s_n-q_n)^2 + s_nL[s'_n + q'_n(1+L) + t'_n + s_n + s_nL(1+L) + q_n], \\
\omega_n &= s_nL[s'_n + q'_n(1+L) + t'_n + s_n + s_nL(1+L) + q_n] \\
&\quad +2s_nL[s'_n(1+L) + t'_nL(1+L) + q_nL] + 2h_n(s_n+q_n).
\end{aligned}$$

Simplifying (3.8), we obtain

$$\begin{aligned}
\|\psi_n - p\|^2 &\leq \frac{m_n}{1-\omega_n}\|\psi_{n-1}-p\|^2 + \frac{2\tau_n(s_n+q_n)}{1-\omega_n} - \frac{2(s_n+q_n)}{1-\omega_n}\Phi(\|\psi_n - p\|) \\
&= \left(1 + \frac{m_n+\omega_n-1}{1-\omega_n}\right)\|\psi_{n-1}-p\|^2 + \frac{2\tau_n(s_n+q_n)}{1-\omega_n} \\
&\quad -\frac{2(s_n+q_n)}{1-\omega_n}\Phi(\|\psi_n - p\|). \qquad (3.9)
\end{aligned}$$

Notice that

$$\begin{aligned}
m_n + \omega_n - 1 &= (s_n+q_n)^2 + 2s_nL[s'_n + q'_n(1+L) + t'_n + s_nL(1+L) + (s_n+q_n)] \\
&\quad +2s_nL[s'_n(1+L) + t'_nL(1+L) + q_nL] + 2(s_n+q_n)(h_n-1).
\end{aligned}$$

Set $\lambda_n = m_n + \omega_n - 1$, then from (3.9) we obtain

$$\begin{aligned}\|\psi_n - p\|^2 &\leq \left(1 + \frac{\lambda_n}{1-\omega_n}\right)\|\psi_{n-1} - p\|^2 + \frac{2\tau_n(s_n+q_n)}{1-\omega_n} \\ &\quad -\frac{2(s_n+q_n)}{1-\omega_n}\Phi(\|\psi_n - p\|). \end{aligned} \tag{3.10}$$

Since from condition (ii) we have $\sum_{n=0}^{\infty}(s_n+q_n)^2 < \infty$, this implies that s_n+q_n, s_n^2 and $s_nq_n \to 0$ as $n \to \infty$ and also recalling that $h_n \to 1$ as $n \to \infty$, then from condition (iv) it follows that

$$\begin{aligned}\omega_n &= s_nL[s'_n + q'_n(1+L) + t'_n + s_n + s_nL(1+L) + q_n] \\ &\quad +2s_nL[s'_n(1+L) + t'_nL(1+L) + q_nL] + 2h_n(s_n+q_n) \to 0 \text{ as } n \to \infty,\end{aligned}$$

thus, there exists a positive n_0 such that $1 - \omega_n \geq \frac{1}{2}$, for any $n \geq n_0$.

Thus, from (3.10) we have

$$\begin{aligned}\|\psi_n - p\|^2 &\leq (1+2\lambda_n)\|\psi_{n-1} - p\|^2 + 4\tau_n(s_n+q_n) \\ &\quad -2(s_n+q_n)\Phi(\|\psi_n - p\|).\end{aligned} \tag{3.11}$$

Since $\Phi(\xi) \geq 0$ for all $\xi \geq 0$, then for all $n \geq n_0$, it follows from (3.11) that

$$\|\psi_n - p\|^2 \leq (1+2\lambda_n)\|\psi_{n-1} - p\|^2 + 4\tau_n(s_n+q_n). \tag{3.12}$$

Since from condition (ii) we have $\sum_{n=0}^{\infty}(s_n+q_n)^2 < \infty$, then this implies that $\sum_{n=0}^{\infty} s_n^2 < \infty$, $\sum_{n=0}^{\infty} s_nq_n < \infty$, together with conditions (iii) and (iv) we have $\sum_{n=0}^{\infty} 2\lambda_n^2 < \infty$ and $\sum_{n=0}^{\infty} 4\tau_n(s_n + q_n) < \infty$.

Notices that (3.12) takes the form

$$\vartheta_{n+1} \leq (1+\Lambda_n)\vartheta_n + \sigma_n, \tag{3.13}$$

where

$$\begin{aligned}\vartheta_{n+1} &= \|\psi_n - p\|^2 \\ \vartheta_n &= \|\psi_{n-1} - p\|^2; \\ \Lambda_n &= 2\lambda_n; \\ \sigma_n &= 4\tau_n(s_n+q_n).\end{aligned}$$

From these, we see that all the conditions of Lemma 2.2 are satisfied, thus we have that $\lim_{n\to\infty}\|\psi_n - p\|$ exists. Therefore, the sequence $\{\|\psi_n - p\|\}$ is bounded. Without loss of generality, we can assume that $\|\psi_n - p\|^2 \leq M_1$, where M_1 is a positive constant.

Now from (3.11) we obtain

$$\begin{aligned}\|\psi_n - p\|^2 &\leq \|\psi_{n-1} - p\|^2 - 2(s_n+q_n)\Phi(\|\psi_n - p\|) + 2\lambda_nM_1 \\ &\quad +4\tau_n(s_n+q_n).\end{aligned} \tag{3.14}$$

Clearly, (3.14) can be written as

$$\gamma_{n+1}^2 \leq \gamma_n^2 - \eta_n\Phi(\gamma_{n+1}) + \varpi_n, \tag{3.15}$$

where

$$\begin{aligned}\gamma_{n+1} &= \|\psi_n - p\|; \\ \gamma_n &= \|\psi_{n-1} - p\|; \\ \eta_n &= 2(s_n+q_n); \\ \varpi_n &= 2\lambda_nM_1 + +4\tau_n(s_n+q_n).\end{aligned}$$

Therefore, all the conditions of Lemma 2.3 are satisfied. Hence,

$$\lim_{n\to\infty} \|\psi_n - p\| = 0. \tag{3.16}$$

This completes the proof of Theorem 3.1. $\square$

From Theorem 3.1 we obtain the following results immediately:

Corollary 3.2. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $G_i : K \to K$ be a uniformly L_g^i–Lipschitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense with sequence $\{a_n^i\} \subset [1, \infty)$, where $a_n^i \to 1$ as $n \to \infty$, for each $i \in I$. Let , where $h_n = \max\{a_n^i : i \in I\}$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^N F(G_i) \neq \emptyset$. Let $\{s_n\}$ and $\{t'_n\}$ be sequences in [0,1]. Let $\{\psi_n\}$ be a sequence generated in (1.17). Put $\ell_n^i = \max\left\{0, \sup_{(\psi,p)\in K\times F(G)} (\langle G_i^n\psi - p, j(\psi - p)\rangle - a_n^i\|\psi - p\|^2 + \Phi_i(\|\psi - p\|))\right\}$. Let $\tau_n = \max\{\ell_n^i : i \in I\}$. Let $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in I\}$, for each $\wp \geq 0$. Assume that the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} s_n = \infty$;

(ii) $\sum_{n=0}^{\infty} s_n^2 < \infty$;

(iii) $\sum_{n=0}^{\infty} s_n(h_n - 1) < \infty$, $\sum_{n=0}^{\infty} \tau_n s_n < \infty$;

(iv) $\sum_{n=0}^{\infty} s_n t'_n < \infty$;

(v) $s_n t'_n L^2 < 1, \forall n \geq 1$.

Then the sequence $\{\psi_n\}$ converges strongly to a point in $\boldsymbol{F}$.

Proof. Take $q_n = s'_n = q'_n = 0$ in Theorem 3.1. $\square$

Corollary 3.3. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $G_i : K \to K$ be a uniformly L_g^i–Lipschitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense with sequence $\{a_n^i\} \subset [1, \infty)$, where $a_n^i \to 1$ as $n \to \infty$ and $H_i : K \to K$ be a uniformly L_h^i–Lipscitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense with sequence $\{b_n^i\} \subset [1, \infty)$, where $b_n^i \to 1$ as $n \to \infty$, for each $i \in I$. Let $h_n = \max\{\mu_n, \nu_n\}$, where $\mu_n = \max\{a_n^i : i \in I\}$ and $\nu_n = \max\{b_n^i : i \in I\}$. Assume that $\boldsymbol{F} = (\bigcap_{i=1}^N F(G_i)) \bigcap (\bigcap_{i=1}^N F(H_i)) \neq \emptyset$. Let $\{s_n\}$ and $\{q_n\}$ be sequences in [0,1] such that $s_n + q_n \leq 1$ for each $n \geq 1$. Let $\{\psi_n\}$ be a sequence generated in (1.18). Put $\ell_n^i = \max\left\{0, \sup_{(\psi,p)\in K\times F(G)} (\langle G_i^n\psi - p, j(\psi - p)\rangle - a_n^i\|\psi - p\|^2 + \Phi_i(\|\psi - p\|))\right\}$ and $\Gamma_n^i = \max\left\{0, \sup_{(\psi,p)\in K\times F(H)} (\langle H_i^n\psi - p, j(\psi - p)\rangle - b_n^i\|\psi - p\|^2 + \Phi_i(\|\psi - p\|))\right\}$. Let $\tau_n = \max\{\ell_n, \Gamma_n\}$, where $\ell_n = \max\{\ell_n^i : i \in I\}$, $\Gamma_n = \max\{\Gamma_n^i : i \in I\}$. Let $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in I\}$, for each $\wp \geq 0$. Assume that the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} (s_n + q_n) = \infty$;

(ii) $\sum_{n=0}^{\infty} (s_n + q_n)^2 < \infty$;

(iii) $\sum_{n=0}^{\infty} (s_n + q_n)(h_n - 1) < \infty$, $\sum_{n=0}^{\infty} \tau_n(s_n + q_n) < \infty$;

(iv) $s_n q_n L^2 < 1$, $\forall n \geq 1$.

Then the sequence $\{\psi_n\}$ *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Take $s'_n = q'_n = t'_n = 0$ in Theorem 3.1. □

Corollary 3.4. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $N \geq 1$ *be a positive integer and* $I = \{1, 2, 3, ..., N\}$. *Let* $H_i : K \to K$ *be a uniformly* L_h^i*–Lipscitzian asymptotically generalized* Φ*-hemicontractive mappings in the intermediate sense with sequence* $\{b_n^i\} \subset [1, \infty)$, *where* $b_n^i \to 1$ *as* $n \to \infty$, *for each* $i \in I$. *Let* $h_n = \max\{b_n^i : i \in I\}$. *Assume that* $\boldsymbol{F} = \bigcap_{i=1}^N F(H_i) \neq \emptyset$. *Let* $\{q_n\}$ *be sequence a in [0,1]. Let* $\{\psi_n\}$ *be a sequence generated in* (1.20). *Put*
$\Gamma_n^i = \max\left\{0, \sup_{(\psi,p) \in K \times F(H)} (\langle H_i^n \psi - p, j(\psi - p)\rangle - b_n^i \|\psi - p\|^2 + \Phi_i(\|\psi - p\|))\right\}$. *Let* $\tau_n = \max\{\Gamma_n^i : i \in I\}$. *Let* $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in I\}$, *for each* $\wp \geq 0$. *Assume that the following conditions are satisfied:*

(i) $\sum\limits_{n=0}^{\infty} q_n = \infty$;

(ii) $\sum\limits_{n=0}^{\infty} q_n^2 < \infty$;

(iii) $\sum\limits_{n=0}^{\infty} q_n(h_n - 1) < \infty$, $\sum\limits_{n=0}^{\infty} \tau_n q_n < \infty$.

Then the sequence $\{\psi_n\}$ *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Take $s_n = s'_n = q'_n = t'_n = 0$ in Theorem 3.1. □

Corollary 3.5. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $N \geq 1$ *be a positive integer and* $I = \{1, 2, 3, ..., N\}$. *Let* $G_i : K \to K$ *be a uniformly* L_h^i*–Lipscitzian asymptotically generalized* Φ*-hemicontractive mappings in the intermediate sense with sequence* $\{a_n^i\} \subset [1, \infty)$, *where* $a_n^i \to 1$ *as* $n \to \infty$, *for each* $i \in I$. *Let* $h_n = \max\{a_n^i : i \in I\}$. *Assume that* $\boldsymbol{F} = \bigcap_{i=1}^N F(G_i) \neq \emptyset$. *Let* $\{s_n\}$ *be sequence a in [0,1]. Let* $\{\psi_n\}$ *be a sequence generated in* (1.23). *Put*
$\ell_n^i = \max\left\{0, \sup_{(\psi,p) \in K \times F(G)} (\langle G_i^n \psi - p, j(\psi - p)\rangle - a_n^i \|\psi - p\|^2 + \Phi_i(\|\psi - p\|))\right\}$. *Let* $\tau_n = \max\{\ell_n^i : i \in I\}$. *Let* $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in I\}$, *for each* $\wp \geq 0$. *Assume that the following conditions are satisfied:*

(i) $\sum\limits_{n=0}^{\infty} s_n = \infty$;

(ii) $\sum\limits_{n=0}^{\infty} s_n^2 < \infty$;

(iii) $\sum\limits_{n=0}^{\infty} s_n(h_n - 1) < \infty$, $\sum\limits_{n=0}^{\infty} \tau_n s_n < \infty$.

Then the sequence $\{\psi_n\}$ *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Take $q_n = s'_n = q'_n = t'_n = 0$ in Theorem 3.1. □

Corollary 3.6. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $N \geq 1$ *be a positive integer and* $I = \{1, 2, 3, ..., N\}$. *Let* $G_i : K \to K$ *be a uniformly* L_g^i*–Lipschitzian asymptotically generalized* Φ*-hemicontractive mappings in the intermediate sense with sequence* $\{a_n^i\} \subset [1, \infty)$, *where* $a_n^i \to 1$ *as* $n \to \infty$. *Let* $h_n = \max\{a_n^i : i \in I\}$. *Assume that* $\boldsymbol{F} = (\bigcap_{i=1}^N F(G_i)) \neq \emptyset$. *Let* $\{s_n\}$ *and* $\{q_n\}$ *be sequences in [0,1] such that* $s_n + q_n \leq 1$ *for each* $n \geq 1$. *Put*
$\ell_n^i = \max\left\{0, \sup_{(\psi,p) \in K \times F(G)} (\langle G_i^n \psi - p, j(\psi - p)\rangle - a_n^i \|\psi - p\|^2 + \Phi_i(\|\psi - p\|))\right\}$. *Let* $\tau_n =$

$\max\{\ell_n^i : i \in I\}$ *and* $\Phi(\wp) = \max\{\Phi_i(\wp) : i \in I\}$, *for each* $\wp \geq 0$. *Assume that the following conditions are satisfied:*

(i) $\sum\limits_{n=0}^{\infty}(s_n + q_n) = \infty$;

(ii) $\sum\limits_{n=0}^{\infty}(s_n + q_n)^2 < \infty$;

(iii) $\sum\limits_{n=0}^{\infty}(s_n + q_n)(h_n - 1) < \infty$, $\sum\limits_{n=0}^{\infty}\tau_n(s_n + q_n) < \infty$;

(iv) $s_n q_n L^2 < 1, \forall n \geq 1$.

Let $\{\psi_n\}$ *be a sequence generated by*

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1 - s_n - q_n)\psi_{n-1} + s_n G_{i(n)}^{k(n)}\psi_{n-1} + q_n G_{i(n)}^{k(n)}\psi_n, \end{cases} \quad \forall n \geq 1 \tag{3.17}$$

where $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ *is some positive integers and* $k(n) \to \infty$ *as* $n \to \infty$.

Then the sequence $\{\psi_n\}$ *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Take $G_i = H_i$ in corollary 3.3. □

Corollary 3.7. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $G : K \to K$ *be a uniformly* L_g*–Lipschitzian asymptotically generalized* Φ*-hemicontractive mappings in the intermediate sense with sequence* $\{h_n\} \subset [1, \infty)$, *where* $h_n \to 1$ *as* $n \to \infty$. *Assume that* $\boldsymbol{F} = F(G) \neq \emptyset$. *Let* $\{s_n\}$ *and* $\{t'_n\}$ *be sequences in [0,1]. Put* $\tau_n = \max\left\{0, \sup\limits_{(\psi,p)\in K\times F(G)}(\langle G^n\psi - p, j(\psi - p)\rangle - h_n\|\psi - p\|^2 + \Phi(\|\psi - p\|))\right\}$. *Assume that the following conditions are satisfied:*

(i) $\sum\limits_{n=0}^{\infty} s_n = \infty$;

(ii) $\sum\limits_{n=0}^{\infty} s_n^2 < \infty$;

(iii) $\sum\limits_{n=0}^{\infty} s_n(h_n - 1) < \infty$, $\sum\limits_{n=0}^{\infty}\tau_n s_n < \infty$;

(iv) $\sum\limits_{n=0}^{\infty} s_n t'_n < \infty$;

(v) $s_n t'_n L^2 < 1, \forall n \geq 1$.

Let $\{\psi_n\}$ *be a sequence generated by*

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1 - s_n)\psi_{n-1} + s_n G^n\zeta_n, \qquad \forall n \geq 1, \\ \zeta_n = (1 - t'_n)\psi_{n-1} + t'_n G^n\psi_n \end{cases} \tag{3.18}$$

Then the sequence $\{\psi_n\}$ *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Take $N = 1$ in corollary 3.2. □

Corollary 3.8. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $G : K \to K$ *be a uniformly* L_g*–Lipschitzian asymptotically generalized* Φ*-hemicontractive mappings in the intermediate sense with sequence* $\{a_n\} \subset [1, \infty)$, *where* $a_n \to 1$ *as* $n \to \infty$ *and* $H : K \to K$ *be a uniformly* L_h*–Lipscitzian asymptotically generalized* Φ*-hemicontractive mappings in the intermediate sense with sequence* $\{b_n\} \subset [1, \infty)$, *where* $b_n \to 1$ *as* $n \to \infty$. *Let* $h_n = \max\{a_n, b_n\}$. *Assume that* $\boldsymbol{F} = F(G) \bigcap F(H) \neq \emptyset$. *Let* $\{s_n\}$ *and* $\{q_n\}$ *be sequences in*

[0,1] such that $s_n + q_n \leq 1$ *for each* $n \geq 1$. *Put*
$\ell_n = \max\left\{0, \sup_{(\psi,p)\in K\times F(G)} (\langle G^n\psi - p, j(\psi - p)\rangle - a_n\|\psi - p\|^2 + \Phi(\|\psi - p\|))\right\}$,
$\Gamma_n = \max\left\{0, \sup_{(\psi,p)\in K\times F(H)} (\langle H^n\psi - p, j(\psi - p)\rangle - b_n\|\psi - p\|^2 + \Phi(\|\psi - p\|))\right\}$. *Let* $\tau_n = \max\{\ell_n, \Gamma_n\}$. *Assume that the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty}(s_n + q_n) = \infty$;

(ii) $\sum_{n=0}^{\infty}(s_n + q_n)^2 < \infty$;

(iii) $\sum_{n=0}^{\infty}(s_n + q_n)(h_n - 1) < \infty$, $\sum_{n=0}^{\infty}\tau_n(s_n + q_n) < \infty$;

(iv) $s_n q_n L^2 < 1$, $\forall n \geq 1$.

Let $\{\psi_n\}$ *be a sequence generated by*

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1 - s_n - q_n)\psi_{n-1} + s_n G^n\psi_{n-1} + q_n H^n\psi_n, \end{cases} \quad \forall n \geq 1. \tag{3.19}$$

Then the sequence $\{\psi_n\}$ *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Take $N = 1$ in Corollary 3.3. □

Corollary 3.9. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $H : K \to K$ *be a uniformly* L_h*–Lipscitzian asymptotically generalized* Φ*-hemicontractive mappings in the intermediate sense with sequence* $\{h_n\} \subset [1, \infty)$, *where* $h_n \to 1$ *as* $n \to \infty$. *Assume that* $\boldsymbol{F} = F(H) \neq \emptyset$. *Let* $\{q_n\}$ *be sequence a in [0,1]. Put*
$\tau_n = \max\left\{0, \sup_{(\psi,p)\in K\times F(H)} (\langle H^n\psi - p, j(\psi - p)\rangle - h_n\|\psi - p\|^2 + \Phi(\|\psi - p\|))\right\}$. *Assume that the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} q_n = \infty$;

(ii) $\sum_{n=0}^{\infty} q_n^2 < \infty$;

(iii) $\sum_{n=0}^{\infty} q_n(h_n - 1) < \infty$, $\sum_{n=0}^{\infty}\tau_n q_n < \infty$.

Let $\{\psi_n\}$ *be a sequence generated by*

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1 - q_n)\psi_{n-1} + q_n H^n\psi_n, \end{cases} \quad \forall n \geq 1, \tag{3.20}$$

Then the sequence $\{\psi_n\}$ *converges strongly to a point in* $\boldsymbol{F}$.

Proof. Take $N = 1$ in Corollary 3.4. □

Corollary 3.10. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $H : K \to K$ *be a uniformly* L_h*–Lipscitzian asymptotically generalized* Φ*-hemicontractive mappings in the intermediate sense with sequence* $\{h_n\} \subset [1, \infty)$, *where* $h_n \to 1$ *as* $n \to \infty$. *Assume that* $\boldsymbol{F} = F(G) \neq \emptyset$. *Let* $\{s_n\}$ *be sequence a in [0,1]. Put*
$\tau_n = \max\left\{0, \sup_{(\psi,p)\in K\times F(G)} (\langle G^n\psi - p, j(\psi - p)\rangle - h_n\|\psi - p\|^2 + \Phi(\|\psi - p\|))\right\}$. *Assume that the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} s_n = \infty$;

(ii) $\sum_{n=0}^{\infty} s_n^2 < \infty$;

(iii) $\sum_{n=0}^{\infty} s_n(h_n - 1) < \infty$, $\sum_{n=0}^{\infty} \tau_n s_n < \infty$.

Let $\{\psi_n\}$ be a sequence generated by

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1 - s_n)\psi_{n-1} + s_n G^n \psi_{n-1}, \end{cases} \quad \forall n \geq 1, \tag{3.21}$$

Then the sequence $\{\psi_n\}$ converges strongly to a point in $\boldsymbol{F}$.

Proof. Take $N = 1$ in Corollary 3.5. □

Corollary 3.11. *Let K be a nonempty closed convex subset of a real Banach space E. Let $G : K \to K$ be a uniformly L_g–Lipschitzian asymptotically generalized Φ-hemicontractive mappings in the intermediate sense with sequence $\{h_n\} \subset [1, \infty)$, where $h_n \to 1$ as $n \to \infty$. Assume that $\boldsymbol{F} = F(G) \neq \emptyset$. Let $\{s_n\}$ and $\{q_n\}$ be sequences in [0,1] such that $s_n + q_n \leq 1$ for each $n \geq 1$. Put*

$\tau_n = \max\left\{0, \sup_{(\psi,p)\in K\times F(G)} (\langle G^n\psi - p, j(\psi - p)\rangle - h_n\|\psi - p\|^2 + \Phi(\|\psi - p\|))\right\}$. *Assume that the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty}(s_n + q_n) = \infty$;

(ii) $\sum_{n=0}^{\infty}(s_n + q_n)^2 < \infty$;

(iii) $\sum_{n=0}^{\infty}(s_n + q_n)(h_n - 1) < \infty$, $\sum_{n=0}^{\infty} \tau_n(s_n + q_n) < \infty$;

(iv) $s_n q_n L^2 < 1$, $\forall n \geq 1$.

Let $\{\psi_n\}$ be a sequence generated by

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1 - s_n - q_n)\psi_{n-1} + s_n G^n \psi_{n-1} + q_n G^n \psi_n, \end{cases} \quad \forall n \geq 1. \tag{3.22}$$

Then the sequence $\{\psi_n\}$ converges strongly to a point in $\boldsymbol{F}$.

Proof. Take $N = 1$ in corollary 3.6. □

Remark 3.12. Under suitable conditions, the sequence $\{\psi_n\}$ defined by (1.16) can also be generalized to the iterative sequences with errors. Thus all the results proved in this paper can also be proved for the iterative process with errors. In this case our main iterative process (1.16) looks like

$$\begin{cases} \psi_0 \in K, \\ \psi_n = (1 - s_n - q_n - c_n)\psi_{n-1} + s_n G_{i(n)}^{k(n)} \zeta_n + q_n H_{i(n)}^{k(n)} \psi_n + c_n u_n, \\ \zeta_n = (1 - s'_n - q'_n - t'_n - c'_n)\psi_{n-1} + s'_n \psi_n + q'_n H_{i(n)}^{k(n)} \psi_{n-1} + t'_n G_{i(n)}^{k(n)} \psi_n + c'_n v_n, \end{cases} \tag{3.23}$$

$\forall n \geq 1$, where $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K, $\{s_n\}, \{q_n\}, \{c_n\}, \{s'_n\}, \{q'_n\}, \{t'_n\}$ and $\{c'_n\}$ are real sequences in [0,1] satisfying $s_n + q_n + c_n \leq 1$ and $s'_n + q'_n + t'_n + c'_n \leq 1$, $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

Remark 3.13. If we set $\tau_n = 0$ in Corollary 3.3, we capture completely the result of Khan et al. [26] which is a generalization and improvement of several results in the literature.

4 Conclusion

Since our iteration process properly includes the iteration processes (1.17)–(1.25) which has been studied by many authors (see for example Khan et al. [26]–[27], Chang [7], Ofoedu [37], Schu [48], Cho et al. [16], Zeng [63]-[64], Kim [31], Chidume [12]-[13], Gu [21]). Also, owing the fact the class of asymptotically generalized Φ–hemicontractive mappings in the intermediate sense properly includes the class of strongly pseudocontraction mappings, ϕ–strongly pseudocontractive mappings, asymptotically generalized Φ–pseudocontractive mappings and asymptotically generalized Φ–hemicontractive mappings which have been considered by several authors (see for example, [7, 10, 11, 21, 22, 30, 45, 57, 61]). Hence, our results improve, extend, generalize and unify the corresponding results in Khan et al. [26]–[27], Chang [7], Ofoedu [37], Schu [48], Cho et al. [16], Zeng [63]-[64], Kim [31], Chidume [12]-[13], Gu [21] and several others in the existing literature in these directions.

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