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M_n–Polynomials of Theta and Wagner Graphs

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Abstract Let *S* be a subset of vertex set of a connected graph *G,V(G)* and *v*be any vertex in *V(G)*which is not belong to *S*such that $|S| = n - 1, n \geq 3$. The max – n – distance between the vertex *v*and the set *S* is the maximum distance between *v* and *u* for all *u* belong to *S*. We find polynomial of two types of graphs are theta and Wagner graphs with respect to based max – n – distance. Also, we study some properties of them.

1 Introduction

The distance is one of the most important topics in graph theory, due to its wide applications in many branches, for example in chemical, computer, and engineering, so many researchers in mathematics and other branches were interested to finding many types of distances which depend on the ordinary distance, the maximum distance and the Steiner distance, (see [1-5]).Dankelmann, etc. defined the distance between two non-empty subsets of vertices in a connected graph G [6], as:
The minimum distance from U to W as:

$$d_{min}\,(U, W) = min\;\{\;d\,(u, w) : u \in U\;,\;w \in W\;\}\;,$$

the average distance from U to W as:

$$d_{avg}\,(U, W) = \frac{1}{|U|\,|W|} \sum_{u \in U\;,\;w \in W} d(u, w),$$

and the maximum distance from U to W as:

$$d_{max}\,(U, W) = max\;\{\;d\,(u, w) : u \in U\;,\;w \in W\;\}\;,$$

where U and W are not necessarily disjoint subsets of vertices of G and $d(u, w)$ is the shortest path between u and w. Since 2004 and until now, many researchers have worked to find polynomials and their indices to the minimum distance between two subsets of vertices, the first subset consist of one vertex and the second subset consist of $n - 1$ vertices, $n \geq 2$, (see [7-11]).

In this paper, we consider the all graphs G are simple,undirected,connected [12].The maximum distance between a singleton vertex$v$, $v \in V(G)$ and $(n-1)-$ subset S, $S \subseteq V(G)$ such that v is not belong to S, is called max – n – distance, is defined by:

$$d_{max}\,(v, S) = max\{d(v, u)\;: u \in\; S\}\;, |S| = n - 1, 2 \leq n \leq p\},$$

where p is the number of vertices(is called order) of a graph G.

When $n = 2$, the max – n – distance is become ordinary distance, [13]. Therefore we take$n \geq 3$. If the vertex v is dominating to all vertices S, then $d_{max}\,(v, S) = 1$, then $d_{max}\,(v, S) \geq 1$, for all vertex v in $V(G\;)$ and $S \subseteq V(G), v \notin S$.

We define the diameter with respect to this distance as:

$diam_{max}(G,n) = \delta_{max}(G,n) = max_{v\in V}\{d_{max}(v,S)\}$, this diameter equal the diameter with respect to ordinary distance , that is :

$$\delta_{max}(G,n) = max_{v\in V}\{d_{max}(v,S)\} = max_{v,u\in V}\{d(v,u)\} = \delta(G).$$

The M_n-polynomial of $G$of order p,is denoted by $M_n(G;x)$, and defined by :

$$M_n(G;x) = \sum_{k=m}^{\delta} C_n(G,k)x^k,$$

and the M_n- index of any graph G can be obtained from M_n-polynomial as follows:

$$M_n(G) = \frac{d}{dx}M_n(G;x)|_{x=1} = \sum_{k=m}^{\delta} kC_n(G,k),$$

where $m = min\{d_{max}(v,S), v \in V - S, \ S \subseteq V\}$ and$C_n(G,k)$ be the number of pairs (v,S),$S \subseteq V(G)$,$|S| = n-1$, $3 \leq n \leq p$,such that$d_{max}(v,S) = k$,for each$m \leq k \leq \delta$.

If $\lambda_k(v)$ is representation the number of pairs (v,S) that max – n – distance is k apart between v and S, then

$$C_n(G,k) = \sum_{k=m}^{\delta_{max}(G,v)} \lambda_k(v).$$

2 The M_n- Polynomials of Theta Graph

Let C_p be a cycle of order $p = 2r$,$r \geq 2$, such that $C_p : v_1, v_2, \ldots, v_{\frac{p}{2}}, v_{\frac{p}{2}+1}, \ldots, v_p, v_1$. Then the **Theta graph** obtained from C_p by adding the edge $v_1 v_{\frac{p}{2}+1}$ denoted by θ_p, see Fig.(2..1).

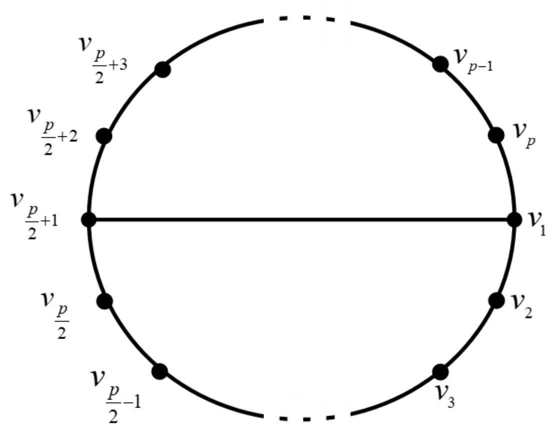


Figure 2.1. Theta graph $\theta_p, p = 2r, r \geq 2$

For finding theM_n-polynomial of the theta graph θ_p required to take two cases for theta graph depending on the order p.

Case One: If $p \equiv 0 \ (mod\ 4)$, then the theta graph $\theta_p, p = 4r, r \geq 2$ has the following properties:

- **Symmetric property:** There are vertices of the graph θ_p have the same max - n - distance are called the symmetric vertices (the symmetric is either horizontal or vertical or both), see Fig.(2.2). The symmetric vertices are:

$$v_1 \equiv v_{\frac{p}{2}+1},$$

$$v_2 \equiv v_{\frac{p}{2}} \equiv v_{\frac{p}{2}+2} \equiv v_p,$$

$$\vdots$$

$$v_{\frac{p}{4}} \equiv v_{\frac{p}{4}+2} \equiv v_{\frac{3p}{4}} \equiv v_{\frac{3p}{4}+2},$$

$$v_{\frac{p}{4}+1} \equiv v_{\frac{3p}{4}+1}.$$

- **Diameter property:** The graph θ_p has the max – n – diameter $p/2$**.**
- **The distance property :**The max – n – distance between the vertex v_i and any subset S of vertices of θ_p is less than and equal $\frac{p}{4} + i - 1$, for all $i = 1, 2, \ldots, \frac{p}{4} + 1$, i.e. $d_{max}(v_i, S) \leq \frac{p}{4} + i - 1, S \subseteq V(\theta_p), |S| = n - 1, n \geq 3$.

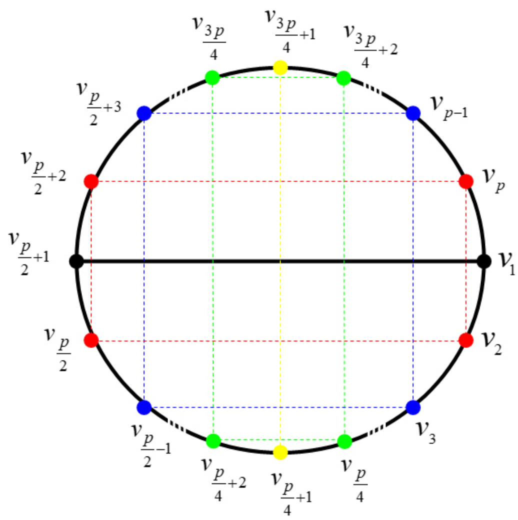


Figure 2.2. Theta graph $\theta_p, p = 4r, r \geq 2$

In following theorem, we obtained M_n-polynomial of theta graph θ_p, when $p \equiv 0(mod4)$.

Theorem 2.1. *Let θ_p be the theta graph of order p , $p = 4r$, $r \geq 3$, then*

$$M_n(\theta_p; x) = \sum_{k=1}^{p/2} C_n(\theta_p, k)\, x^k,$$

where

$$C_n(\theta_p, 1) = 2\binom{3}{n-1} + (p-2)\binom{2}{n-1},$$

$$C_n(\theta_p, 2) = 2\left[\binom{7}{n-1} - \binom{3}{n-1}\right] + 4\binom{5}{n-1} + (p-6)\binom{4}{n-1} - (p-2)\binom{2}{n-1},$$

$$C_n(\theta_p, k) = \begin{cases} 2\binom{4k-1}{n-1} + 4\binom{4k-3}{n-1} + 2\binom{4k-5}{n-1} + (p-4k+2)\binom{2k}{n-1} \\ -4\binom{2k-1}{n-1} - (p-4k+6)\binom{2k-2}{n-1}, \ when\ 3 \leq k \leq \frac{p}{4}, \\ 4\binom{p-1}{n-1} + 2\binom{2k}{n-1} - 4\binom{2k-1}{n-1} - 2\binom{2k-2}{n-1}, \ when \frac{p}{4} + 1 \leq k \leq \frac{p}{2} - 1, \end{cases}$$

$$C_n\left(\theta_p, \frac{p}{2}\right) = 2\binom{p-2}{n-2}.$$

Proof. From clearly that:

$$C_n(\theta_p,1)=2\binom{3}{n-1}+(p-2)\binom{2}{n-1} \tag{2.1}$$

Let $\lambda_k(v)$ be the number of pairs (v,S) which the max-n- distance between v and S at k such that $v\notin S$, $S\subseteq V(\theta_p)$, $|S|=n-1$, $n\geq 3$.
When $k=2$, then $d(v_1,u)=2$, when $u\in\{v_3\ ,\ v_{\frac{p}{2}}\ ,\ v_{\frac{p}{2}+2}\ ,\ v_{p-1}\}$and $d(v_1,u)=1$, when $u\in\{v_2\ ,\ v_{\frac{p}{2}+1}\ ,\ v_p\ \}$.
Then $\lambda_2(v_1)=\sum_{j=1}^{4}\binom{4}{j}\binom{3}{n-j-1}=\binom{7}{n-1}-\binom{3}{n-1}$.
And $d(v_2,u)=2$, when $u\in\{v_4\ ,\ v_{\frac{p}{2}+1}\ ,\ v_p\}$ and $d(v_2,u)=1$, when $u\in\{v_1\ ,\ v_3\ \}$.
Hence $\lambda_2(v_2)=\sum_{j=1}^{3}\binom{3}{j}\binom{2}{n-j-1}=\binom{5}{n-1}-\binom{2}{n-1}$.
And also, $d(v_i,u)=2$, for all $i=3,4,\ldots,\frac{p}{4}+1$ when $u\in\{v_{i+2}\ ,\ v_{i-2}\}$ and $d(v_i,u)=1$, when $u\in\{v_{i+1}\ ,\ v_{i-1}\ \}$.
Hence $\lambda_2(v_i)=\sum_{j=1}^{2}\binom{2}{j}\binom{2}{n-j-1}=\binom{4}{n-1}-\binom{2}{n-1}$, for all $i=3,4,\ldots,\frac{p}{4}+1$.
Thus from previously and symmetric property, we have:

$$C_n(\theta_p,2)=2\binom{7}{n-1}-2\binom{3}{n-1}+4\binom{5}{n-1}-4\binom{2}{n-1}$$

$$+(p-8)\left[\binom{4}{n-1}-\binom{2}{n-1}\right]+2\binom{4}{n-1}-2\binom{2}{n-1}$$

$$=2\left[\binom{7}{n-1}-\binom{3}{n-1}\right]+4\binom{5}{n-1}+(p-6)\binom{4}{n-1}-(p-2)\binom{2}{n-1} \tag{2.2}$$

Now, to find $C_n(\theta_p,k)$, $3\leq k\leq\frac{p}{4}$, then there are three cases:
<u>Case I :</u> If $i<k$, then there are four vertices $\{v_{i+k}\ ,\ v_{\frac{p}{2}+1-k+i}\ ,\ v_{\frac{p}{2}+1-i+k}\ ,\ v_{p-k+i}\}$ lying at a distance of k from v_i and $(4k-2i-3)$ vertices $\{v_{i+1}\ ,\ v_{i+2}\ ,\ \ldots\ ,\ v_{i+k-1}\ ;\ v_p\ ,\ v_{p-1}\ ,\ \ldots\ ,\ v_{p-k+i+1}\ ;v_{\frac{p}{2}-k+i+2}\ ,\ \ldots,v_{\frac{p}{2}},\ v_{\frac{p}{2}+1},\ \ldots\ ,\ v_{\frac{p}{2}-i+k}\}\cup S_1$ lying at a distance less than k to v_i, $i=1,\ 2\ ,\ \ldots\ ,\ k-1$, where $S_1=\emptyset$, when $i=1$, otherwise $S_1=\{v_{i-1},\ v_{i-2},\ldots,v_1\}$.
Then: $\lambda_k(v_i)=\sum_{j=1}^{4}\binom{4}{j}\binom{4k-2i-3}{n-j-1}=\binom{4k-2i+1}{n-1}-\binom{4k-2i-3}{n-1}$, for $i=1,\ 2\ ,\ \ldots\ ,\ k-1$.
<u>Case II :</u> If $i=k$, then there are three vertices $\{v_{2k}\ ,\ v_{\frac{p}{2}+1}\ ,\ v_p\}$ lying at a distance of k from v_k and $(2k-2)$ vertices $\{\ v_1,\ldots,\ v_{k-1}\ ;\ v_{k+1}\ ,\ \ldots,\ \ v_{2k-1}\}$ lying at a distance less than k to v_k.
Then:

$$\lambda_k(v_k)=\sum_{j=1}^{3}\binom{3}{j}\binom{2k-2}{n-j-1}=\binom{2k+1}{n-1}-\binom{2k-2}{n-1}.$$

<u>Case III :</u> If $i>k$, then there are only two vertices $\{v_{i+k}\ ,\ \ v_{i-k}\}$ lying at a distance of k from v_i and there are $(2k-2)$ vertices $\{\ v_{i-k+1},\ldots,v_{i-1}\ ;\ v_{i+1}\ ,\ \ v_{i+2}\ ,\ \ldots\ ,\ \ v_{i+k-1}\}$ lying at a distance less than k to v_i , $i=k+1\ ,\ \ k+\ 2\ ,\ \ldots\ ,\ \ \frac{p}{4}+1$.
Then:
$\lambda_k(v_i)=\sum_{j=1}^{2}\binom{2}{j}\binom{2k-2}{n-j-1}=\binom{2k}{n-1}-\binom{2k-2}{n-1}$, for $i=k+1\ ,\ \ k+\ 2\ ,\ \ldots\ ,\ \ \frac{p}{4}+1$.
From three cases and symmetric property, then for all $3\leq k\leq p/4$, we have:

$$C_n(\theta_p,k)=2\left[\binom{4k-1}{n-1}-\binom{4k-5}{n-1}\right]+4\sum_{i=2}^{k-1}\left[\binom{4k-2i+1}{n-1}-\binom{4k-2i-3}{n-1}\right]$$

$$+4\left[\binom{2k+1}{n-1}-\binom{2k-2}{n-1}\right]+(p-4k+2)\left[\binom{2k}{n-1}-\binom{2k-2}{n-1}\right]$$

$$=2\binom{4k-1}{n-1}+4\binom{4k-3}{n-1}+2\binom{4k-5}{n-1}+(p-4k+2)\binom{2k}{n-1}$$

$$-(p-4k+6)\binom{2k-2}{n-1}-4\binom{2k-1}{n-1}. \tag{2.3}$$

Now , to find $C_n(\theta_p,k)$ for all $\frac{p}{4}+1\le k\le\frac{p}{2}-1$. Then also, there are three cases for all $1\le i\le\frac{p}{4}+1$.

Cases I : If $i<k-\frac{p}{4}+1$,then we have from distance property

$$\lambda_k(v_i)=0.$$

Case II : If $k-\frac{p}{4}+1\le i\le\frac{p}{4}$, then there are two vertices$\{v_{\frac{p}{2}+k-i+1}\ ,\ v_{p-k+i}\}$ lying at a distance of k from v_i and $(2k+\frac{p}{2}-2i-1)$ vertices $\{v_1,\ v_2\ ,\dots\ ,v_{\frac{p}{2}+k-i}\ ;\ v_p,\ v_{p-1}\ ,\dots\ ,v_{p-k+i+1}\}$ lying at a distance less than k to v_i , $i=k-\frac{p}{4}+1\ ,\ \dots\ ,\ \frac{p}{4}$. Hence

$$\lambda_k(v_i)=\sum_{j=1}^{2}\binom{2}{j}\binom{2k+\frac{p}{2}-2i-1}{n-j-1}=\binom{2k+\frac{p}{2}-2i+1}{n-1}-\binom{2k+\frac{p}{2}-2i-1}{n-1}.$$

Case III : If $i=\frac{p}{4}+1$, then there are two vertices$\{v_{i+k}\ ,\ v_{p+i-k}\}$ lying at a distance of k from $v_{\frac{p}{4}+1}$ and $(2k-2)$ vertices $\{v_1,\ v_2\ ,\dots\ ,v_{i-1}\ ;\ v_{i+1},\ v_{i+2}\ ,\dots\ ,v_{i+k-1}\}\cup S_2$ lying at a distance less than k to $v_{\frac{p}{4}+1}$,where $S_2=\emptyset$,when $k=\frac{p}{4}+1$, otherwise $S_2=\{v_p\ ,\ v_{p-1},\dots,v_{p+i-k+1}\}$. Hence

$$\lambda_k\left(v_{\frac{p}{4}+1}\right)=\sum_{j=1}^{2}\binom{2}{j}\binom{2k-2}{n-j-1}=\binom{2k}{n-1}-\binom{2k-2}{n-1}.$$

From previously and symmetry property , then for all $\frac{p}{4}+1\le k\le\frac{p}{2}-1$, we have

$$C_n(\theta_p,k)=4\sum_{i=k-\left(\frac{p}{4}-1\right)}^{\frac{p}{4}}\left[\binom{2k+\frac{p}{2}-2i+1}{n-1}-\binom{2k+\frac{p}{2}-2i-1}{n-1}\right]+2\left[\binom{2k}{n-1}-\binom{2k-2}{n-1}\right]$$

$$=4\binom{p-1}{n-1}+2\binom{2k}{n-1}-4\binom{2k-1}{n-1}-2\binom{2k-2}{n-1} \tag{2.4}$$

Finally, when $k=\frac{p}{2}$, then $\lambda_{\frac{p}{2}}(v_i)=0$, for all $1\le i\le\frac{p}{4}$, and there is only one vertex $\{v_{\frac{3p}{4}+1}\}$ lying at a distance $\frac{p}{2}$ from $v_{\frac{p}{4}+1}$ and there are $(p-2)$ vertices $V(\theta_p)-\{v_{\frac{p}{4}+1},v_{\frac{3p}{4}+1}\}$ lying at a distance less than $\frac{p}{2}$. Hence

$$\lambda_{\frac{p}{2}}\left(v_{\frac{p}{4}+1}\right)=\binom{p-2}{n-2}.$$

From this and symmetric property, we have

$$C_n\left(\theta_p,\frac{p}{2}\right)=2\binom{p-2}{n-2} \tag{2.5}$$

From (2.1)-(2.5), we have $C_n(\theta_p,k)$, for all $1\le k\le\frac{p}{2}$.

□

Remark 2.2. :

- $M_n(\theta_4\,;x)=2\left[\binom{3}{n-1}+\binom{2}{n-1}\right]x+2\binom{2}{n-2}x^2.$
- $M_n(\theta_8\,;x)=\left[2\binom{3}{n-1}+6\binom{2}{n-1}\right]x+\left[2\binom{7}{n-1}+4\binom{5}{n-1}+2\binom{4}{n-1}-2\binom{3}{n-1}-6\binom{2}{n-1}\right]x^2$

$$+\left[4\binom{7}{n-1}+2\binom{6}{n-1}-4\binom{5}{n-1}-2\binom{4}{n-1}\right]x^3+2\binom{6}{n-2}x^4.$$

Corollary 2.3. *The max – n –Wiener index of theta graph θ_p of order $p = 4r$, $r \geq 3$ is*

$$M_n(\theta_p) = p\binom{p-2}{n-2} + 4\binom{7}{n-1} + 8\binom{5}{n-1} + 2(p-6)\binom{4}{n-1} - p\binom{2}{n-1}$$

$$-2\binom{2}{n-2} + \sum_{k=3}^{p/2} kC_n(\theta_p, k),$$

where $C_n(\theta_p, k)$ as mentioned in Theorem 2.1.

□

Case two: If$p \not\equiv 0 \ (mod\ 4)$, then the theta graph θ_p , $p = 4r + 2$, $r \in \mathbb{N}$ has the following properties:

- **Symmetric property :** There are vertices of the graph θ_p have the same max – n – distance are called the symmetric vertices (the symmetric are either horizontal or horizontal and vertical), see Fig.2..3. The symmetric vertices are:

$$v_1 \equiv v_{\frac{p}{2}+1},$$

$$v_2 \equiv v_{\frac{p}{2}} \equiv v_{\frac{p}{2}+2} \equiv v_p,$$

$$\vdots$$

$$v_{\lfloor \frac{p}{4} \rfloor+1} \equiv v_{\lfloor \frac{p}{4} \rfloor+2} \equiv v_{\lfloor \frac{3p}{4} \rfloor+1} \equiv v_{\lfloor \frac{3p}{4} \rfloor+2}.$$

- **Diameter property:** The graph θ_p has the max – n– diameter $\frac{p}{2}$ **.**
- **The distance property :**The max- n - distance between the vertex v_i and any subset S of vertices of θ_p is less than and equal $\lfloor \frac{p}{4} \rfloor + i$, for all $i = 1, 2, \ldots, \lfloor \frac{p}{4} \rfloor + 1$, i.e. $d_{max}(v_i, S) \leq \lfloor \frac{p}{4} \rfloor + i$ and $S \subseteq V(\theta_p)$, $|S| = n - 1$,$n \geq 3$.

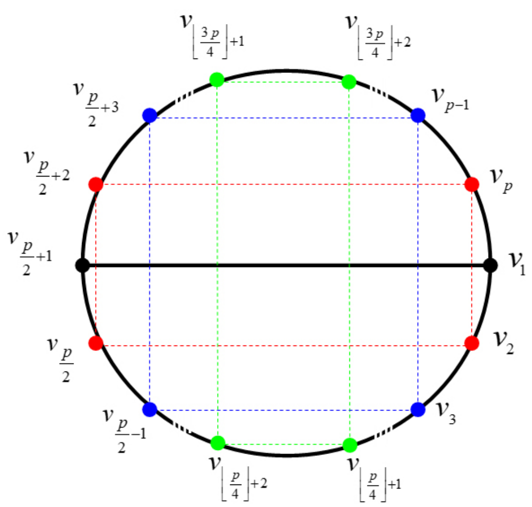


Figure 2.3. Theta graph $\theta_p, p = 4r + 2, r \in N$

In following theorem, we obtained M_n-polynomial of theta graph θ_p , when $p \not\equiv 0 \ (mod\ 4)$

Theorem 2.4. *Let θ_p be the theta graph of order p, $p = 4r + 2$, $r \geq 3$, then*

$$M_n(\theta_p; x) = \sum_{k=1}^{\frac{p}{2}} C_n(\theta_p, k)\, x^k,$$

where

$$C_n(\theta_p, 1) = 2\binom{3}{n-1} + (p-2)\binom{2}{n-1},$$

$$C_n(\theta_p, 2) = 2\binom{7}{n-1} + 4\binom{4}{n-2} - 2\binom{3}{n-1} + 4\left\lfloor\frac{p}{4}\right\rfloor\left[\binom{4}{n-1} - \binom{2}{n-1}\right],$$

$$C_n(\theta_p, k) = \begin{cases} 2\binom{4k-1}{n-1} + 4\binom{4k-3}{n-1} + 2\binom{4k-5}{n-1} \\ +4\left(\left\lfloor\frac{p}{4}\right\rfloor - k + 1\right)\left[\binom{2k}{n-1} - \binom{2k-2}{n-1}\right] \\ -4\left[\binom{2k-1}{n-1} + \binom{2k-2}{n-1}\right], & when\ 3 \le k \le \left\lfloor\frac{p}{4}\right\rfloor \\ \\ 2\left[\binom{4k-3}{n-1} - \binom{4k-5}{n-1}\right] + 4\sum_{i=3}^{k-1}\binom{4k-2i-1}{n-2} \\ +4\left[\binom{4k-4}{n-1} + \binom{2k-1}{n-2} - \binom{2k-2}{n-1}\right], & when k = \left\lfloor\frac{p}{4}\right\rfloor + 1 \\ \\ 4\binom{p-1}{n-1} - 4\binom{2k-2}{n-1}, & when \left\lfloor\frac{p}{4}\right\rfloor + 2 \le k \le \frac{p}{2} - 1 \end{cases}$$

$$C_n\left(\theta_p, \frac{p}{2}\right) = 4\binom{p-2}{n-2}.$$

Proof. From clearly that:

$$C_n(\theta_p, 1) = 2\binom{3}{n-1} + (p-2)\binom{2}{n-1} \tag{2.6}$$

Let $\lambda_k(v)$ be the number of pairs $(v,\ S)$ which the max-n- distance between v and S at k such that $v \notin S$, $S \subseteq V(\theta_p)$, $|S| = n-1$, $n \ge 3$.
When $k = 2$, then $d(v_1, u) = 2$ when $u \in \{v_3\ ,\ v_{\frac{p}{2}}\ ,\ v_{\frac{p}{2}+2}\ ,\ v_{p-1}\}$ and $d(v_1, u) = 1$ when $u \in \{v_2\ ,\ v_{\frac{p}{2}+1}\ ,\ v_p\ \}$.
Hence $\lambda_2(v_1) = \sum_{j=1}^{4}\binom{4}{j}\binom{3}{n-j-1} = \binom{7}{n-1} - \binom{3}{n-1}$.
And $d(v_2, u) = 2$ when $u \in \{v_4\ ,\ v_{\frac{p}{2}+1}\ ,\ v_p\}$ and $d(v_2, u) = 1$ when $u \in \{v_1\ ,\ v_3\ \}$.
Then $\lambda_2(v_2) = \sum_{j=1}^{3}\binom{3}{j}\binom{2}{n-j-1} = \binom{5}{n-1} - \binom{2}{n-1}$.
And $d(v_i, u) = 2$ when $u \in \{v_{i+2}\ ,\ v_{i-2}\}$ and $d(v_i, u) = 1$ when $u \in \{v_{i+1}\ ,\ v_{i-1}\ \}$.
Hence $\lambda_2(v_i) = \sum_{j=1}^{2}\binom{2}{j}\binom{2}{n-j-1} = \binom{4}{n-1} - \binom{2}{n-1}$, for all $i = 3,\ 4\ ,\ \ldots\ ,\ \left\lfloor\frac{p}{4}\right\rfloor + 1$.
Thus from previously and symmetric property , we have

$$C_n(\theta_p, 2) = 2\binom{7}{n-1} + 4\binom{5}{n-1} + 4\left(\left\lfloor\frac{p}{4}\right\rfloor - 1\right)\binom{4}{n-1} - 2\binom{3}{n-1} - 4\left\lfloor\frac{p}{4}\right\rfloor\binom{2}{n-1}$$

$$= 2\binom{7}{n-1} + 4\binom{4}{n-2} - 2\binom{3}{n-1} + 4\left\lfloor\frac{p}{4}\right\rfloor\left[\binom{4}{n-1} - \binom{2}{n-1}\right] \tag{2.7}$$

Now , to find $C_n(\theta_p, k)$, $3 \le k \le \left\lfloor\frac{p}{4}\right\rfloor$, then there are three cases for all $1 \le i \le \left\lfloor\frac{p}{4}\right\rfloor + 1$, we have
Case I : If $i < k$, then we have $d(v_i, u) = k$ when $u \in \{v_{i+k}\ ,\ v_{\frac{p}{2}-k+i+1}\ ,\ v_{\frac{p}{2}-i+k+1}\ ,\ v_{p-k+i}\}$ and $d(v_i, u) < k$ when $u \in \{v_{i+1}\ ,\ v_{i+2}\ ,\ \ldots\ ,\ v_{i+k-1}\ ;\ v_{\frac{p}{2}-k+i+2}, v_{\frac{p}{2}-k+i+3}, \ldots, v_{\frac{p}{2}+1}, \ldots, v_{\frac{p}{2}-i+k}\ ;\ v_p\ , v_{p-1}\ , \ldots, v_{p-k+i+1}\ \} \cup S_1$, where $S_1 = \emptyset$ when $i = 1$ otherwise $S_1 = \{v_{i-1}, v_{i-2}\ ,\ \ldots\ , v_1\ \}$.
Hence

$$\lambda_k(v_i) = \sum_{j=1}^{4}\binom{4}{j}\binom{4k-2i-3}{n-j-1} = \binom{4k-2i+1}{n-1} - \binom{4k-2i-3}{n-1}; i = 1, 2,\ \ldots\ ,\ k-1.$$

Case II : If $i = k$, then $d(v_i, u) = k$ when $u \in \{v_{i+k} \ , \ v_{\frac{p}{2}+1} \ , \ \ v_p\}$ and $d(v_i, u) < k$ when $u \in \{v_{i+1} \ , \ \ v_{i+2} \ , \ \ldots \ , \ \ v_{i+k-1} \ ; \ v_{i-1} \ , \ v_{i-2} \ , \ \ ... \ \ , v_1\}$. Hence

$$\lambda_k(v_i) = \sum_{j=1}^{3} \binom{3}{j} \binom{2k-2}{n-j-1} = \binom{2k+1}{n-1} - \binom{2k-2}{n-1}, i = k.$$

Case III : If $i > k$, then we have $d(v_i, u) = k$ when $u \in \{v_{i+k} \ , \ v_{i-k}\}$ and $d(v_i, u) < k$ when $u \in \{v_{i+1} \ , \ \ v_{i+2} \ , \ \ldots \ , \ \ v_{i+k-1} \ ; \ v_{i-1} \ , \ v_{i-2} \ , \ \ ... \ \ , v_{i-k+1} \ \}$. Hence

$$\lambda_k(v_i) = \sum_{j=1}^{2} \binom{2}{j} \binom{2k-2}{n-j-1} = \binom{2k}{n-1} - \binom{2k-2}{n-1}, i = k+1 \ , \ k+2 \ , \ \ldots \ , \ \left\lfloor \frac{p}{4} \right\rfloor + 1 \ .$$

From three cases and symmetric property, we have:

$$C_n(\theta_p, k) = 2\left[\binom{4k-1}{n-1} - \binom{4k-5}{n-1}\right] + 4\sum_{i=2}^{k-1}\left[\binom{4k-2i+1}{n-1} - \binom{4k-2i-3}{n-1}\right]$$

$$+4\left[\binom{2k+1}{n-1} - \binom{2k-2}{n-1}\right] + 4\left(\left\lfloor \frac{p}{4} \right\rfloor - k + 1\right)\left[\binom{2k}{n-1} - \binom{2k-2}{n-1}\right]$$

$$= 2\binom{4k-1}{n-1} + 4\binom{4k-3}{n-1} + 2\binom{4k-5}{n-1} + 4\left(\left\lfloor \frac{p}{4} \right\rfloor - k + 1\right)\left[\binom{2k}{n-1} - \binom{2k-2}{n-1}\right]$$

$$-4\left[\binom{2k-1}{n-1} + \binom{2k-2}{n-1}\right] \tag{2.8}$$

Now , to find $C_n(\theta_p, k)$, for $k = \left\lfloor \frac{p}{4} \right\rfloor + 1$, we note that three cases:
Case I : If $i = 1$, then $d(v_1, u) = k$when $u \in \{v_{\lfloor \frac{p}{4} \rfloor + 2} \ , \ v_{\lfloor \frac{3p}{4} \rfloor + 1}\}$ and $d(v_1, u) < k$
When $u \in V(\theta_p) - \{v_1 \ , \ \ v_{\lfloor \frac{p}{4} \rfloor + 2} \ , \ v_{\lfloor \frac{3p}{4} \rfloor + 1}\}$. Hence

$$\lambda_k(v_1) = \sum_{j=1}^{2} \binom{2}{j} \binom{4k-5}{n-j-1} = \binom{4k-3}{n-1} - \binom{4k-5}{n-1}.$$

Case II : If $2 \leq i \leq \left\lfloor \frac{p}{4} \right\rfloor$, then we have $d(v_i, u) = k$when $u \in \{v_{i+k} \ , \ v_{3k-i} \ , \ \ v_{p-k+i}\}$ and $d(v_i, u) < k$ when $u \in \{ \ v_1 \ , v_2 \ , \ \ldots , v_{i-1} \ ; \ v_{i+1} \ , \ v_{i+2} \ , \ \ldots \ , \ \ v_{i+k-1} \ ; v_{i+k+1}, \ v_{i+k+2} \ , \ \ldots , v_{3k-i-1} \ ;$ $v_p, \ v_{p-1} \ , \ \ldots , v_{p-k+i+1}\}$. Hence

$$\lambda_{\lfloor \frac{p}{4} \rfloor + 1}(v_i) = \sum_{j=1}^{3} \binom{3}{j} \binom{4k-2i-3}{n-j-1} = \binom{4k-2i}{n-1} - \binom{4k-2i-3}{n-1}; 2 \leq i \leq \left\lfloor \frac{p}{4} \right\rfloor.$$

Case III : If $i = \left\lfloor \frac{p}{4} \right\rfloor + 1$, then we have $d(v_i, u) = k$ when $u \in \{v_{i+k} \ , \ \ v_p\}$ and $d(v_i, u) < k$ when $u \in \{v_{i+1} \ , \ \ v_{i+2} \ , \ \ldots \ , \ \ v_{i+k-1} \ ; \ v_{i-1} \ , \ v_{i-2} \ , \ \ ... \ , v_1 \ \}$. Hence

$$\lambda_k(v_i) = \sum_{j=1}^{2} \binom{2}{j} \binom{2k-2}{n-j-1} = \binom{2k}{n-1} - \binom{2k-2}{n-1}, i = \left\lfloor \frac{p}{4} \right\rfloor + 1.$$

From previously and symmetric property, we have

$$C_n(\theta_p, k) = 2\left[\binom{4k-3}{n-1} - \binom{4k-5}{n-1}\right] + 4\sum_{j=2}^{\lfloor \frac{p}{4} \rfloor}\left[\binom{4k-2i}{n-1} - \binom{4k-2i-3}{n-1}\right]$$

$$+4\left[\binom{2k}{n-1} - \binom{2k-2}{n-1}\right]$$

$$= 2\left[\binom{4k-3}{n-1} - \binom{4k-5}{n-1}\right] + 4\sum_{i=3}^{\lfloor \frac{p}{4} \rfloor}\binom{4k-2i-1}{n-2} + 4\left[\binom{4k-4}{n-1} + \binom{2k-1}{n-2} - \binom{2k-2}{n-1}\right], k = \left\lfloor \frac{p}{4} \right\rfloor + 1 \tag{2.9}$$

To find $C_n(\theta_p, k)$, $\lfloor \frac{p}{4} \rfloor + 2 \le k \le \frac{p}{2} - 1$, then there are three cases for all $1 \le i \le \lfloor \frac{p}{4} \rfloor + 1$.

Case I : If $i < k - \lfloor \frac{p}{4} \rfloor$, then we have from distance property

$\lambda_k(v_i) = 0$, for all $i = 1, 2, \ldots, k - \lfloor \frac{p}{4} \rfloor - 1$.

Case II : If $i = k - \lfloor \frac{p}{4} \rfloor$, then we have $d(v_i, u) = k$ when $u = v_{\lfloor \frac{3p}{4} \rfloor + 1}$ and $d(v_i, u) < k$ when $u \in V(\theta_p) - \{v_i, v_{\lfloor \frac{3p}{4} \rfloor + 1}\}$. Hence

$$\lambda_k(v_i) = \binom{p-2}{n-2}, i = k - \left\lfloor \frac{p}{4} \right\rfloor.$$

Case III : If $k - \lfloor \frac{p}{4} \rfloor + 1 \le i \le \lfloor \frac{p}{4} \rfloor + 1$, then we have $d(v_i, u) = k$ when $u \in \{v_{\frac{p}{2}+k-i+1}, v_{p+i-k}\}$ and $d(v_i, u) < k$ when $u \in \{v_1, \ldots, v_{i-1}; v_{i+1}, \ldots, v_{\frac{p}{2}+k-i}; v_p, v_{p-1}, \ldots, v_{p+i-k+1}\}$.
Hence

$$\lambda_k(v_i) = \sum_{j=1}^{2}\binom{2}{j}\binom{2k+\frac{p}{2}-2i-1}{n-j-1} = \binom{2k+\frac{p}{2}-2i+1}{n-1} - \binom{2k+\frac{p}{2}-2i-1}{n-1},$$

$$k - \left\lfloor \frac{p}{4} \right\rfloor + 1 \le i \le \left\lfloor \frac{p}{4} \right\rfloor + 1.$$

From three cases and symmetric property, we have

$$C_n(\theta_p, k) = 4\binom{p-2}{n-2} + 4\sum_{i=k-\lfloor \frac{p}{4} \rfloor + 1}^{\lfloor \frac{p}{4} \rfloor + 1}\left[\binom{2k+\frac{p}{2}-2i+1}{n-1} - \binom{2k+\frac{p}{2}-2i-1}{n-1}\right]$$

$$= 4\binom{p-1}{n-1} - 4\binom{2k-2}{n-1}, for\ all \left\lfloor \frac{p}{4} \right\rfloor + 2 \le k \le \frac{p}{2} - 1 \tag{2.10}$$

Finally, when $k = \frac{p}{2}$, then we have from distance property

$\lambda_{\frac{p}{2}}(v_i) = 0$, for all $i = 1, 2, \ldots, \lfloor \frac{p}{4} \rfloor$, and for $i = \lfloor \frac{p}{4} \rfloor + 1$, we have $d(v_{\lfloor \frac{p}{4} \rfloor + 1}, u) = \frac{p}{2}$ when $u = v_{\lfloor \frac{3p}{4} \rfloor + 1}$ and $d(v_{\lfloor \frac{p}{4} \rfloor + 1}, u) < \frac{p}{2}$ when $u \in V(\theta_p) - \{v_{\lfloor \frac{p}{4} \rfloor + 1}, v_{\lfloor \frac{3p}{4} \rfloor + 1}\}$. Hence

$$\lambda_{\frac{p}{2}}\left(v_{\lfloor \frac{p}{4} \rfloor + 1}\right) = \binom{p-2}{n-2} \tag{2.11}$$

Thus from this and symmetric property, we have

$C_n(\theta_p, \frac{p}{2}) = 4\binom{p-2}{n-2}$.

From (2.6) - (2.11), we have $C_n(\theta_p, k)$ for all $1 \le k \le \frac{p}{2}$. □

Remark 2.5. :

- $M_n(\theta_6; x) = \left[2\binom{3}{n-1} + 4\binom{2}{n-1}\right]x + \left[2\binom{5}{n-1} + 4\binom{4}{n-1} - 2\binom{3}{n-1} - 4\binom{2}{n-1}\right]x^2 + 4\binom{4}{n-2}x^3$.
- $M_n(\theta_{10}; x) = \left[2\binom{3}{n-1} + 8\binom{2}{n-1}\right]x + \left[2\binom{7}{n-1} + 4\binom{5}{n-1} + 4\binom{4}{n-1} - 2\binom{3}{n-1} - 8\binom{2}{n-1}\right]x^2$

$$+\left[2\binom{9}{n-1} + 4\binom{8}{n-1} - 2\binom{7}{n-1} + 4\binom{6}{n-1} - 4\binom{5}{n-1} - 4\binom{4}{n-1}\right]x^3$$

$$+\left[4\binom{8}{n-2} + 4\binom{8}{n-1} - 4\binom{6}{n-1}\right]x^4 + 4\binom{8}{n-2}x^5.$$

Corollary 2.6. *The max – n –Wiener index of theta graph θ_p of order , $p = 4r + 2$, $r \geq 3$is :*

$$M_n(\theta_p) = 4\binom{7}{n-1} + 8\binom{5}{n-1} + 8\left(\left\lfloor \frac{p}{4} \right\rfloor + 1\right)\binom{4}{n-1} - 2\binom{3}{n-1} + \left(p - 8\left\lfloor \frac{p}{4} \right\rfloor - 2\right)\binom{2}{n-1}$$

$+2p\binom{p-2}{n-2} + \sum_{k=3}^{p/2} kC_n(\theta_p, k)$ *, where*$C_n(\theta_p, k)$ *as mentioned in Theorem 2.4.*

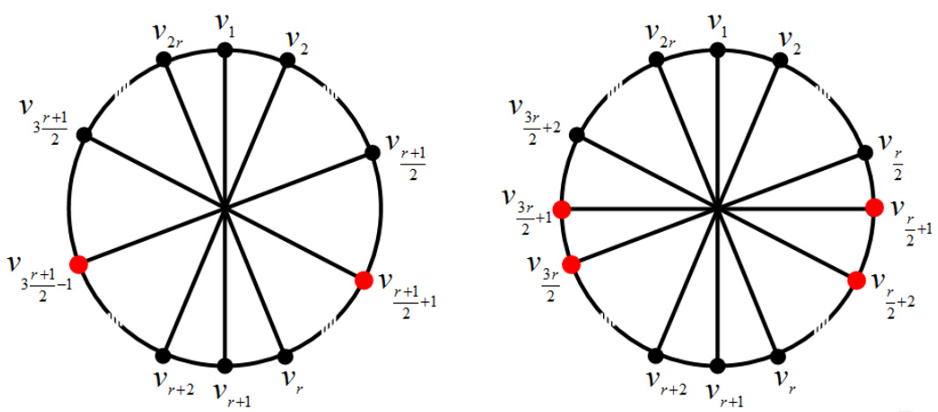
3 The M_n-Polynomial of Wagner Graph G_{2r}

Let C_{2r} be a cycle of order $2r$, $r \geq 2$ such that $C_{2r}\ : v_1, v_2, \ldots, v_{2r}, v_1$. Then the **Wagner graph** $G_{2r}$is the graph obtained from C_{2r} by adding r edges $v_i\, v_{i+r}$, $i = 1, 2, \ldots, r$.

Some Properties of G_{2r} :

(i) **The same M_n- polynomial of vertex property:** The graph G_{2r} is a cubic graph and all vertex of Wagner graph has the same M_n- polynomial of vertex, i.e. $M_n(v, G_{2r}; x) = M_n(u, G_{2r}; x)$ for all $u, v \in V(G_{2r})$.

(ii) **The diameter property:** The graph G_{2r} has the max– n –diameter is $\lfloor \frac{r+1}{2} \rfloor$and the vertices that lying at a distance of max –n –diameter from v_1 are $v_{\frac{r}{2}+1}$, $v_{\frac{r}{2}+2}$, $v_{\frac{3r}{2}}$, $v_{\frac{3r}{2}+1}$ when r is an even and $v_{\frac{r+1}{2}+1}$, $v_{3\frac{r+1}{2}-1}$when r is an odd.See Fig. 3.1.



r is an odd r is an even

Figure 3.1. Wagner Graph

Theorem 3.1. *For all* $3 \leq n \leq 2r$, $r \geq 5$ *, we have:*

$$M_n(G_{2r}; x) = 2r\binom{3}{n-1}x + 2r\sum_{k=2}^{\delta-1}\left[\binom{4k-1}{n-1} - \binom{4k-5}{n-1}\right]x^k + 2r\left[\binom{\alpha}{n-1} - \binom{4\delta-5}{n-1}\right]x^\delta,$$

where$\alpha = 4\delta - 1$ *, if r is an even number and* $\alpha = 4\delta - 3$ *, if r is an odd number.*

Proof. : From clearly that

$$C_n(G_{2r}, 1) = 2r\binom{3}{n-1}$$

To proof $C_n(G_{2r}, k)$ for all $2 \leq k \leq \delta - 1$,we will find the coefficients $C_n(v_1, G_{2r}, k)$ and using the same M_n- polynomial of vertex property to get the required result.
Let $S \subseteq V(G_{2r})$ be a subset of $(n-1)$ elements,3 $\leq n \leq 2r$ and let $\lambda_k(v_1)$ be the number of subset Ssuch that the max-n-distance between v_1 and S equal to k.

Now, we note that for all $2 \le k \le \delta - 1$ there are four vertices lying at a distance k from v_1 and there are $(4k-5)$ vertices lying at a distance less than $k$atv_1, i.e.

$d(v_1, u) = k$, $u \in \{v_{k+1}\ ,\ v_{r-k+2}\ ,\ v_{r+k}\ ,\ v_{2r-k+1}\}$ and

$d(v_1, u) < k$, $u \in \{v_2,\ v_3,\ \ldots\ ,\ v_k\ ;\ v_{r-k+3},\ v_{r-k+4}\ ,\ \ldots\ ,\ v_{r+k-1}\ ;\ v_{2r},\ v_{2r-1}\ ,\ \cdots\ ,v_{2r-k+2}\}$.

Hence $C_n(v_1, G_{2r}\ , k) = \lambda_k(v_1) = \sum_{j=1}^{4}\binom{4}{j}\binom{4k-5}{n-j-1}$;

$$= \binom{4k-1}{n-1} - \binom{4k-5}{n-1};\ 2 \le k \le \delta - 1.$$

And by using the same M_n- polynomial of vertex property, we get

$$C_n(G_{2r}\ , k) = 2r\left[\binom{4k-1}{n-1} - \binom{4k-5}{n-1}\right]; 2 \le k \le \delta - 1.$$

Now, to find $C_n(G_{2r}\ , \delta)$, we note that there are $(4\delta - 5)$ vertices lying at a distance less than $\delta$from v_1 and using the diameter property , we have:

$$C_n(v_1, G_{2r}\ , \delta) = \begin{cases} \sum_{j=1}^{4}\binom{4}{j}\binom{4\delta-5}{n-j-1} & , if\ r\ is\ an\ even\ number, \\ \sum_{j=1}^{2}\binom{2}{j}\binom{4\delta-5}{n-j-1} & , if\ r\ is\ an\ odd\ number, \end{cases}$$

$$= \begin{cases} \binom{4\delta-1}{n-1} - \binom{4\delta-5}{n-1} & , if\ r\ is\ an\ even\ number, \\ \binom{4\delta-3}{n-1} - \binom{4\delta-5}{n-1} & , if\ r\ is\ an\ odd\ number. \end{cases}$$

And by using the same M_n- polynomial of vertex property, we get the required result. □

Corollary 3.2. *The max– n – Wiener index of G_{2r} is :*

$$M_n(G_{2r}) = 2r\binom{3}{n-1} + 2r\sum_{k=2}^{\delta-1} k\left[\binom{4k-1}{n-1} - \binom{4k-5}{n-1}\right] + 2r\delta\left[\binom{\alpha}{n-1} - \binom{4\delta-5}{n-1}\right].$$

where $\alpha = 4\delta - 1$, if r is an even number and $\alpha = 4\delta - 3$, if r is an odd number.

Remark 3.3. (i) $M_n(G_4\ ; x) = 4\binom{3}{n-1}x$, $G_4 \equiv K_4$.

(ii) $M_n(G_6\ ; x) = 6\binom{3}{n-1}x + 6[\binom{5}{n-1} - \binom{3}{n-1}]x^2$.

(iii) $M_n(G_8\ ; x) = 8\binom{3}{n-1}x + 8[\binom{7}{n-1} - \binom{3}{n-1}]x^2$.

4 Conclusion

We found the general formulas of polynomials of theta and Wagner graphs with respect to based max – n – distance between the vertex v in $V(G)$ and the subset S of vertices of $V(G)$ has $(n-1)$ – vertices, ($S \subseteq V(G)$,$|S| = n - 1$,$n \ge 3$)such that v is not belong to S.

We can obtain the Hosoya polynomial and Wiener index, [14]from M_n – polynomials and max – n –Wiener index when $n = 2$respectively for theta and Wagner graphs by:

$$H(G; x) = \frac{1}{2}M_2(G; x) \quad (4.1)$$

$$W(G) = \frac{d}{dx}H(G; x)|_{x=1} \quad (4.2)$$

From (4.1) and Theorems 2.1, 2.4 and 3.1, we can be obtain the following corollary when:

Corollary 4.1. *:*

(i) $H(\theta_p;x) = (p+1)\,x + (p+4)\,x^2 + \sum_{k=3}^{p/4}(p+4k-4)\,x^k$
$+\sum_{k=\frac{p}{4}+1}^{\frac{p}{2}-1}(2p-4k+2)x^k + x^{p/2}$, *if* $p \equiv 0(mod\,4)$, $p \geq 12$.

(ii) $H(\theta_p;x) = (p+1)\,x + (p+4)\,x^2 + \sum_{k=3}^{(p-2)/4}(p+4k-4)\,x^k + \frac{3}{2}(p-2)\,x^{(p+2)/4}$
$+\sum_{k=(p+6)/4}^{\frac{p}{2}-1}(2p-4k+2)x^k + 2x^{p/2}$, *if* $p \equiv 2(mod\,4)$,$p \geq 14$.

(iii) $H(G_{2r};x) = 3rx + 4r\sum_{k=2}^{\delta-1}x^k + 2r\begin{cases} 2x^{r/2} & ;\ if\ r\ is\ an\ even\ ,\ r \geq 6\ , \\ x^{(r+1)/2} & ;if\ r\ is\ an\ odd\ ,\ r \geq 5\ . \end{cases}$

From (4.2) and Corollary 4.1, we have

Corollary 4.2. *:*

(i) $W(\theta_p) = \frac{1}{96}(9p^3 + 18p^2 - 72p + 96)$, *if* $p \equiv 0(mod\,4)$, $p \geq 4$.

(ii) $W(\theta_p) = \frac{1}{96}(9p^3 + 18p^2 - 36p + 24)$, *if* $p \equiv 2(mod\,4)$, $p \geq 6$.

(iii) $W(G_{2r}) = \frac{1}{2}\begin{cases} r^2 + 2r - 2 & ;\ if\ r\ is\ an\ even\ ,\ r \geq 4, \\ r^2 + 2r - 1 & ;if\ r\ is\ an\ odd\ r \geq 3. \end{cases}$

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