Palestine Journal of Mathematics

Vol. 11(1)(2022) , 87–103

On $\ell-$regular partition triples with designated summands

M. S. Mahadeva Naika and Harishkumar T

Communicated by V. Lokesha

MSC 2010 Classifications: 11P83, 05A15, 05A17.

Keywords and phrases: Congruences, Dissections, Partitions with designated summands.

The authors are thankful to the referee for his/her useful comments. The second author would like to thank the Ministry of Tribal Affairs, Govt. of India for providing financial assistance under NFST, ref. no. 201718-NFST-KAR-00136 dated 07.06.2018.

Abstract. Andrews, Lewis and Lovejoy investigated a new class of partitions with designated summands by taking ordinary partitions and tagging exactly one of each part size. Let $B_\ell(n)$ denote the number of ℓ-regular partition triples of n with designated summands. In this work, we establish many infinite families of congruences modulo powers of 2 and 3 for $B_\ell(n)$. For example, for each $n \geq 1$ and $\beta \geq 0$,

$$B_3\left(12\cdot 5^{2\beta+2}n + a_1\cdot 5^{2\beta+1}\right) \equiv 0 \pmod 9,$$

where $a_1 \in \{22, 34, 46, 58\}$.

1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. An ℓ-regular partition is a partition in which none of the part is divisible by ℓ.

Andrews, Lewis and Lovejoy [1] investigated a new class of partitions with designated summands by taking ordinary partitions and tagging exactly one of each part size. The total number of partitions of n with designated summands is denoted by $PD(n)$ and the generating function for $PD(n)$ is given by

$$\sum_{n=0}^{\infty} PD\left(n\right)q^n = \frac{f_6}{f_1f_2f_3}, \tag{1.1}$$

where

$$f_n := \prod_{j=1}^{\infty}(1-q^{nj}), n \geq 1. \tag{1.2}$$

For example, $PD(4) = 10$, namely

$$4',\ \ 3'+1',\ \ 2'+2,\ \ 2+2',\ \ 2'+1'+1,\ \ 2'+1+1',\ \ 1'+1+1+1,\ \ 1+1'+1+1,$$
$$1+1+1'+1,\ \ 1+1+1+1'.$$

Mahadeva Naika and Gireesh [8] studied $PD_3(n)$, the number of partitions of n with designated summands whose parts are not divisible by 3 and the generating function is given by

$$\sum_{n=0}^{\infty} PD_3\left(n\right)q^n = \frac{f_6^2f_9}{f_1f_2f_{18}}. \tag{1.3}$$

They obtained many congruences modulo 4, 9, 12, 36, 48 and 144 for $PD_3(n)$. For example, for each nonnegative integer n and $\alpha \geq 0$,

$$PD_3\left(4\times 3^{\alpha+1}n + 10\times 3^{\alpha}\right) \equiv 0 \pmod 9.$$

Andrews et al. [1], Baruah and Ojah [2] studied $PDO(n)$, the number of partitions of n with designated summands in which all parts are odd. The generating function for $PDO(n)$ is given by

$$\sum_{n=0}^{\infty} PDO\left(n\right)q^{n} = \frac{f_4 f_6^2}{f_1 f_3 f_{12}}. \tag{1.4}$$

Mahadeva Naika and Shivashankar [13] established many congruences for $BPD(n)$, the number of bipartitions of n with designated summands and the generating function is given by

$$\sum_{n=0}^{\infty} BPD\left(n\right)q^{n} = \frac{f_6^2}{f_1^2 f_2^2 f_3^2}. \tag{1.5}$$

For more details, one can see [4, 9, 10, 11, 12, 15].

Motivated by the above work, in this paper, we define $B_\ell(n)$, the number of ℓ-regular partition triples of n with designated summands. The generating function for $B_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} B_\ell\left(n\right)q^{n} = \frac{f_6^3 f_\ell^3 f_{2\ell}^3 f_{3\ell}^3}{f_1^3 f_2^3 f_3^3 f_{6\ell}^3}. \tag{1.6}$$

Also, we establish many infinite families of congruences modulo powers of 2 and 3 for $B_\ell(n)$. For example, for each $n \geq 1$ and $\beta \geq 0$,

$$B_3\left(12 \cdot 5^{2\beta+2} n + a_1 \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{9},$$

where $a_1 \in \{22, 34, 46, 58\}$.

2 Preliminary results

In this section, we list few dissection formulas which are useful in proving our main results.

Lemma 2.1. *The following* 2*-dissection holds:*

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}. \tag{2.1}$$

For a proof, we can see [3, p. 40, Entry 25].

Lemma 2.2. *The following* 2*-dissection holds:*

$$\frac{f_9}{f_1} = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}. \tag{2.2}$$

The identity (2.2) was obtained by Xia and Yao [17].

Lemma 2.3. *The following* 3*-dissections hold:*

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \tag{2.3}$$

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}, \tag{2.4}$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}. \tag{2.5}$$

Lemma 2.3 was proved by Hirschhorn and Sellers [7].

Lemma 2.4. *The following* 2*-dissections hold:*

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \tag{2.6}$$

$$\frac{f_1^2}{f_3^2} = \frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}. \tag{2.7}$$

Xia and Yao [16] proved (2.6) by employing an addition formula for theta functions. Replacing q by $-q$ in (2.6) and using the fact that $(-q;-q)_\infty = \dfrac{f_2^3}{f_1f_4}$, we obtain (2.7).

Lemma 2.5. *The following* 2*-dissections hold:*

$$\frac{1}{f_1f_3} = \frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2} + q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}, \tag{2.8}$$

$$f_1f_3 = \frac{f_2f_8^2f_{12}^4}{f_4^2f_6f_{24}^2} - q\frac{f_4^4f_6f_{24}^2}{f_2f_8^2f_{12}^2}. \tag{2.9}$$

The equation (2.8) was proved by Baruah and Ojah [2]. Replacing q by $-q$ in (2.8) and using the fact that $(-q;-q)_\infty = \dfrac{f_2^3}{f_1f_4}$, we get (2.9).

Lemma 2.6. *The following* 3*-dissection holds:*

$$f_1f_2 = \frac{f_6f_9^4}{f_3f_{18}^2} - qf_9f_{18} - 2q^2\frac{f_3f_{18}^4}{f_6f_9^2}. \tag{2.10}$$

For a proof, we can see [6].

Lemma 2.7. *The following* 3*-dissection holds:*

$$f_1^3 = \frac{f_6f_9^6}{f_3f_{18}^3} - 3qf_9^3 + 4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}. \tag{2.11}$$

For a proof, see [3, p.345].

Lemma 2.8. *The following* 2*-dissection holds:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3f_6^2}{f_2^2f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{2.12}$$

The identity (2.12) was obtained by Hirschhorn et al. [5].

Lemma 2.9. *[14, p. 212] We have the following* 5*-dissection formula*

$$f_1 = f_{25}\left(a(q^5) - q - q^2/a(q^5)\right), \tag{2.13}$$

where

$$a := a(q) := \frac{(q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty}. \tag{2.14}$$

Lemma 2.10. *[3, p.303, Entry 17(v)] We have*

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{2.15}$$

where $A(q) = f(-q^3,-q^4)$, $B(q) = f(-q^2,-q^5)$ *and* $C(q) = f(-q,-q^6)$.

Lemma 2.11. *For positive integers* k *and* m, *we have*

$$f_{3k}^m \equiv f_k^{3m} \pmod 3, \tag{2.16}$$

$$f_{3k}^{3m} \equiv f_k^{9m} \pmod 9, \tag{2.17}$$

$$f_{2k}^m \equiv f_k^{2m} \pmod 2, \tag{2.18}$$

$$f_{2k}^{2m} \equiv f_k^{4m} \pmod 4 \tag{2.19}$$

and

$$f_{2k}^{4m} \equiv f_k^{8m} \pmod 8. \tag{2.20}$$

3 Congruences modulo 4 and 8 for $B_2(n)$

Theorem 3.1. *For all* $n \geq 0$ *and* $\alpha, \beta \geq 0$, *we have*

$$B_2(12n+11) \equiv 0 \pmod{8}, \tag{3.1}$$

$$B_2(24n+19) \equiv 0 \pmod{8}, \tag{3.2}$$

$$B_2\left(4 \cdot 3^{2\alpha+2}n + 3^{2\alpha+3}\right) \equiv 5^{\alpha+1} \cdot B_2(4n+3) \pmod{8}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} B_2\left(24 \cdot 5^{2\beta}n + 7 \cdot 5^{2\beta}\right) q^n \equiv 4f_1^7 \pmod{8}, \tag{3.4}$$

$$\sum_{n=0}^{\infty} B_2\left(24 \cdot 5^{2\beta+1}n + 11 \cdot 5^{2\beta+1}\right) q^n \equiv 4qf_5^7 \pmod{8}, \tag{3.5}$$

$$B_2\left(24 \cdot 5^{2\beta+2}n + a_1 \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{8}, \tag{3.6}$$

where $a_1 \in \{11, 59, 83, 107\}$.

Proof. Setting $\ell = 2$ in (1.6), we find that

$$\sum_{n=0}^{\infty} B_2(n)\, q^n = \frac{f_4^3 f_6^6}{f_1^3 f_3^3 f_{12}^3}. \tag{3.7}$$

Employing (2.8) into (3.7), we obtain

$$\sum_{n=0}^{\infty} B_2(2n)\, q^n = \frac{f_4^6 f_6^{12}}{f_1^6 f_3^6 f_{12}^6} + 3q\frac{f_2^{12} f_{12}^2}{f_1^{10} f_3^2 f_4^2} \tag{3.8}$$

and

$$\sum_{n=0}^{\infty} B_2(2n+1)\, q^n = 3\frac{f_2^6 f_4^2 f_6^6}{f_1^8 f_3^4 f_{12}^2} + q\frac{f_2^{18} f_{12}^6}{f_1^{12} f_4^6 f_6^6}. \tag{3.9}$$

Invoking (2.20) into (3.9), we find that

$$\sum_{n=0}^{\infty} B_2(2n+1)\, q^n \equiv 3\frac{f_2^2 f_3^4 f_4^2 f_6^2}{f_{12}^2} + q\frac{f_1^4 f_2^2 f_6^2 f_{12}^2}{f_4^2} \pmod{8}. \tag{3.10}$$

Substituting (2.1) into (3.10), we obtain

$$\sum_{n=0}^{\infty} B_2(4n+1)\, q^n \equiv 3\frac{f_1^2 f_2^2 f_6^8}{f_{12}^4} + 4q\frac{f_1^4 f_3^2 f_4^4 f_6^2}{f_2^4} \pmod{8} \tag{3.11}$$

and

$$\sum_{n=0}^{\infty} B_2(4n+3)\, q^n \equiv \frac{f_2^8 f_3^2 f_6^2}{f_4^4} + 4q\frac{f_1^2 f_2^2 f_3^4 f_{12}^4}{f_6^4} \pmod{8}. \tag{3.12}$$

The equation (3.12) becomes

$$\sum_{n=0}^{\infty} B_2(4n+3)\, q^n \equiv f_3^2 f_6^2 + 4qf_1^2 f_2^2 f_{12}^3 \pmod{8}. \tag{3.13}$$

Utilizing (2.10) into (3.13), we get

$$\sum_{n=0}^{\infty} B_2(4n+3)\, q^n \equiv f_3^2 f_6^2 + 4q\frac{f_6^2 f_9^8 f_{12}^3}{f_3^2 f_{18}^4} + 4q^3 f_9^2 f_{12}^3 f_{18}^2 \pmod{8}. \tag{3.14}$$

Extracting the terms involving q^{3n+2} from both sides of the above equation, we arrive at (3.1).

The equation (3.14) implies

$$\sum_{n=0}^{\infty} B_2\left(12n+3\right)q^n \equiv f_1^2 f_2^2 + 4q f_3^2 f_4^3 f_6^2 \pmod{8}. \tag{3.15}$$

Using (2.10) and (2.11) into (3.15) and then comparing the terms involving q^{3n+2} on both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} B_2\left(36n+27\right)q^n \equiv 5 f_3^2 f_6^2 + 4q f_1^2 f_2^2 f_{12}^3 \pmod{8}. \tag{3.16}$$

In view of the congruences (3.13) and (3.16), we see that

$$B_2(36n+27) \equiv 5 \cdot B_2(4n+3) \pmod{8}. \tag{3.17}$$

Using the above relation and by induction on α, we arrive at (3.3).

The equation (3.14) implies

$$\sum_{n=0}^{\infty} B_2\left(12n+7\right)q^n \equiv 4 f_2^7 \pmod{8}. \tag{3.18}$$

Extracting the terms involving q^{2n+1} from both sides of the equation (3.18), we get (3.2).

The equation (3.18) implies

$$\sum_{n=0}^{\infty} B_2\left(24n+7\right)q^n \equiv 4 f_1^7 \pmod{8}, \tag{3.19}$$

which is $\beta = 0$ case of (3.4). Suppose that the congruence (3.4) is true for $\beta \geq 0$, we have

$$\sum_{n=0}^{\infty} B_2\left(24 \cdot 5^{2\beta} n + 7 \cdot 5^{2\beta}\right)q^n \equiv 4 f_1^7 \pmod{8}. \tag{3.20}$$

Substituting (2.13) into (3.20), we arrive at

$$\sum_{n=0}^{\infty} B_2\left(24 \cdot 5^{2\beta+1} n + 11 \cdot 5^{2\beta+1}\right)q^n \equiv 4q f_5^7 \pmod{8}, \tag{3.21}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(24 \cdot 5^{2\beta+2} n + 7 \cdot 5^{2\beta+2}\right)q^n \equiv 4 f_1^7 \pmod{8}, \tag{3.22}$$

which implies that the congruence (3.4) is true for $\beta + 1$. Hence, by induction, the congruence (3.4) holds for all integer $\beta \geq 0$.

Employing (2.13) into (3.4) and then comparing the coefficients of q^{5n+2} on both sides of the resultant equation, we arrive at (3.5).

Extracting the terms involving q^{5n+i} for $i = 0, 2, 3, 4$ from the equation (3.5), we obtain (3.6). □

Theorem 3.2. *For all* $n \geq 0$ *and* $\alpha, \beta \geq 0$, *we have*

$$B_2(36n+35) \equiv 0 \pmod{8}, \tag{3.23}$$

$$B_2(72n+57) \equiv 0 \pmod{8}, \tag{3.24}$$

$$B_2\left(4 \cdot 3^{2\alpha+2} n + 3^{2\alpha+2}\right) \equiv 5^{\alpha+1} \cdot B_2(4n+1) \pmod{8}, \tag{3.25}$$

$$\sum_{n=0}^{\infty} B_2\left(72\cdot 5^{2\beta}n+21\cdot 5^{2\beta}\right)q^n \equiv 4f_1^7 \pmod 8, \tag{3.26}$$

$$\sum_{n=0}^{\infty} B_2\left(72\cdot 5^{2\beta+1}n+33\cdot 5^{2\beta+1}\right)q^n \equiv 4qf_5^7 \pmod 8, \tag{3.27}$$

$$B_2\left(72\cdot 5^{2\beta+2}n+a_2\cdot 5^{2\beta+1}\right) \equiv 0 \pmod 8, \tag{3.28}$$

where $a_2 \in \{33, 177, 249, 321\}$.

Proof. From the equation (3.11), we arrive at

$$\sum_{n=0}^{\infty} B_2(4n+1)q^n \equiv 3f_1^2f_2^2+4qf_3^2f_4^3f_6^2 \pmod 8. \tag{3.29}$$

Substituting (2.10) and (2.11) into (3.29), we obtain

$$\sum_{n=0}^{\infty} B_2(12n+1)q^n \equiv 3\frac{f_2^2}{f_1^2}+4q\frac{f_3^9}{f_1} \pmod 8, \tag{3.30}$$

$$\sum_{n=0}^{\infty} B_2(12n+5)q^n \equiv 2\frac{f_2f_3f_6}{f_1}+4f_2f_8+4q\frac{f_6^6}{f_2} \pmod 8 \tag{3.31}$$

and

$$\sum_{n=0}^{\infty} B_2(12n+9)q^n \equiv 7f_3^2f_6^2+4qf_1^2f_2^2f_{12}^3 \pmod 8. \tag{3.32}$$

Using (2.10) into (3.32), we find that

$$\sum_{n=0}^{\infty} B_2(12n+9)q^n \equiv 7f_3^2f_6^2+4qf_6^7+4q^3f_9^2f_{12}^3f_{18}^2 \pmod 8. \tag{3.33}$$

Extracting the terms involving q^{3n+2} from both sides of the above equation, we arrive at (3.23).

The congruence (3.33) implies

$$\sum_{n=0}^{\infty} B_2(36n+9)q^n \equiv 7f_1^2f_2^2+4qf_3^2f_4^3f_6^2 \pmod 8. \tag{3.34}$$

In view of the congruences (3.29) and (3.34), we get

$$B_2(36n+9) \equiv 5\cdot B_2(4n+1) \pmod 8. \tag{3.35}$$

Using the above relation and by induction on α, we arrive at (3.25).

From the congruence (3.33), we obtain

$$\sum_{n=0}^{\infty} B_2(36n+21)q^n \equiv 4f_2^7 \pmod 8. \tag{3.36}$$

Collecting the coefficients of q^{2n+1} from both sides of the above equation, we arrive at (3.24).

The congruence (3.36) implies

$$\sum_{n=0}^{\infty} B_2(72n+21)q^n \equiv 4f_1^7 \pmod 8, \tag{3.37}$$

which is $\beta = 0$ case of (3.26). The rest of the proofs of the identities (3.26)-(3.28) are similar to the proofs of the identities (3.4)-(3.6). So, we omit the details. □

Theorem 3.3. *Let $a_3 \in \{11, 59, 83, 107, 131, 155\}$, then for all $n \geq 0$ and $\beta, \gamma \geq 0$, we have*

$$\sum_{n=0}^{\infty} B_2\left(24 \cdot 7^{2\gamma} n + 5 \cdot 7^{2\gamma}\right) q^n \equiv 2 f_1 f_4 \pmod 4, \tag{3.38}$$

$$\sum_{n=0}^{\infty} B_2\left(24 \cdot 7^{2\gamma+1} n + 11 \cdot 7^{2\gamma+1}\right) q^n \equiv 2q f_7 f_{28} \pmod 4, \tag{3.39}$$

$$B_2\left(24 \cdot 7^{2\gamma+2} n + a_3 \cdot 7^{2\gamma+1}\right) \equiv 0 \pmod 4, \tag{3.40}$$

$$B_2\left(12 \cdot 5^{2\beta+2} n + 5^{2\beta+2}\right) \equiv 3^{\beta+1} \cdot B_2(12n+1) \pmod 4, \tag{3.41}$$

$$B_2\left(60(5n+i)+25\right) \equiv 0 \pmod 4, \tag{3.42}$$

where $i = 1, 2, 3, 4$.

Proof. From the equation (3.31), we arrive at

$$\sum_{n=0}^{\infty} B_2\left(12n+5\right) q^n \equiv 2\frac{f_2 f_3^3}{f_1} \pmod 4. \tag{3.43}$$

Substituting (2.12) into (3.43) and then collecting the terms involving q^{2n} from the resultant equation, we get

$$\sum_{n=0}^{\infty} B_2\left(24n+5\right) q^n \equiv 2 f_1 f_4 \pmod 4, \tag{3.44}$$

which is $\gamma = 0$ case of (3.38). Suppose that the congruence (3.38) is true for $\gamma \geq 0$. Substituting (2.15) into (3.38), we arrive at

$$\sum_{n=0}^{\infty} B_2\left(24 \cdot 7^{2\gamma+1} n + 11 \cdot 7^{2\gamma+1}\right) q^n \equiv 2q f_7 f_{28} \pmod 4, \tag{3.45}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(24 \cdot 7^{2\gamma+2} n + 5 \cdot 7^{2\gamma+2}\right) q^n \equiv 2 f_1 f_4 \pmod 4, \tag{3.46}$$

which implies that the congruence (3.38) is true for $\gamma + 1$. So, by induction, the congruence (3.38) holds for all integer $\gamma \geq 0$.

Employing (2.15) into (3.38) and then extracting the terms involving q^{7n+3} from both sides of the resultant equation, we get (3.39).

From the equation (3.39), we arrive at (3.40).

The congruence (3.30) reduces to

$$\sum_{n=0}^{\infty} B_2\left(12n+1\right) q^n \equiv 3 f_1^2 \equiv 3 f_{25}^2 \left(a(q^5) - q - q^2/a(q^5)\right)^2 \pmod 4, \tag{3.47}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(60n+25\right) q^n \equiv f_5^2 \pmod 4, \tag{3.48}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(300n+25\right) q^n \equiv f_1^2 \pmod 4. \tag{3.49}$$

In view of the congruences (3.47) and (3.49), we see that

$$B_2(300n+25) \equiv 3 \cdot B_2(12n+1) \pmod 4. \tag{3.50}$$

Using the above relation and by induction on β, we arrive at (3.41).

Extracting the terms involving q^{5n+i} for $i = 1, 2, 3, 4$ from the equation (3.48), we obtain (3.42). □

Theorem 3.4. *For all* $n \geq 0$ *and* $\alpha \geq 0$, *we have*

$$\sum_{n=0}^{\infty} B_2\left(4 \cdot 3^{2\alpha} n + 2 \cdot 3^{2\alpha}\right) q^n \equiv f_1^4 f_2^4 + 3q f_3^4 f_6^4 \pmod{8}. \tag{3.51}$$

Proof. Invoking (2.20) into (3.8), we find that

$$\sum_{n=0}^{\infty} B_2(2n)\, q^n \equiv \frac{f_1^2 f_3^2 f_4^6}{f_2^4 f_{12}^2} + 3q \frac{f_4^2 f_{12}^2}{f_1^2 f_3^2} \pmod{8}. \tag{3.52}$$

Substituting (2.8) and (2.9) into (3.52), we get

$$\sum_{n=0}^{\infty} B_2(4n)\, q^n \equiv \frac{f_2^2 f_4^4 f_6^6}{f_1^2 f_3^2 f_{12}^4} + q \frac{f_2^{14} f_3^2 f_{12}^4}{f_1^6 f_4^4 f_6^6} + 6q \frac{f_2^6 f_6^6}{f_1^6 f_3^6} \pmod{8} \tag{3.53}$$

and

$$\sum_{n=0}^{\infty} B_2(4n+2)\, q^n \equiv 6\frac{f_2^8}{f_1^4} + 3\frac{f_4^4 f_6^{12}}{f_1^4 f_3^8 f_{12}^4} + 3q \frac{f_2^{12} f_{12}^4}{f_1^8 f_3^4 f_4^4} \pmod{8}. \tag{3.54}$$

The equation (3.54) reduces to

$$\sum_{n=0}^{\infty} B_2(4n+2)\, q^n \equiv f_1^4 f_2^4 + 3q f_3^4 f_6^4 \pmod{8}, \tag{3.55}$$

which is $\alpha = 0$ case of (3.51). Suppose that the congruence (3.51) is true for $\alpha \geq 0$. Employing (2.10) into (3.51) and then collecting the coefficients of q^{3n}, q^{3n+1} and q^{3n+2}, we obtain

$$\sum_{n=0}^{\infty} B_2\left(4 \cdot 3^{2\alpha+1} n + 2 \cdot 3^{2\alpha}\right) q^n \equiv f_1^4 + 4q \frac{f_2 f_3^3 f_6^3}{f_1} \pmod{8}, \tag{3.56}$$

$$\sum_{n=0}^{\infty} B_2\left(4 \cdot 3^{2\alpha+1} n + 2 \cdot 3^{2\alpha+1}\right) q^n \equiv 3 f_1^4 f_2^4 + 4 f_1^3 f_3^3 + q f_3^4 f_6^4 \pmod{8} \tag{3.57}$$

and

$$\sum_{n=0}^{\infty} B_2\left(4 \cdot 3^{2\alpha+1} n + 10 \cdot 3^{2\alpha}\right) q^n \equiv 6 \frac{f_2^2 f_3^2 f_6^2}{f_1^2} \pmod{8}. \tag{3.58}$$

Substituting (2.10) and (2.11) into (3.57) and then collecting the terms involving q^{3n+1} from the resultant equation, we arrive at

$$\sum_{n=0}^{\infty} B_2\left(4 \cdot 3^{2\alpha+2} n + 2 \cdot 3^{2\alpha+2}\right) q^n \equiv f_1^4 f_2^4 + 3q f_3^4 f_6^4 \pmod{8}, \tag{3.59}$$

which implies that the congruence (3.51) is true for $\alpha + 1$. By induction, the congruence (3.51) holds for all integer $\alpha \geq 0$. □

Theorem 3.5. *Let* $a_4 \in \{46, 94, 142, 238\}$ *and* $a_5 \in \{14, 62, 158, 206\}$, *then for all* $n \geq 0$ *and* $\alpha, \beta \geq 0$, *we have*

$$B_2\left(16 \cdot 3^{2\alpha+1} n + 14 \cdot 3^{2\alpha}\right) \equiv 0 \pmod{8}, \tag{3.60}$$

$$B_2(16 \cdot 3^{2\alpha+1} n + 46 \cdot 3^{2\alpha}) \equiv 0 \pmod{8}, \tag{3.61}$$

$$\sum_{n=0}^{\infty} B_2\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} n + 38 \cdot 3^{2\alpha} \cdot 5^{2\beta}\right) q^n \equiv 4 f_1 f_6^3 \pmod{8}, \tag{3.62}$$

$$\sum_{n=0}^{\infty} B_2\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} n + 46 \cdot 3^{2\alpha} \cdot 5^{2\beta+1}\right) q^n \equiv 4q^3 f_5 f_{30}^3 \pmod{8}, \tag{3.63}$$

$$B_2\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}n+a_4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\right)\equiv 0 \pmod 8, \tag{3.64}$$

$$\sum_{n=0}^{\infty} B_2\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}n+22\cdot 3^{2\alpha}\cdot 5^{2\beta}\right)q^n\equiv 4f_2f_3^3 \pmod 8, \tag{3.65}$$

$$\sum_{n=0}^{\infty} B_2\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+14\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\right)q^n\equiv 4q^2f_{10}f_{15}^3 \pmod 8, \tag{3.66}$$

$$B_2\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}n+a_5\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\right)\equiv 0 \pmod 8. \tag{3.67}$$

Proof. Substituting (2.1) and (2.12) into (3.56), we get

$$\sum_{n=0}^{\infty} B_2\left(8\cdot 3^{2\alpha+1}n+2\cdot 3^{2\alpha}\right)q^n\equiv \frac{f_2^2}{f_1^2}+4q\frac{f_3^3f_6^3}{f_1} \pmod 8 \tag{3.68}$$

and

$$\sum_{n=0}^{\infty} B_2\left(8\cdot 3^{2\alpha+1}n+14\cdot 3^{2\alpha}\right)q^n\equiv 4f_2^7+4\frac{f_2^3f_3^3}{f_1} \pmod 8. \tag{3.69}$$

Using (2.12) into (3.69), we obtain

$$\sum_{n=0}^{\infty} B_2\left(16\cdot 3^{2\alpha+1}n+14\cdot 3^{2\alpha}\right)q^n\equiv 4f_1^7+4\frac{f_1f_2^3f_3^2}{f_6} \pmod 8 \tag{3.70}$$

and

$$\sum_{n=0}^{\infty} B_2\left(16\cdot 3^{2\alpha+1}n+38\cdot 3^{2\alpha}\right)q^n\equiv 4f_1f_6^3 \pmod 8. \tag{3.71}$$

From the equation (3.70), we arrive at (3.60).

The congruence (3.71) is $\beta=0$ case of (3.62). Suppose that the congruence (3.62) is true for $\alpha,\beta\geq 0$, we have

$$\sum_{n=0}^{\infty} B_2\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}n+38\cdot 3^{2\alpha}\cdot 5^{2\beta}\right)q^n\equiv 4f_1f_6^3 \pmod 8. \tag{3.72}$$

Utilizing (2.13) into (3.72), we get

$$\sum_{n=0}^{\infty} B_2\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+46\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\right)q^n\equiv 4q^3f_5f_{30}^3 \pmod 8, \tag{3.73}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}n+38\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\right)q^n\equiv 4f_1f_6^3 \pmod 8, \tag{3.74}$$

which implies that the congruence (3.62) is true for $\beta+1$. Hence, by induction, the congruence (3.62) holds for all integers $\alpha,\beta\geq 0$.

Utilizing (2.13) into (3.62) and then collecting the coefficients of q^{5n+4} from both sides of the resultant equation, we get (3.63).

Extracting the terms involving q^{5n+i} for $i=0,1,2,4$ from the equation (3.63), we obtain (3.64).

Substituting (2.6) into (3.58) and then collecting the coefficients of q^{2n+1} from the resultant equation, we get

$$\sum_{n=0}^{\infty} B_2\left(8\cdot 3^{2\alpha+1}n+22\cdot 3^{2\alpha}\right)q^n\equiv 4f_4f_6^3 \pmod 8. \tag{3.75}$$

Extracting the terms involving q^{2n+1} from both sides of the equation (3.75), we arrive at (3.61).
The congruence (3.75) implies

$$\sum_{n=0}^{\infty} B_2\left(16\cdot 3^{2\alpha+1}n+22\cdot 3^{2\alpha}\right)q^n \equiv 4f_2f_3^3 \pmod 8, \tag{3.76}$$

which is $\beta=0$ case of (3.65). Suppose that the congruence (3.65) is true for $\beta\geq 0$, we have

$$\sum_{n=0}^{\infty} B_2\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}n+22\cdot 3^{2\alpha}\cdot 5^{2\beta}\right)q^n \equiv 4f_2f_3^3 \pmod 8. \tag{3.77}$$

Substituting (2.13) into (3.77), we arrive at

$$\sum_{n=0}^{\infty} B_2\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+14\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\right)q^n \equiv 4q^2f_{10}f_{15}^3 \pmod 8, \tag{3.78}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}n+22\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\right)q^n \equiv 4f_2f_3^3 \pmod 8, \tag{3.79}$$

which implies that the congruence (3.65) is true for $\beta+1$. Hence, by induction, the congruence (3.65) holds for all integers $\alpha,\beta\geq 0$.

Substituting (2.13) into (3.65) and then collecting the coefficients of q^{5n+1} from the resultant equation, we arrive at (3.66).

Extracting the terms involving q^{5n+i} for $i=0,1,3,4$ from both sides of the equation (3.66), we obtain (3.67). □

Theorem 3.6. *Let* $a_6\in\{38,62,86,110,134,158\}$, *then for all* $n\geq 0$ *and* $\alpha,\beta,\gamma\geq 0$, *we have*

$$\sum_{n=0}^{\infty} B_2\left(8\cdot 3^{2\alpha+1}\cdot 5^{4\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{4\beta}\cdot 7^{2\gamma}\right)q^n \equiv f_1^2 \pmod 4, \tag{3.80}$$

$$B_2\left(8\cdot 3^{2\alpha+1}\cdot 5^{4\beta+1}(5n+i)+2\cdot 3^{2\alpha}\cdot 5^{4\beta+2}\right) \equiv 0 \pmod 4, \tag{3.81}$$

$$\sum_{n=0}^{\infty} B_2\left(8\cdot 3^{2\alpha+1}\cdot 5^{4\beta}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha}\cdot 5^{4\beta}\cdot 7^{2\gamma+2}\right)q^n \equiv f_7^2 \pmod 4, \tag{3.82}$$

$$B_2\left(8\cdot 3^{2\alpha+1}\cdot 5^{4\beta}\cdot 7^{2\gamma+2}n+a_6\cdot 3^{2\alpha+1}\cdot 5^{4\beta}\cdot 7^{2\gamma+1}\right) \equiv 0 \pmod 4, \tag{3.83}$$

where $i=1,2,3,4$.

Proof. The equation (3.68) becomes

$$\sum_{n=0}^{\infty} B_2\left(8\cdot 3^{2\alpha+1}n+2\cdot 3^{2\alpha}\right)q^n \equiv f_1^2 \pmod 4, \tag{3.84}$$

which is $\beta=\gamma=0$ case of (3.80). Suppose that the congruence (3.80) is true for $\alpha,\beta\geq 0$ and $\gamma=0$. Substituting (2.13) into (3.80) with $\gamma=0$ and then comparing the coefficients of q^{5n+2} on both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} B_2\left(8\cdot 3^{2\alpha+1}\cdot 5^{4\beta+1}n+2\cdot 3^{2\alpha}\cdot 5^{4\beta+2}\right)q^n \equiv 3f_5^2 \pmod 4, \tag{3.85}$$

which implies (3.81)
and

$$\sum_{n=0}^{\infty} B_2\left(8\cdot 3^{2\alpha+1}\cdot 5^{4\beta+2}n+2\cdot 3^{2\alpha}\cdot 5^{4\beta+2}\right)q^n \equiv 3f_1^2 \pmod 4. \tag{3.86}$$

Again, using (2.13) into (3.86), we arrive at

$$\sum_{n=0}^{\infty} B_2\left(8\cdot 3^{2\alpha+1}\cdot 5^{4\beta+3}n+2\cdot 3^{2\alpha}\cdot 5^{4\beta+4}\right)q^n \equiv f_5^2 \pmod 4, \tag{3.87}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(8\cdot 3^{2\alpha+1}\cdot 5^{4\beta+4}n+2\cdot 3^{2\alpha}\cdot 5^{4\beta+4}\right)q^n \equiv f_1^2 \pmod 4, \tag{3.88}$$

which implies that the congruence (3.80) is true for $\beta+1$ with $\gamma=0$. By induction, the congruence (3.80) holds for $\alpha,\beta\geq 0$ with $\gamma=0$. Suppose that the congruence (3.80) is true for $\alpha,\beta,\gamma\geq 0$. Using (2.15) into (3.80), we arrive at

$$\sum_{n=0}^{\infty} B_2\left(8\cdot 3^{2\alpha+1}\cdot 5^{4\beta}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha}\cdot 5^{4\beta}\cdot 7^{2\gamma+2}\right)q^n \equiv f_7^2 \pmod 4, \tag{3.89}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(8\cdot 3^{2\alpha+1}\cdot 5^{4\beta}\cdot 7^{2\gamma+2}n+2\cdot 3^{2\alpha}\cdot 5^{4\beta}\cdot 7^{2\gamma+2}\right)q^n \equiv f_1^2 \pmod 4, \tag{3.90}$$

which implies that the congruence (3.80) is true for $\gamma+1$. Hence, by induction, the congruence (3.80) holds for all integers $\alpha,\beta,\gamma\geq 0$.

Employing (2.15) into (3.80) and then collecting the coefficients of q^{7n+4} from both sides of the resultant equation, we obtain (3.82).

From the congruence (3.82), we arrive at (3.83). □

Theorem 3.7. *For all* $n\geq 0$ *and* $\alpha\geq 0$, *we have*

$$B_2\left(8\cdot 3^{2\alpha+2}n+4\cdot 3^{2\alpha+2}\right)\equiv B_2(8n+4) \pmod 8, \tag{3.91}$$

$$B_2\left(16\cdot 3^{2\alpha+2}n+8\cdot 3^{2\alpha+2}\right)\equiv B_2(16n+8) \pmod 8. \tag{3.92}$$

Proof. The equation (3.53) reduces to

$$\sum_{n=0}^{\infty} B_2(4n)\,q^n \equiv \frac{f_2^2f_4^4}{f_1^2f_3^2f_6^2}+7qf_1^2f_2^2f_3^2f_6^2 \pmod 8. \tag{3.93}$$

Employing (2.8) and (2.9) into (3.93), we get

$$\sum_{n=0}^{\infty} B_2(8n)\,q^n \equiv \frac{f_2^2f_4^4}{f_1^2f_3^2f_6^2}+3qf_1^2f_2^2f_3^2f_6^2 \pmod 8 \tag{3.94}$$

and

$$\sum_{n=0}^{\infty} B_2(8n+4)\,q^n \equiv f_1^4f_2^4+7qf_3^4f_6^4 \pmod 8. \tag{3.95}$$

Substituting (2.10) into (3.95) and then collecting the coefficients of q^{3n+1} from the resultant equation, we arrive at

$$\sum_{n=0}^{\infty} B_2(24n+12)\,q^n \equiv 7f_1^4f_2^4+4f_1^3f_3^3+qf_3^4f_6^4 \pmod 8. \tag{3.96}$$

Using (2.10) and (2.11) in the above equation and then extracting the terms involving q^{3n+1} from the resultant equation, we obtain

$$\sum_{n=0}^{\infty} B_2(72n+36)\,q^n \equiv f_1^4f_2^4+7qf_3^4f_6^4 \pmod 8. \tag{3.97}$$

In view of the congruences (3.95) and (3.97), we find that

$$B_2(72n+36) \equiv B_2(8n+4) \pmod 8. \tag{3.98}$$

Using the above relation and by induction on α, we arrive at (3.91).

Using (2.8) and (2.9) into (3.94), we get

$$\sum_{n=0}^{\infty} B_2(16n)\, q^n \equiv \frac{f_2^2 f_4^4}{f_1^2 f_3^2 f_6^2} + 3q f_1^2 f_2^2 f_3^2 f_6^2 \pmod 8 \tag{3.99}$$

and

$$\sum_{n=0}^{\infty} B_2(16n+8)\, q^n \equiv 5 f_1^4 f_2^4 + 3q f_3^4 f_6^4 \pmod 8. \tag{3.100}$$

In view of the congruences (3.94) and (3.99), we get

$$B_2(16n) \equiv B_2(8n) \pmod 8. \tag{3.101}$$

Substituting (2.10) into (3.100) and then comparing the coefficients of q^{3n+1} on both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} B_2(48n+24)\, q^n \equiv 3 f_1^4 f_2^4 + 4 f_1^3 f_3^3 + 5q f_3^4 f_6^4 \pmod 8. \tag{3.102}$$

Employing (2.10) and (2.11) in the above equation and then extracting the terms involving q^{3n+1} from the resultant equation, we get

$$\sum_{n=0}^{\infty} B_2(144n+72)\, q^n \equiv 5 f_1^4 f_2^4 + 3q f_3^4 f_6^4 \pmod 8. \tag{3.103}$$

In view of the congruences (3.100) and (3.103), we obtain

$$B_2(144n+72) \equiv B_2(16n+8) \pmod 8. \tag{3.104}$$

Using the above relation and by induction on α, we arrive at (3.92). □

4 Congruences modulo 3 and 9 for $B_3(n)$

Theorem 4.1. *For all $n \geq 0$ and $\alpha \geq 0$, we have*

$$B_3(6n+1) \equiv 0 \pmod 3, \tag{4.1}$$

$$B_3(6n+5) \equiv 0 \pmod 3, \tag{4.2}$$

$$B_3\left(2\cdot 3^{\alpha+3} n + 3^{\alpha+3}\right) \equiv B_3(18n+9) \pmod 9. \tag{4.3}$$

Proof. Setting $\ell = 3$ in (1.6), we find that

$$\sum_{n=0}^{\infty} B_3(n)\, q^n = \frac{f_6^6 f_9^3}{f_1^3 f_2^3 f_{18}^3}. \tag{4.4}$$

Employing (2.2) into (4.4), we obtain

$$\sum_{n=0}^{\infty} B_3(2n)\, q^n = \frac{f_3^3 f_6^9}{f_1^9 f_{18}^3} + 3q \frac{f_2^4 f_3^7 f_6 f_{18}}{f_1^{11} f_9^2} \tag{4.5}$$

and

$$\sum_{n=0}^{\infty} B_3(2n+1)\, q^n = 3 \frac{f_2^2 f_3^5 f_6^5}{f_1^{10} f_9 f_{18}} + q \frac{f_2^6 f_3^9 f_{18}^3}{f_1^{12} f_6^3 f_9^3}. \tag{4.6}$$

Invoking (2.16) and (2.17) into (4.6), we find that

$$\sum_{n=0}^{\infty} B_3\left(2n+1\right)q^n \equiv 3\frac{f_2^2 f_6^2}{f_1 f_3} + q\frac{f_2^6 f_{18}^3}{f_1^3 f_3^3 f_6^3} \pmod 9. \tag{4.7}$$

Substituting (2.4) into (4.7), we have

$$\sum_{n=0}^{\infty} B_3\left(6n+1\right)q^n \equiv 3f_1 f_3 + 3q\frac{f_2 f_6^5}{f_1 f_3} \pmod 9, \tag{4.8}$$

$$\sum_{n=0}^{\infty} B_3\left(6n+3\right)q^n \equiv 3\frac{f_2^2 f_6^2}{f_1 f_3} + \frac{f_3^6}{f_1^6} + q\frac{f_6^9}{f_1^3 f_2^3 f_3^3} \pmod 9 \tag{4.9}$$

and

$$\sum_{n=0}^{\infty} B_3\left(6n+5\right)q^n \equiv 3\frac{f_3^2 f_6^3}{f_1^2 f_2} \pmod 9. \tag{4.10}$$

From the equations (4.8) and (4.10), we arrive at (4.1) and (4.2) respectively.

The equation (4.9) becomes

$$\sum_{n=0}^{\infty} B_3\left(6n+3\right)q^n \equiv 3\frac{f_2^2 f_6^2}{f_1 f_3} + f_1^3 f_3^3 + q\frac{f_2^6 f_6^6}{f_1^3 f_3^3} \pmod 9. \tag{4.11}$$

Using (2.4) and (2.11) into (4.11) and then extracting the terms involving q^{3n+1} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} B_3\left(18n+9\right)q^n \equiv 3\frac{f_2^2 f_6^2}{f_1 f_3} + 7f_1^3 f_3^3 + q\frac{f_2^6 f_6^6}{f_1^3 f_3^3} \pmod 9. \tag{4.12}$$

Again, using (2.4) and (2.11) in the above equation and then collecting the coefficients of q^{3n+1} from the resultant equation, we obtain

$$\sum_{n=0}^{\infty} B_3\left(54n+27\right)q^n \equiv 3\frac{f_2^2 f_6^2}{f_1 f_3} + 7f_1^3 f_3^3 + q\frac{f_2^6 f_6^6}{f_1^3 f_3^3} \pmod 9. \tag{4.13}$$

In view of the congruences (4.12) and (4.13), we see that

$$B_3(54n+27) \equiv B_3(18n+9) \pmod 9. \tag{4.14}$$

Using the above relation and by induction on α, we arrive at (4.3). □

Theorem 4.2. *Let* $a_7 \in \{22, 34, 46, 58\}$ *and* $a_8 \in \{44, 68, 92, 116\}$, *then for all* $n \geq 1$ *and* $\alpha, \beta \geq 0$, *we have*

$$B_3(12n+10) \equiv 0 \pmod 9, \tag{4.15}$$

$$B_3(24n+20) \equiv 0 \pmod 9, \tag{4.16}$$

$$B_3\left(2\cdot 3^{\alpha+1}n\right) \equiv B_3(2n) \pmod 9, \tag{4.17}$$

$$B_3\left(3\cdot 2^{2\alpha+3}n + 2^{2\alpha+3}\right) \equiv B_3(6n+2) \pmod 9, \tag{4.18}$$

$$\sum_{n=1}^{\infty} B_3\left(12\cdot 5^{2\beta}n + 2\cdot 5^{2\beta}\right)q^n \equiv 3f_1 f_3 \pmod 9, \tag{4.19}$$

$$\sum_{n=1}^{\infty} B_3\left(12\cdot 5^{2\beta+1}n + 2\cdot 5^{2\beta+2}\right)q^n \equiv 3f_5 f_{15} \pmod 9, \tag{4.20}$$

$$B_3\left(12\cdot 5^{2\beta+2}n+a_7\cdot 5^{2\beta+1}\right)\equiv 0 \pmod 9, \tag{4.21}$$

$$B_3\left(3\cdot 2^{2\alpha+3}n+2^{2\alpha+4}\right)\equiv B_3(6n+4) \pmod 9, \tag{4.22}$$

$$\sum_{n=1}^{\infty} B_3\left(24\cdot 5^{2\beta}n+4\cdot 5^{2\beta}\right)q^n\equiv 6f_1f_3 \pmod 9, \tag{4.23}$$

$$\sum_{n=1}^{\infty} B_3\left(24\cdot 5^{2\beta+1}n+4\cdot 5^{2\beta+2}\right)q^n\equiv 6f_5f_{15} \pmod 9, \tag{4.24}$$

$$B_3\left(24\cdot 5^{2\beta+2}n+a_8\cdot 5^{2\beta+1}\right)\equiv 0 \pmod 9. \tag{4.25}$$

Proof. Invoking (2.16) and (2.17) into (4.5), we find that

$$\sum_{n=0}^{\infty} B_3(2n)q^n\equiv 1+3q\frac{f_2f_6^2f_{18}}{f_1^2f_3^2} \pmod 9, \tag{4.26}$$

which implies

$$\sum_{n=1}^{\infty} B_3(2n)q^n\equiv 3q\frac{f_2f_6^2f_{18}}{f_1^2f_3^2} \pmod 9. \tag{4.27}$$

Substituting (2.5) into (4.27), we obtain

$$\sum_{n=1}^{\infty} B_3(6n)q^n\equiv 3q\frac{f_2^4f_6^4}{f_1^8} \pmod 9, \tag{4.28}$$

$$\sum_{n=1}^{\infty} B_3(6n+2)q^n\equiv 3\frac{f_2^6f_3^6}{f_1^{10}f_6^2} \pmod 9 \tag{4.29}$$

and

$$\sum_{n=1}^{\infty} B_3(6n+4)q^n\equiv 6\frac{f_2^5f_3^3f_6}{f_1^9} \pmod 9. \tag{4.30}$$

The equation (4.28) can be written as

$$\sum_{n=1}^{\infty} B_3(6n)q^n\equiv 3q\frac{f_2f_6^2f_{18}}{f_1^2f_3^2} \pmod 9. \tag{4.31}$$

In view of the congruences (4.27) and (4.31), we get

$$B_3(6n)\equiv B_3(2n) \pmod 9. \tag{4.32}$$

Using the above relation and by induction on α, we arrive at (4.17).

The congruence (4.29) reduces to

$$\sum_{n=1}^{\infty} B_3(6n+2)q^n\equiv 3\frac{f_3^3}{f_1} \pmod 9. \tag{4.33}$$

Employing (2.12) into (4.33), we obtain

$$\sum_{n=1}^{\infty} B_3(12n+2)q^n\equiv 3f_1f_3 \pmod 9 \tag{4.34}$$

and

$$\sum_{n=1}^{\infty} B_3(12n+8)q^n\equiv 3\frac{f_6^3}{f_2} \pmod 9. \tag{4.35}$$

Extracting the terms involving q^{2n+1} from the equation (4.35), we get (4.16).

The equation (4.35) implies

$$\sum_{n=1}^{\infty} B_3\left(24n+8\right)q^n \equiv 3\frac{f_3^3}{f_1} \pmod 9. \tag{4.36}$$

In view of the congruences (4.33) and (4.36), we find that

$$B_3(24n+8) \equiv B_3(6n+2) \pmod 9. \tag{4.37}$$

Using the above relation and by induction on α, we arrive at (4.18).

The congruence (4.34) is $\beta = 0$ case of (4.19). Suppose that the congruence (4.19) is true for $\beta \geq 0$ and using (2.13) into (4.19), we get

$$\sum_{n=1}^{\infty} B_3\left(12\cdot 5^{2\beta+1}n + 2\cdot 5^{2\beta+2}\right)q^n \equiv 3f_5 f_{15} \pmod 9, \tag{4.38}$$

which implies

$$\sum_{n=1}^{\infty} B_3\left(12\cdot 5^{2\beta+2}n + 2\cdot 5^{2\beta+2}\right)q^n \equiv 3f_1 f_3 \pmod 9, \tag{4.39}$$

which implies that the congruence (4.19) is true for $\beta+1$. So, by induction, the congruence (4.19) holds for all integer $\beta \geq 0$.

Employing (2.13) into (4.19) and then collecting the coefficients of q^{5n+4} from both sides of the resultant equation, we obtain (4.20).

From the equation (4.20), we arrive at (4.21).

The equation (4.30) reduces to

$$\sum_{n=1}^{\infty} B_3\left(6n+4\right)q^n \equiv 6\frac{f_6^3}{f_2} \pmod 9. \tag{4.40}$$

Extracting the terms involving q^{2n+1} from both sides of the above equation, we get (4.15).

The equation (4.40) implies

$$\sum_{n=1}^{\infty} B_3\left(12n+4\right)q^n \equiv 6\frac{f_3^3}{f_1} \pmod 9. \tag{4.41}$$

The rest of the proofs of the identities (4.22)-(4.25) are similar to the proofs of the identities (4.18)-(4.21). So, we omit the details. □

5 Congruences modulo 3 and 9 for $B_9(n)$

Theorem 5.1. *For all $n \geq 0$ and $\alpha \geq 0$, we have*

$$B_9(3n+1) \equiv 0 \pmod 3, \tag{5.1}$$

$$B_9(3n+2) \equiv 0 \pmod 3, \tag{5.2}$$

$$B_9(9n+6) \equiv 0 \pmod 9, \tag{5.3}$$

$$B_9(3^{\alpha+3}n) \equiv B_9(9n) \pmod 9. \tag{5.4}$$

Proof. Letting $\ell = 9$ in (1.6), we find that

$$\sum_{n=0}^{\infty} B_9\left(n\right)q^n = \frac{f_6^3 f_9^3 f_{18}^3 f_{27}^3}{f_1^3 f_2^3 f_3^3 f_{54}^3}. \tag{5.5}$$

Invoking (2.17) into (5.5), we see that

$$\sum_{n=0}^{\infty} B_9(n)\, q^n \equiv \frac{f_1^6 f_6^3 f_9^3 f_{18}^3 f_{27}^3}{f_2^3 f_3^6 f_{54}^3} \pmod 9. \tag{5.6}$$

Using (2.3) into (5.6), we obtain

$$\sum_{n=0}^{\infty} B_9(3n)\, q^n \equiv \frac{f_1^3 f_2^3 f_3^6 f_9^3}{f_{18}^3} + q f_1^6 f_3^6 \pmod 9, \tag{5.7}$$

$$\sum_{n=0}^{\infty} B_9(3n+1)\, q^n \equiv 3\frac{f_2^2 f_3^2 f_9^4}{f_1^2 f_{18}^2} \pmod 9 \tag{5.8}$$

and

$$\sum_{n=0}^{\infty} B_9(3n+2)\, q^n \equiv 3\frac{f_2 f_3^2 f_9^3}{f_1 f_{18}} \pmod 9. \tag{5.9}$$

From the equations (5.8) and (5.9), we arrive at (5.1) and (5.2) respectively.
Employing (2.10) and (2.11) into (5.7), we obtain (5.3) and

$$\sum_{n=0}^{\infty} B_9(9n)\, q^n \equiv \frac{f_1^3 f_2^3 f_3^{15}}{f_{18}^3} + 2q f_1^6 f_3^6 + q^2 f_2^6 f_6^6 \pmod 9. \tag{5.10}$$

Again, using (2.10) and (2.11) into (5.10) and then collecting the coefficients of q^{3n} from both sides, we get

$$\sum_{n=0}^{\infty} B_9(27n)\, q^n \equiv \frac{f_1^3 f_2^3 f_3^{15}}{f_{18}^3} + 2q f_1^6 f_3^6 + q^2 f_2^6 f_6^6 \pmod 9. \tag{5.11}$$

In view of the congruences (5.10) and (5.11), we see that

$$B_9(27n) \equiv B_9(9n) \pmod 9. \tag{5.12}$$

Using the above relation and by induction on α, we arrive at (5.4). □

References

[1] G. E. Andrews, R. P. Lewis and J. Lovejoy, Partitions with designated summands, *Acta Arith.* **105**, 51–66 (2002).

[2] N. D. Baruah and K. K. Ojah, Partitions with designated summands in which all parts are odd, *Integers* **15**, #A9 (2015).

[3] B. C. Berndt, *Ramanujan's Notebooks Part III*, Springer-Verlag, New York (1991).

[4] W. Y. C. Chen, K. Q. Ji, H. T. Jin and E. Y. Y. Shen, On the number of partitions with designated summands, *J. Number Theory* **133**, 2929–2938 (2013).

[5] M. D. Hirschhorn, F. Garvan and J. Borwein, Cubic analogues of the Jacobian cubic theta function $\theta(z, q)$, *Canad. J. Math.* **45**, 673–694 (1993).

[6] M. D. Hirschhorn and J. A. Sellers, A congruence modulo 3 for partitions into distinct non-multiples of four, *J. Integer Seq.* **17**, Article 14.9.6 (2014).

[7] M. D. Hirschhorn and J. A. Sellers, Arithmetic properties of partitions with odd distinct, *Ramanujan J.* **22**, 273–284 (2010).

[8] M. S. Mahadeva Naika and D. S. Gireesh, Congruences for 3-regular partitions with designated summands, *Integers* **16**, #A25 (2016).

[9] M. S. Mahadeva Naika, Harishkumar T and Y. Veeranna, On $(3, 4)$-regular bipartitions with designated summands, *Proc. Jang. Math. Soc.* **23 (4)**, 465–478 (2020).

[10] M. S. Mahadeva Naika, B. Hemanthkumar and H. S. Sumanth Bharadwaj, Congruences modulo small powers of 2 and 3 for partitions into odd designated summands, *J. Integer Seq.* **20**, Article 17.4.3 (2017).

[11] M. S. Mahadeva Naika and S. Shivaprasada Nayaka, Congruences for $(2, 3)$-regular partition with designated summands, *Note di Mate.* **36 (2)**, 99–123 (2016).

[12] M. S. Mahadeva Naika and S. Shivaprasada Nayaka, Arithmetic properties of 3-regular bipartitions with designated summands, *Matematicki Vesnik* **69 (3)**, 192–206 (2017).

[13] M. S. Mahadeva Naika and C. Shivashankar, Arithmetic properties of bipartitions with designated summands, *Bol. Soc. Mat. Mex.* **24 (1)**, 37–60 (2018).

[14] S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge (1927), reprinted by Chelsea, New York (1962), reprinted by the American Mathematical Society, Providence, RI (2000).

[15] E. X. W. Xia, Arithmetic properties of partitions with designated summands, *J. Number Theory* **159**, 160–175 (2016).

[16] E. X. W. Xia and O. X. M. Yao, Analogues of Ramanujan's partition identities, *Ramanujan J.* **31**, 373–396 (2013).

[17] E. X. W. Xia and O. X. M. Yao, Some modular relations for the Göllnitz Gordon functions by an even-odd method, *J. Math. Anal. Appl.* **387**, 126–138 (2012).

Author information

M. S. Mahadeva Naika, Department of Mathematics, Central College Campus, Bengaluru City University, Bengaluru-560 001, Karnataka, India.
E-mail: `msmnaika@rediffmail.com`

Harishkumar T, Department of Mathematics, Central College Campus, Bangalore University, Bengaluru-560 001, Karnataka, India.
E-mail: `harishhaf@gmail.com`

Received: June 30, 2020.
Accepted : November 21, 2020.