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Z-ElEMENTS AND z_j-ElEMENTS IN MULTIPLICATIVE LATTICES.

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Abstract. In this paper we introduced and studied the concepts of z-elements and z_j-elements as a generalization of a z-ideal and z_j ideal. Various properties and characterizations of z-elements and z_j-elements are obtained. It is shown that the Jacobson radical which is meet of all maximal elements is a z-element and is contained in every z-element.

1 Introduction

A multiplicative lattice L is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element $a \in L$ is called proper if $a < 1$. A proper element p of L is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. If $a \in L$, $b \in L$, $(a : b)$ is the join of all elements c in L such that $cb \leq a$. A proper element p of L is said to be primary if $ab \leq p$ implies $a \leq p$ or $b^n \leq p$ for some positive integer n. If $a \in L$, then $\sqrt{a}= \vee \{ x \in L_* / x^n \leq a, n \in Z_+ \}$. An element $a \in L$ is called a radical element if$a = \sqrt{a}$. Radical element is also called as a semi-prime element. An element $a \in L$ is called compact if $a \leq \bigvee_\alpha b_\alpha$ implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} ... \vee b_{\alpha_n}$ for some finite subset $\{\alpha_1, \alpha_2 ... \alpha_n\}$. An element m of L is called maximal element if $m \nleq x$ for any other $x \in L$. Throughout this paper, L denotes a compactly generated multiplicative lattice with 1 compact in which every finite product of compact element is compact. We shall denote by L_*, the set of compact elements of L. A nonempty subset of L_* is called a filter if the following conditions are satisfied.

i)$x, y \in F$ implies $xy \in F$

ii)$x \in F$, $x \leq y$ implies $y \in F$

Let $F(L_*)$ denotes a set of all filters of L. For a nonempty subset $\{F_\alpha\} \subseteq F(L_*)$, define $\Cup F_\alpha$= $\{x \geq f_1 f_2 ... f_n, f_i \in F_{\alpha_i}$ for some $i = 1, 2...n\}$. Then it is observed that, $F(L_*) = \langle F(L_*), \Cup, \cap \rangle$ is a complete distributive lattice with $\Cup$ as the supremum and the set theoretic $\cap$ as the infimum. For $a \in L_*$ the smallest filter containing a is denoted by $[a)$ and it is given by $[a)$= $\{x \in L_* / x \geq a^n$ for some non-negative integer $n\}$. For a filter $F \in F(L_*)$ we denote,

$0_F = \vee \{x \in L_* / xs = 0$, for some $s \in F\}$.

A lattice L is called semi-complemented if for any element $a \in L$, $(a \neq 1)$ there exists a non-zero element $b \in L$ such that $ab = 0$. A lattice L is said to be dual semi-complemented if for every element $a \in L$, $(a \neq 0)$ there exists a non-zero element $b \neq 1$ such that $a \vee b = 1$.

A lattice L with 0 is section semi-complemented if $a \nleq b$ then there exists $c \in L$ such that $0 < c \leq a$ and $b \wedge c$ =0.

For all these definitions one can refer R.P.Dilworth[11], F.Alarcon, Jayaram and Anderson [7].

2 Z-Elements in multiplicative lattices.

The concept of z-ideals was first introduced by Kohls[5] which played an important role in studying the ideal theory of $C(X)$, the ring of continuous real valued functions on compactly regular Hausdroff space X: See Gillman and M. Jerison [8]. Mason [9]studied z-ideals in general commutative rings. He proved that maximal ideals , minimal prime ideals and some other deals in commutative rings are z-ideals. As a generalization of z-ideals the concept of z^0-ideals

is introduced and studied in $C(X)$. In [4] Huijsman and De-Pagter studied z^0-ideals under the name of d-ideals in Riesz spaces. Speed [13]introduced and studied the concept of Baer ideals in commutative Baer rings which are essentially z^0-ideals (equivalently d-ideals) and characterized regular rings and quasi regular rings. Jayaram [6], Anderson, Jayaram and Phiri [3]defined the concept of Baer ideals for lattice and multiplicative lattices respectively. The analogous concept of z-ideals is introduced by Kavishwar and Joshi[12].
We introduce the concept of z-elements in compactly generated multiplicative lattices in which 1 is compact and every finite product of compact elements is compact.
Let L denote a compactly generated multiplicative lattice with largest element 1 compact in which every finite product of compact elements is compact. Let $\mu = Max(L)$ denote the set of all maximal elements in a lattice L and $\mu(a) = \{m \in \mu | a \leq m\}$ for $a \in L$.For $a \in L$ the meet of all maximal elements in L containing a is denoted by M_a i.e. $M_a = \wedge(\mu(a)) = \wedge\{m \in \mu | a \leq m\}$.
We introduce the concept of z-element in multiplicative lattices which is a generalization of z-ideals in a commutative ring.
Now we prove the properties of z-elements in multiplicative lattices.

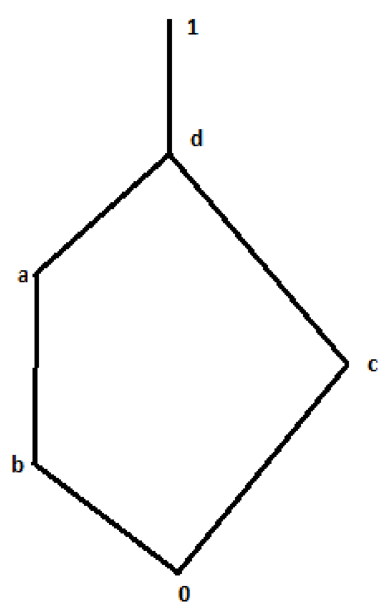
Definition 2.1. An element h of L is called a z-element if $\mu(b) \subseteq \mu(a)$ and $b \leq h$ implies $a \leq h$.

Lemma 2.2. *Every maximal element in L is a z- element.*

Proof. Let m be a maximal element of L and $\mu(a) \subseteq \mu(b)$, $a \leq m$. Since $a \leq m$ we have $m \in \mu(a)$. But $\mu(a) \subseteq \mu(b)$ implies $m \in \mu(b)$. Thus $b \leq m$. Hence m is a z- element. □

Lemma 2.3. *Let m be a unique maximal element of L such that $h \lneqq m$, then h is not a z-element.*

Proof. Since $h \lneqq m$ there exists $x \leq m$ such that $x \nleq h$. Let $i \leq h$. Since m is a unique maximal element, we have $\mu(i) = \mu(x)$ and $x \nleq h$. Thus $\mu(i) \subseteq \mu(x), i \leq h$ but $x \nleq h$. Hence h is not a z-element. □



[Diagram (1)]

Ex.1)- Consider the lattice shown in diagram (1) with the trivial multiplication $x.y = 0 = y.x$, for each $x \neq 1 \neq y$ and $x.1 = x = 1.x$ for every $x \in L$. Then it is easy to show that L is a

multiplicative lattice.
d being a maximal is a z-element.
2)In the above diagram, a is not a z-element. Because $\mu(b) \subseteq \mu(c), b \leq a$,$\mu(b) = \mu(c) = \{d\}$ but $c \nleq a$.
3)Let $R = Z$. Then $L(Z)$ the lattice of ideals of Z is a multiplicative lattice. $< 4 >$ is not a z-element in $L(Z)$. Obviously $< 2 >, < 3 >, < 5 >$ being maximal elements are z-elements.

Here is a characterization of dual semi complemented lattices in terms of maximal elements.

Lemma 2.4. *A lattice L is dual semi complemented if and only if* $\wedge\{m|m \in Max(L)\} = 0$*.*

Proof. Let L be a dual semi complemented lattice. Suppose $\wedge\{m|m \in Max(L)\} \neq 0$, $a \leq \wedge\{m|m \in Max(L)\}$ and $a \neq 0$. Since L is dual semi complemented there exists $b \neq 1$ such that $a \vee b = 1$. This implies that $b \nleq \wedge\{m|m \in Max(L)\}$. Since $b \neq 1$, there exists a maximal element m_1 such that $b \leq m_1$. But $a \leq \wedge\{m|m \in Max(L)\}$ implies $a \leq m_1$. Thus $1 = a \vee b \leq m_1$, a contradiction. Hence $\wedge\{m|m \in Max(L)\} = 0$. Conversely, suppose $\wedge\{m|m \in Max(L)\} = 0$. We show that L is dual semi complemented. Let $0 \neq a \in L$. Since $a \neq 0$, we have $a \nleq \wedge\{m|m \in Max(L)\} = 0$. Then there exists a maximal element m_1 such that $a \nleq m_1$. Therefore $a \vee m_1 = 1$. Hence m is dual semi complemented. □

Using the above characterization we show that the least element of L is a z-element.

Lemma 2.5. *Let L be a dual semi-complemented lattice. Then 0 is a z- element.*

Proof. Let $\mu(a) \subseteq \mu(b)$ and $a \leq 0$. Then $a = 0$ and $\mu(a) = \mu(0) = max(L)$. So $\mu(b) = max(L)$ and $b \leq \wedge\{m|m \in Max(L)\} = 0$ implies $b = 0$. Hence 0 is a z- element. □

Lemma 2.6. *Let $a, b \in L$. Then the following statements hold:- 1)$M_{a\wedge b}$ = $M_a \wedge M_b$ = M_{ab}. 2)$\mu(b) \subseteq \mu(a)$ then $\mu(b \wedge c) \subseteq \mu(a \wedge c)$ and $\mu(bc) \leq \mu(ac)$ for any $c \in L$.*

Proof. 1)We have, $M_{(a\wedge b)} = \wedge\{m \in \mu|(a \wedge b) \leq m\}$. Let $x \leq M_{(a\wedge b)}$ and $x \nleq M_a \wedge M_b$. Then $x \nleq M_a$ or $x \nleq M_b$. Suppose $x \nleq M_b = \wedge(\mu(b)) = \wedge\{m \in \mu|b \leq m\}$. Without loss of generality assume that, $x \nleq m_1$ for some maximal element m_1 such that $b \leq m_1$. But then, $x \leq M_{(a\wedge b)}$ $\leq m_1$, a contradiction. Hence $x \leq M_a \wedge M_b$ and $M_{(a\wedge b)} \leq M_a \wedge M_b$. Let $x \leq M_a \wedge M_b$ and $x \nleq M_{(a\wedge b)}$. Then there exists a maximal element m_2 such that $(a \wedge b) \leq m_2$ but $x \nleq m_2$. Since m_2 is a maximal element, m_2 is prime and $ab \leq m_2$ implies $a \leq m_2$ or $b \leq m_2$. Without loss of generality, assume that $a \leq m_2$. Then $x \leq M_a \leq m_2$. This contradicts the fact that $x \nleq m_2$. Hence $M_a \wedge M_b \leq M_{(a\wedge b)}$ and $M_{(a\wedge b)} = M_a \wedge M_b$.
To prove that $M_a \wedge M_b = M_{ab}$. Let $x \leq M_{ab}$ and $x \nleq M_a \wedge M_b$. Then $x \nleq M_a$ or $x \nleq M_b$. Suppose $x \nleq M_a$. Then there exists a maximal element m_1 such that $x \nleq m_1$ but $a \leq m_1$. Then $x \leq M_{ab} \leq m_1$, a contradiction. So $x \leq M_a \wedge M_b$and $M_{ab} \leq M_a \wedge M_b$. Let $x \leq M_a \wedge M_b$, but $x \nleq M_{ab}$. This implies there exists a maximal element m_2 such that $ab \leq m_2$ but $x \nleq m_2$. Since m_2 is prime, $ab \leq m_2$ implies $a \leq m_2$ or $b \leq m_2$. If $a \leq m_2$, $x \leq M_a \leq m_2$, a contradiction and hence $M_a \wedge M_b = M_{ab}$.
2) To prove that if $\mu(b) \subseteq \mu(a)$ then $\mu(b \wedge c) \subseteq \mu(a \wedge c)$ for any $c \in L$. Let m be a maximal element such that $m \in \mu(b \wedge c)$. Since m is also a prime element, $bc \leq m$ implies $b \leq m$ or $c \leq m$. If $c \leq m$, $a \wedge c \leq m$ and $m \in \mu(a \wedge c)$. If $b \leq m$, $m \in \mu(b) \subseteq \mu(a)$ and we have, $a \leq m$. This shows that $(a \wedge c) \leq m$ and $m \in \mu(a \wedge c)$. Thus $\mu(b \wedge c) \subseteq \mu(a \wedge c)$ for any $c \in L$.
To prove that $\mu(bc) \leq \mu(ac)$.Let $m \in \mu(bc)$. Since m is prime $bc \leq m$ implies $b \leq m$ or $a \leq m$. If $c \leq m$, $ac \leq m$ and $m \in \mu(ac)$. If $b \leq m$, $m \in \mu(b) \subseteq \mu(a)$ and we have $a \leq m$ and hence $ac \leq m$ so that $m \in \mu(ac)$. Thus $\mu(bc) \leq \mu(ac)$. □

Lemma 2.7. *Every element i is contained in the least z-element namely, $i_z = \wedge\{j \geq i|$ j is z-element $\}$ is the smallest z-element containing i.*

Proof. Let $\mu(b) \subseteq \mu(a)$ and $b \leq i_z$. Let j_1 be an arbitrary z-element such that $j_1 \geq i$. Since $b \leq j_1$, j_1 is a z-element and $\mu(b) \subseteq \mu(a)$ we have $a \leq j_1$. Thus $a \leq \wedge$ $\{j|$ j is a z-element such that $j \geq i\}$ = i_z. Hence i_z is a z-element. Now let j be any z-element containing i. Let $x \leq i_z$. Then clearly, $x \leq j$ and hence $i_z \leq j$. □

Lemma 2.8. *Let L be a multiplicative lattice and i,j be any two elements of L. Then the following statements hold:- 1)If $i \leq j$, then $i_z \leq j_z$*
2) $(i_z)_z$ = i_z.

Proof.1)Let $i \leq j$ and $x \leq i_z$= $\wedge\{k|$ k is a z-element and $k \geq i\}$.
If $x \nleq j_z$ =$\wedge\{q|$ q is a z-element and $q \geq j\}$ then there exist z-element q_1 such that $x \nleq q_1$ and $j \leq q_1$. This together with $i \leq j$, implies $i \leq q_1$. But $x \leq i_z$ and $i \leq q_1$ for some z-element q_1 gives $i_z \leq q_1$. Therefore $x \leq q_1$, a contradiction. Hence $i_z \leq j_z$.
2)Clearly, $i_z \leq (i_z)_z$. Let $x \leq (i_z)_z$ = $\wedge\{q|q \geq i_z$ q is a z-element $\}$. We know that i_z is a z-element and $i_z \leq i_z$. But $(i_z)_z$ is the least z element containing i_z. So $(i_z)_z \leq i_z$ and we have i_z = $(i_z)_z$. □

Lemma 2.9. *Let L be a multiplicative lattice and $a, b \in L$, then $a \leq M_b$ if and only if $M_a \leq M_b$ if and only if $\mu(b) \subseteq \mu(a)$.*

Proof. Let $M_a \leq M_b$. Obviously, $a \leq M_a$ implies $a \leq M_b$. Conversely $a \leq M_b$ implies $a \leq \wedge\{ m \in \mu | b \leq m\}$.
Let $x \leq M_a$ = $\wedge\{ m|a \leq m \in \mu\}$. Let m_1 be any maximal element with $b \leq m_1$. Then $a \leq m_1$. This gives $x \leq m_1$ and hence $x \leq M_b$. Thus $M_a \leq M_b$. Obviously, $M_a \leq M_b$ if and only if $\mu(b) \subseteq \mu(a)$. □

In the next result we characterize z-elements.

Lemma 2.10. *Let i be an element of L, then the following statements are equivalent:-*
1)i is a z-element.
2)$\mu(a) = \mu(b)$ and $b \leq i$ implies $a \leq i$.
3)$M_a \leq i$, for all $a \leq i$.
4)$M_b \leq M_a, a \leq i$ implies $b \leq i$.

Proof. (1) implies (2) is obvious.
(2)implies (3)
Let $x \leq M_a$. Then by lemma (2.9), $M_x \leq M_a$. Hence M_x = $M_x \wedge M_a = M_{a \wedge x}$(by lemma (2.6)). This gives $\mu_x = \mu(a \wedge x)$. If $a \leq i$, then $(a \wedge x) \leq i$. But by (2) $x \leq i$.
(3)implies (4)
Assume that $M_a \leq i$, for all $a \leq i$. We assume that $M_b \leq M_a, a \leq i$. Then $M_a \leq i$ and $b \leq M_b \leq M_a$ implies $b \leq i$.
(4)implies (1)
We assume that $M_b \leq M_a$ and $a \leq i$ implies $b \leq i$. We show that i is a z-element. Let $\mu(b) \subseteq \mu(a)$ and $b \leq i$. Then $M_a \leq M_b$ and we have $b \leq i$, by hypothesis $a \leq i$. Hence i is a z-element. □

Separation lemma for z-element:-

Such type of Separation lemma is obtained by Anderson[2] and for z-ideals in lattices by Kavishwar and Joshi[12].

Lemma 2.11. *Let L be a multiplicative lattice. Suppose $t \nleq i$, for all $t \in S$, where i is z-element and S is multiplicatively closed in L. Then there exists a prime z-element p such that $i \leq p$ and $t \nleq p$, for all $t \in S$.*

Proof. Let $\mathscr{F}$ = $\{j|$ j a z-element such that $i \leq j$ and $t \nleq j, \forall t \in S\}$. Then $\mathscr{F} \neq \emptyset$, since atleast $i \in \mathscr{F}$ and $\mathscr{F}$ is a poset with respect to $\leq$. Let $\mathscr{C}$ be a chain in $\mathscr{F}$ and m = $\vee\{j|j \in \mathscr{C}\}$. We show

that m is a z-element.
Assume that $\mu(a) \subseteq \mu(b)$ and $a \leq m$. Then $a \leq j$ for some $j \in \mathscr{C}$. But j is a z-element and $\mu(a) \subseteq \mu(b)$ and $a \leq j$ implies $b \leq j$ and hence $b \leq m$. Hence m is a z-element. Obviously $j \leq m$ for all $j \in \mathscr{C}$. That is m is an upper bound of $\mathscr{C}$ and $m \in \mathscr{F}$. Thus $\mathscr{F}$ is a poset in which every chain has an upper bound in $\mathscr{F}$. Hence by Zorn's lemma, there exists a maximal element $p \in \mathscr{F}$ and clearly p is a z-element such that $i \leq p$ and $t \nleq p$, $\forall t \in S$. We claim that p is a prime element. Let $ab \leq p$ and $a \nleq p$, $b \nleq p$. Then $(p \vee a) > p$, $(p \vee b) > p$. Since p is a maximal element with respect to $t \nleq p, \forall t \in S$, it follows that there exists $t_1, t_2 \in S$ such that $t_1 \leq (p \vee a)$ and $t_2 \leq (p \vee b)$. Since S is a multiplicatively closed set $t_1 t_2 \in S$. Also $t_1 t_2 \leq (p \vee a)(p \vee b)$ $\leq (p \vee ab) \leq p$. This contradicts the fact that $t \nleq p$, $\forall t \in S$. Hence p is a prime z-element. □

Lemma 2.12. *Let L be a distributive multiplicative lattice. Then every strongly irreducible element is a z-element if and only if every element is a z-element.*

Proof. Obviously, if every element is a z-element every strongly irreducible element is a z-element. Conversely, suppose every strongly irreducible element is a z-element. Let i be any element, $\mu(b) \subseteq \mu(a), b \leq i$. Suppose $a \nleq i$. Let $\mathscr{P} = \{x | i \leq x, a \nleq x\}$. Then $\mathscr{P}$ is a partially ordered set with respect to $\leq$ and $\mathscr{P} \neq \phi$, since $i \in \mathscr{P}$. Let $\mathscr{C}$ be a chain in $\mathscr{P}$. Then $b = \vee\{y | y \in \mathscr{C}\}$ is an upper bound of $\mathscr{C}$ in $\mathscr{P}$. Hence by Zorn's lemma $\mathscr{P}$ has a maximal element p such that $i \leq p$ and $a \nleq p$. We show that p is strongly irreducible. Suppose $x \wedge y \leq p$, $x \nleq p$, $y \nleq p$. Since p is a maximal with respect to $a \nleq p$ we have $p \vee x > a$, $p \vee y > a$. Hence $a \leq (p \vee x) \wedge (p \vee y)$ that is $a \leq p$, a contradiction. Hence p is a strongly irreducible element. Clearly $b \leq p$ and $\mu(b) \subseteq \mu(a)$ implies $a \leq p$, since a strongly irreducible element p is a z-element. This is a contradiction. Hence $a \leq i$ and i is a z-element. □

Lemma 2.13. *Let L be SSC lattice such that* $\wedge\{m | m \in max(L)\}$ *= 0. Then every element is a z-element.*

Proof. Let i be an element of SSC lattice L. Let $\mu(b) \subseteq \mu(a)$ and $b \leq i$. Suppose $a \nleq i$. Then there exists $c \neq 0$ such that $c \leq a$ and $c \wedge i$ = 0. This gives $b \wedge c$ = 0. Thus max (L) = $\mu(b \wedge c) \subseteq \mu(a \wedge c)$, by lemma (2.6). Then $c = a \wedge c \leq \wedge\{m |\ m \in max(L)\}$ = 0. Therefore $c = 0$, a contradiction. Thus $a \leq i$ and hence i is a z-element. □

Let a be an element of L. Then $(0 : a) = \vee\{x \in L | xa = 0\}$. In this case $(0 : a)$ is also denoted by $a^{\perp}$, i.e. $a^{\perp} = \vee\{x \in L | xa = 0\}$.
We shall denote $(0 : a)$ by $a^{\perp}$ and obtained its property in terms of maximal elements.

Lemma 2.14. *Let L be a dual semi complemented lattice.*
Then $a^{\perp} = \wedge\{m \in max(L) | a \nleq m\}$ *for any* $a \in L$.

Proof. Let $x \leq a^{\perp}$. Then $ax = 0$. Let $m \in max(L)$. As every maximal element is prime it follows that m is a prime element. If $a \nleq m$ then $ax = 0 \leq m$ implies $x \leq m$. Thus $a^{\perp} \leq \wedge\{m \in max(L) | a \nleq m\}$. Conversely, suppose $x \leq \wedge\{m \in max(L) | a \nleq m\}$ and $x \nleq a^{\perp}$. Hence $ax \neq 0$. This shows that $ax \nleq \wedge\{m \in max(L) | a \nleq m\}$ = 0, by lemma (2.4). Therefore there exists a maximal element m_1 such that $ax \nleq m_1$, where $a \nleq m_1$ with $x \nleq m_1$. This contradicts the fact that $x \leq \wedge\{m \in max(L) | a \nleq m\}$. Hence $x \leq a^{\perp}$ and $\wedge\{m \in max(L) | a \nleq m\} \leq a^{\perp}$. Therefore $a^{\perp} = \wedge\{m \in max(L) | a \nleq m\}$ for any $a \in L$. □

We generalize the following concepts in lattices for multiplicative lattices.

Definition(2.14 (a)):- An element $i \in L$ is said to be **closed element** if $i^{\perp\perp} = i$.
Definition(2.14 (b)):- An element i of a lattice L is called a **zero element** if there exists a proper filter F such that $i = \vee\{F^0\}$ where $F^0 = \{x \in L | xy = 0, \text{for some } y \in F\}$.
Definition(2.14 (c)):-For an element i and a prime element p of a lattice L we define $i(p)$ as follows, $i(p) = \vee\{x \in L | xy \leq i, \text{for some } y \nleq p\}$. If $i = 0$, then $i(p)$ is denoted by $0(p)$.
Definition(2.14 (d)):-An element i of a lattice L is called **dense** if $i^{\perp} = 0$.
Definition(2.14 (e)):-An element i of a lattice L is called **non-dense** if $i^{\perp} \neq 0$.

Under which condition an element is a z-element is proved in the next result.

Lemma 2.15. *Let L be a lattice without zero divisors such that $\wedge\{m|m \in max(L)\} = 0$. If i is an element of L satisfying any one of the following conditions then i is a z-element.*
1) If i is a non-dense prime element.
2) If i is a closed element.
3) If i is a zero element.
4) If $i = 0(p)$ for any prime element p.
5) If $i = a^{\perp}$ for any element $a \in L$.

Proof.1) Let i be a non-dense element and $\mu(b) \subseteq \mu(a), b \leq i$. Since i is a non-dense element, $i^{\perp} = \vee\{x \in L|ix = 0\} \neq 0$. Then there exists a non-zero element $x \leq i^{\perp}$ such that $ix = 0$. In particular, $xb = 0(b \leq i)$. Since $\mu(b) \subseteq \mu(a)$, by lemma (2.6), we have, $max(L) = \mu(bx) \subseteq \mu(ax)$. Thus $(ax) \leq m$, for all $m \in max(L)$. Hence $ax = 0$ as $\wedge\{m|m \in max(L) = 0$. This implies that $(ax) \leq i$ and since i is a prime element $a \leq i$ or $x \leq i$. If $x \leq i$ then $x \leq i^{\perp}$ implies $x^2 \leq i.i^{\perp} = 0$ i.e. $x^2 = 0, (x \neq 0)$. This contradicts the fact that L has no divisors of zero. Hence $ax \leq i \Rightarrow a \leq i$. Thus i is a z-element.
2) Let i be a closed element i.e. $i = i^{\perp\perp}$ and $\mu(b) \subseteq \mu(a), b \leq i$. Now $b \leq i = i^{\perp\perp} \Rightarrow bx \leq b \wedge x = 0$, for all $x \leq i^{\perp} = \vee\{y|iy = 0\}$. Since $\mu(b) \subseteq \mu(a)$, we have $max(L) = \mu(by) \subseteq \mu(ay)$ $for y \leq i^{\perp}$. Hence $(ay) = \wedge\{m \in max(L)\} = 0$. Therefore $ay = 0$ for all $y \leq i^{\perp}$. Then $ai^{\perp} = 0$ and hence $a \leq i^{\perp\perp} = i$, since i is a closed element. Thus $\mu(b) \subseteq \mu(a), b \leq i$ implies $a \leq i$. So i is a z-element.
3) Let i be a zero element. Then $i = \vee\{F^0\}$ = $\vee\{x \in L|xy = 0$ for some $y \in F\}$, for some proper filter F. Let $\mu(b) \subseteq \mu(a)$ and $b \leq i$. Since $b \leq i$. We have, $by = 0$ for some $y \in F$. Now $\mu(b) \subseteq \mu(a) \Rightarrow max(L) = \mu(by) \subseteq \mu(ay)$, by (2.6). Hence $ay \leq m$, for all $m \in max(L)$. Thus $ay \leq \wedge\{m|m \in max(L)\} = 0$. Hence $ay = 0$ for some $y \in F$. Thus $a \leq \vee\{F^0\} = i$. Hence i is a z-element.
4) Suppose $i = 0(p)$ for some prime element p, where $0(p) = \vee\{x \in L|xy = 0$, for some $y \nleq p\}$. Then $F = L - (p]$ is a filter. Also $i = \vee\{x \in L|xy = 0$, for some $y \in F\}$ = F^0. Now the result follows by (3).
5)Let $i = a^{\perp} = \vee\{x|ax = 0\}$, $\mu(b) \subseteq \mu(a), b \leq i$. Now $b \leq i = a^{\perp}$ implies $ba = 0$. So $bc = 0$ for all $c \leq a$. Since $\mu(b) \subseteq \mu(a)$ we have $\mu(bc) \subseteq \mu(ac)$[by (2.6)]. But $bc = 0$ implies $Max(L) = \mu(bc) \subseteq \mu(ac)$. This gives, $ac \leq \wedge\{m|m \in Max(L)\} = 0$ and hence $ac = 0$ when $c \leq a$. Hence $aa = 0$ and $a \leq a^{\perp} = i$. Therefore i is a z-element. □

3 z_j-Elements in multiplicative lattices.

Kavishwar and Joshi have studied z_j-ideals on the lines of Alibad, Azarpanah and Taherifar[1]. We extend this concept to z_j-elements in compactly generated multiplicative lattices.

Definition 3.1. Let i and j be the two elements of L. The element i is said to be a z_j-element if $M_a \wedge j \leq i$, for all $a \leq i$ where $M_a = \wedge\{m|a \leq m\}$.
Ex. 1)From the diagram (1) b is a z_j-element for $j = c$.
2) a is not a z-element but a is a z_j-element for $j = b$.

Note:- Clearly if $j \leq i$ then i is always a z_j-element and hence an element i is always a z_i-element. Further if $j = 1$ then z_1 element is nothing but a z-element. □

Lemma 3.2. *If i is a z-element then i is a z_j-element for any element j of a lattice L.*

Proof. Let $a \leq i$ and $x \leq M_a \wedge j$. Then $x \leq M_a$ implies $M_x \leq M_a$. Since i is a z-element, $\mu(a) \subseteq \mu(x)$ and $a \leq i$ implies $x \leq i$. Thus $M_a \wedge j \leq i$, for all $a \leq i$. Hence i is a z_j-element. □

Definition 3.3. An element $x \in L$ is called semi primary if $\sqrt{x}$ is a prime element.
An element $a \in L$ is called semi prime if $\sqrt{a} = a$.

Lemma 3.4. *Let i be a semi prime element and j be any element of L. Then the following statements hold:-*
1) If i is a z_j-element (z-element) and p is a minimal prime containing i, then p is also a z_j-element (z-element).
2) A prime element p in L is a z_j-element if and only if p is either a z-element or $j \leq p$.

Proof. 1) Let p be minimal prime containing i and suppose $x \leq p$. We claim that $M_x \wedge j \leq p$. Since $x \leq p$ there exists $y \nleq p$ such that $x^n y \leq i$, for some integer $n \geq 1$. (See [10]). Since i is a z_j-element $M_{x^n y} \wedge j$ = $M_{x^n} \wedge M_y \wedge M_j \leq i \leq p$ by (2.6). Also note that $M_{x^n} = M_x$ for any positive integer n. Since $y \nleq p$, $M_y \nleq p$ and p is a prime element gives $M_x \wedge j \leq p$. Thus $M_x \wedge j \leq p$, for all $x \leq p$. Hence p is a z_j- element.
2) Let p be a prime z_j- element such that $j \nleq p$. Suppose $\mu(b) \subseteq \mu(a)$ and $b \leq p$. Since p is a z_j- element, we have $M_b \wedge j \leq p$. This together with $j \nleq p$ implies $M_b \leq p$. But $M_a \leq M_b$ gives $a \leq M_a \leq p$. Hence p is a z-element. Conversely assume that p is a z-element or $j \leq p$. Suppose $j \leq p$. Let $a \leq p$. Then $M_a \wedge j \leq p$. This holds for all $a \leq p$. Hence p is a z_j-element. Now suppose $j \nleq p$ and p is a z-element. By lemma (3.2), it follows that p is a z_j-element. □

Lemma 3.5. *Let i be a semi-prime element, j be any element and p,q be prime elements of L. Then the following statements hold:-*
1)If $i \wedge p$ is a z_j-element then either i or p is a z_j-element.
2) If $p \wedge q$ is a z_j-element and p and q are not comparable then p and q are z_j-elements.

Proof. 1) Let $i \wedge p$ be a z_j-element. If $i \leq p$, then clearly i is a z_j-element. Now suppose $i \nleq p$. Let $b \leq p$. Then there exists an element $a \leq i$ but $a \nleq p$. Hence $ab \leq p$, $ab \leq a \leq i$ implies $ab \leq i \wedge p$. Since $i \wedge p$ is a z_j-element. $ab \leq i \wedge p$ implies $M_{a \wedge b} \wedge j \leq i \wedge p$. By (2.6), $M_a \wedge M_b \wedge j \leq p$. Since p is prime and $M_a(M_b \wedge j) \leq p$ and $M_a \nleq p$, we have $M_b \wedge j \leq p$ for all $b \leq p$. Hence p is a z_j- element.
2) Suppose $p \wedge q$ is a z_j- element, where p and q are not comparable, so that $p \nleq q$ and $q \nleq p$. Now $p \nleq q$ implies q is a z_j- element (by 1) and $q \nleq p$ implies p is a z_j- element (by 1). □

Lemma 3.6. *Let i and j be two elements of L. Then* $(i \wedge j)_z$ = $i_z \wedge j_z$.

Proof. Clearly $i_z \wedge j_z$ is an element containing $i \wedge j$. Since i_z is the smallest z-element containing i and j_z is the smallest z-element containing j. It follows that $i_z \wedge j_z \geq (i \wedge j)$. To show that $i_z \wedge j_z$ is a z-element, let $\mu(b) \subseteq \mu(a)$ and $b \leq i_z \wedge j_z$ so $b \leq i_z$ $b \leq j_z$. Since i_z, j_z are z-elements, we have $a \leq i_z, a \leq j_z$. Hence $i_z \wedge j_z$ is a z-element and $i \wedge j \leq i_z \wedge j_z$. To prove that $(i \wedge j)_z$ = $i_z \wedge j_z$, it is enough to show that $i_z \wedge j_z$ is the smallest z-element containing $(i \wedge j)$. To see this let k be a z-element such that $(i \wedge j) \leq k$. If each element of L is a radical element i.e. $\sqrt{a} = a$ for each a in L then $\sqrt{k} = k$. In this case $k = \wedge\{p | p$ is minimal prime containing k$\}$. Since for each $p \in Min(k)$, we have $(i \wedge j) \leq p$, it follows that $i \leq p$ or $j \leq p$. By lemma (3.4) each $p \in Min(k)$ is a z-element. Using this fact along with i_z is the smallest z-element containing i, it follows that $i_z \wedge j_z \leq p$, for each $p \in Min(k)$. Therefore $(i_z \wedge j_z) \leq \wedge \{p | p \in Min(k)\} = k$. It follows that $i_z \wedge j_z$ is the smallest z-element containing $i \wedge j$. Hence $(i \wedge j)_z$= $i_z \wedge j_z$. □

Now we characterize z_j-elements in different ways.
We write $Min(i)$= the set of all minimal primes containing i.

Lemma 3.7. *Let i be a semi prime element of a lattice L and j be an element of L. Then the following statements are equivalent:-1)i is a z_j-element.*
2)$i_z \wedge j \leq i$ (equivalently $i_z \wedge j = i \wedge j$)
3) If there is a z-element k containing i, then $k \wedge j \leq i$.
4)For each $a \leq i$ and $b \leq j$,if $M_b \leq M_a$, then $b \leq i$.

Proof. (1)implies (2)
Let i be a semi prime z_j-element. Then $i = \wedge\{p|p$ is a minimal prime such that $i \leq p\}$. Hence $i_z = \{\wedge_{p\in Min(i)}p\}_z \leq (\wedge_{p\in Min(i)}p_z)$. By lemma (3.4) $p_z = p$ or $j \leq p$. Hence in any case we have, $i_z \wedge j \leq (\wedge_{p\in Min(i)}p_z) \wedge j$= $[\wedge_{p\in Min(i)}p] \wedge j = i \wedge j \leq i$.
(2) implies (3)
Assume that $i_z \wedge j \leq i$ (equivalently $i_z \wedge j = i \wedge j$). Take $k = i_z$ then i_z is the smallest z-element containing i and hence $i_z \wedge j \leq i$(by hypothesis).
(3)implies (4)
Assume that if there is a z- element k containing i then $k \wedge j \leq i$. Let $a \leq i$,$b \leq j$ and $M_b \leq M_a$. By (3) there exists a z-element k containing i such that $k \wedge j \leq i$. Then by lemma (2.10)$M_a \leq k$. Clearly , $b \leq M_b \leq M_a$. Hence $b \leq M_b \wedge j \leq k \wedge j \leq i$. Thus $b \leq i$.
(4) implies (1)
Assume that for each $a \leq i$ and $b \leq j$ if $M_b \leq M_a$ then $b \leq i$. Let $a \leq i$ and $x \leq M_a \wedge j$. Then by lemma (2.9) $x \leq M_a$ implies $M_x \leq M_a$. Now $a \leq i$, $x \leq j$ and $M_x \leq M_a$ implies $x \leq i$ (by assumption). Thus $M_a \wedge j \leq i$, for all $a \leq i$. Hence i is a z_j-element. □

Lemma 3.8. *The following statements hold in L:-*
1)If $i = i_1 \wedge i_2$, $j = j_1 \wedge j_2$ and i_1 is a z_{j_1}- element, i_2 is a z_{j_2}- element, then i is a z_j-element.
2) If $j \leq k$ and i is a z_k- element then i is also a z_j-element.
3) The meet of z_j-elements is a z_j-element and meet of z-elements is a z-element.
4) If $i \leq j$,i is a z_j-element and j is a z_k-element then i is a z_k-element.

Proof. 1) Let $c = i = i_1 \wedge i_2$. Since i_1, i_2 are z_{j_1} and z_{j_2}-elements respectively, we have $M_c \wedge j_1 \leq i_1$ and $M_c \wedge j_2 \leq i_2$. This gives $M_c \wedge j_1 \wedge j_2 \leq i_1 \wedge i_2$ for all $c \leq i_1 \wedge i_2$. This shows that i is a z_j-element.
2)Suppose $j \leq k$ and i is a z_k-element. Let $a \leq i$. Then $M_a \wedge k \leq i$(since i is a z_k-element.) Let $a \leq j$. Then $a \leq k$ and by hypothesis, $M_a \wedge k \leq i$. Then $M_a \wedge j \leq i$, for all $a \leq i$ and hence i is a $z_j - element$.
3)Let $i_k (k \in \triangle)$ be collection of all z_j-elements. Let $a \leq \wedge_{k\in\triangle} i_k = i$. Then $a \leq i_k$ for all $k \in \triangle$. We have $M_a \wedge j \leq i_k$, since each i_k is a z_j-element. Hence $M_a \wedge j \leq \wedge_{k\in\triangle} i_k = i$ for all $a \leq i$. Hence $i = \wedge_{k\in\triangle} i_k$ is a z_j-element. Let $h_i, i \in \triangle$ be the collection of z-elements of L and $h = \wedge_{i\in\triangle} h_i$. Let $\mu(b) \subseteq \mu(a)$ and $b \leq h$. Then $b \leq h_i$ for each $i \in \triangle$. As each h_i is a z-element, we have $a \leq h_i$ for each $i \in \triangle$. Hence $a \leq \wedge_{i\in\triangle} h_i = h$ and $h = \wedge_{i\in\triangle} h_i$ is a z-element.
4) Let $i \leq j$ where i is a z_j-element and j is a z_k-element. Let $a \leq i$. Since i is a z_j-element and j is a z_k-element, we have $M_a \wedge j \leq i$ and $M_a \wedge k \leq j$. This gives $M_a \wedge k \leq M_a \wedge j \leq i$ for all $a \leq i$. This shows that i is a z_k-element. □

Now we obtain the property of a Jacobson radical[7],and establish the relation between the Jacobson radical and z-element.

Lemma 3.9. *The Jacobson radical $j = \wedge_{m\in Max(L)} m$ is a z-element and is contained in every z-element.*

Proof. The proof follows from lemma (2.2) and(3) of lemma (3.8). □

Lemma 3.10. *Let L be a multiplicative lattice. Then i is a z_j-element if and only if $i \wedge j$ is a z_j-element.*

Proof. Let i be a z_j-element and $a \leq i \wedge j$. Then $a \leq i$ and i is a z_j-element implies $M_a \wedge j \leq i$. Also $M_a \wedge j \leq j$. Hence $M_a \wedge j \leq (i \wedge j)$ for all $a \leq (i \wedge j)$. Hence $(i \wedge j)$ is a z_j-element. Conversely assume that $(i \wedge j)$ is a z_j-element. Let $a \leq i$ and $x \leq M_a \wedge j$. Then $a \wedge x \leq i$ and $a \wedge x \leq x \leq j$ implies $a \wedge x \leq (i \wedge j)$. Since $(i \wedge j)$ is a z_j-element, we have $M_{a\wedge x} \wedge j \leq (i \wedge j)$. By lemma (2.6),$M_{a\wedge x} = M_a \wedge M_x$. Since $x \leq M_a$, we have $M_x \leq M_a$. Now $M_x \wedge j \leq M_a \wedge j$. So $M_x \wedge j \leq (i \wedge j)$. Now $x \leq M_x \wedge j \leq i \wedge j \leq i$. Thus $M_a \wedge j \leq i$ for all $a \leq i$. Hence i is a z_j-element. □

Finally we obtain the characterization of z_j-element and some properties of z-elements and z_j-elements.

Lemma 3.11. *Let i,j,k be elements of a distributive lattice L. Then the following statements hold:-*
1) An element i of L is a z_j-element if and only if i is $z_{i\vee j}$-element.
2)If j is a z-element then i is a z_j-element if and only if $i \wedge j$ is a z-element.
3)$i \wedge j$ is both z_i-element and z_j-element if and only if i is a z_j-element and j is z_i-element.
4)If m is a maximal element then $i \wedge m$ is a z-element if and only if i is a z-element.
5)$i_z \wedge j$ is the smallest z_j-element containing $i \wedge j$.
6) $i \leq k$,$i_z = k_z$, i is a z_j-element, then k is also a z_j-element.

Proof.1) Let i be a z_j-element. Then $M_a \wedge j \leq i$, for all $a \leq i$. Clearly $M_a \wedge i \leq i$. Since L is distributive , we have , $M_a \wedge (i \vee j) = (M_a \wedge i) \vee (M_a \wedge j) \leq i$, for all $a \leq i$. Hence i is a $z_{i\vee j}$-element. Conversely, Suppose i is a $z_{i\vee j}$-element. Let $a \leq i$. Then $a \leq (i \vee j)$ and $M_a \wedge (i \vee j) \leq i$ i.e. $(M_a \wedge i) \vee (M_a \wedge j) \leq i$. Hence $M_a \wedge j \leq i$ for all $a \leq i$. Hence i is a z_j-element.
2) Let j be a z-element. Assume that i is a z_j-element. We show that $i \wedge j$ is a z-element. Let $\mu(b) \subseteq \mu(a)$, $b \leq (i \wedge j)$. Since $\mu(b) \subseteq \mu(a)$, $b \leq j$ and j is a z-element, we have $a \leq j$. Since $M_a \leq M_b, b \leq i$, by (2.10), $a \leq i$. Now $a \leq M_a \wedge j$ and $M_a \wedge j \leq i$, since i is a z_j-element and $a \leq i$. Thus $a \leq (i \wedge j)$. Hence $(i \wedge j)$ is a z-element. Conversely assume that $i \wedge j$ is a z-element. We show that i is a z_j-element. Let $a \leq i$ and $x \leq M_a \wedge j$. Then $x \leq M_a$ implies $M_x \leq M_a$. But $\mu(x) \subseteq \mu(a), a \leq i$ implies $x \leq i$(by 2.10). Hence $M_a \wedge j \leq i$ for all $a \leq i$ and i is a z_j-element.
3)The proof follows by (3.10).
4)Let m be the maximal element of L. Let $(i \wedge m)$ be a z-element. We show that i is a z-element. If $i \leq m$, then $i = i \wedge m$ and hence i is a z-element. Suppose $i \nleq m$. Then i is a z_m-element by (2). Then i is a $z_{i\vee m}$-element by (1)i.e. i is a z_1-element. Let $\mu(b) \subseteq \mu(a), a \leq i$. Then $M_b \leq M_a \wedge 1 \leq i$ (since $a \leq i$ and i is a z_1-element). Hence $b \leq i$ and i is a z-element.
5) We know that if i is a z-element then i is a z_j-element for any element j. As i_z is a z-element. it follows that i_z is a z_j-element. We know that (by lemma 3.10), i is a z_j -element if and only if $(i \wedge j)$ is a z_j-element. Hence $i_z \wedge j$ is a z_j-element and $i \wedge j \leq i_z \wedge j$. Let k be any z_j-element such that $i \wedge j \leq k$. Then $i_z \wedge j = i_z \wedge j_z \wedge j = (i \wedge j)_z \wedge j \leq k_z \wedge j \leq k$. (By lemma 3.7). Hence $i_z \wedge j$ is the smallest z_j-element containing $i \wedge j$.
6)By (3.7),since i is a z_j-element $i_z \wedge j \leq i$. By hypothesis, $k_z \wedge j = i_z \wedge j \leq i \leq k$. Hence again by (3.7), k is a z_j-element. □

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