Some geometric properties on $h$-exponential change of Finsler metric

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Abstract. In this paper, we study the $\nu$-curvature of Finsler space characterised by $h$-exponential change of Finsler metric and derive some results on $C$-reducibility for the change.

1 Introduction

There are two important transformations in Finsler geometry: the conformal change and the $\beta$-change. In 1984, C. Shibata [10] has dealt with $\beta$-change of Finsler metric. For a $\beta$-change of Finsler metric, the differential 1-form $\beta$ play very important role. The $\beta$-change of Riemannian metric gives $(\alpha, \beta)$-metric which has many application in physics, mechanics, seismology, biology, informatics and control theory [1, 2, 3, 4]. The $\beta$-change has many classes such as Randers change, Kropina change etc. These changes are finite in nature i.e. numbers of terms are finite. An important of class of $\beta$-change is exponential change which is infinite in nature i.e. number of terms are infinite.

In 2006, YU Yao-yong and YOU Ying [12] studied a Finsler space with metric function given by exponential change of Riemannian metric. In 2012, H. S. Shukla et.al. [11] considered a Finsler space $F^n = (M^n, L)$, whose Fundamental metric function is an exponential change of Finsler metric function given by

$$L = e^{\beta}$$

(1.1)

where $\beta = b_i(x,y)y^i$ is 1-form on manifold $M^n$. Present authors have also discussed hypersurface of Finsler space characterised by $h$-exponential change of Finsler metric [6].

In the present paper, we consider a Finsler space $^*F^n = (M^n, ^*L)$, whose metric function $^*L$, an $h$-exponential change of metric, is given by

$$^*L = e^{\beta}$$

(1.2)

where $\beta = b_i(x,y)y^i$ and $b_i$ is an $h$-vector. Authors obtain the $\nu$-curvature tensor for the Finsler space characterised by $h$-exponential change of metric and derive some results on $C$-reducibility.

2 Preliminaries

Let $F^n = (M^n, L)$ be an $n$-dimensional Finsler space equipped with the Fundamental function $L(x,y)$. The metric tensor, angular metric tensor and Cartan tensor are defined by $g_{ij} = \frac{1}{2}\partial_i \partial_j L^2$, $h_{ij} = g_{ij} - l_i l_j$ and $C_{ijk} = \frac{1}{2}\partial_i g_{jk}$ respectively, where $\partial_k = \frac{\partial}{\partial x_k}$. The Cartan connection is given by $CT = (F^i_j, N^i_k, C^i_{jk})$. The $h$- and $v$-covariant derivatives $X_{i|j}$ and $X_{i|j}$ of a covariant vector field $X_i$ are defined by [9]

$$X_{i|j} = \partial_j X_i - N^r_j \partial_r X_i - X_r F^r_{ij},$$

(2.1)

and

$$X_{i|j} = \partial_j X_i - X_r C^r_{ij},$$

(2.2)

where $\partial_k = \frac{\partial}{\partial x_k}$. 
H. Izumi [7] introduced the concept of an $h$-vector $b_i(x, y)$ which is $v$-covariant constant with respect to the Cartan connection and satisfies $L C^h_i j b_h = \rho h_{ij}$, where $\rho$ is a non-zero scalar function and $C^h_{jk}$ are components of Cartan tensor. Thus if $b_i$ is an $h$-vector then

\[(i) \ b_{i|k} = 0, \quad (ii) \ L C^h_i j b_h = \rho h_{ij}. \quad (2.3)\]

From the above definition, we have

\[L \partial_j b_i = \rho h_{ij}, \quad (2.4)\]

which shows that $b_i$ is a function of directional argument also. H. Izumi [7] proved that the scalar $\rho$ is independent of directional argument.

A Finsler space $F^n = (M^n, L)$ with $n \geq 3$ is said to be Quasi-C-reducible if Cartan tensor $C_{ijk}$ satisfies [8]

\[C_{ijk} = Q_{ij} C_k + Q_{jk} C_i + Q_{ki} C_j, \quad (2.5)\]

where $Q_{ij}$ is symmetric indicatory tensor. A Finsler space $F^n = (M^n, L)$ with $n \geq 3$ is said to be $C$-reducible if Cartan tensor $C_{ijk}$ satisfies [8]

\[C_{ijk} = \frac{1}{(n + 1)} (h_{ij} C_k + h_{jk} C_i + h_{ki} C_j), \quad (2.6)\]

where $C_i = g^{hk} C_{ijk}$.

The $v$-curvature tensor $S_{hijk}$ of a Finsler space with respect to Cartan space $C \Gamma$ is defined by [8]

\[S_{hijk} = C_{ijr} C^r_{hk} - C_{ikr} C^r_{hj}. \quad (2.7)\]

A Finsler space $F^n = (M^n, L)$ is said to be $S$-3 like Finsler space if the $v$-curvature tensor has the form [8]

\[L^2 S_{hijk} = S(h_{hj} h_{ik} - h_{hk} h_{ij}), \quad (2.8)\]

where scalar $S$ is function of $x$ alone. A Finsler space $F^n = (M^n, L)$ is said to be $S$-4 like Finsler space if there exists a symmetric and indicatory tensor $K_{ij}$ such that the $v$-curvature tensor has the form [8]

\[L^2 S_{hijk} = S(h_{hj} K_{ik} + h_{ik} K_{hj} - j/k), \quad (2.9)\]

where $-j/k$ means interchange of $j$ and $k$ and subtract the quantities within the bracket.

We use following notations $L_i = \partial_i L = l_i$, $L_{ij} = \partial_i \partial_j L$, $L_{ijk} = \partial_i \partial_j \partial_k L$. The quantities corresponding to ${^*}F^n$ is denoted by asterisk over that quantity. From (1.2), we have

\[\text{*}L_i = e^\tau (m_i + l_i), \quad (2.10)\]

\[\text{*}L_{ij} = e^\tau (1 + \rho - \tau) L_{ij} + \frac{e^\tau}{L} m_i m_j, \quad (2.11)\]

\[\text{*}L_{ijk} = e^\tau (1 + \rho - \tau) L_{ijk} + (\rho - \tau) \frac{e^\tau}{L^2} [m_i L_{jk} + m_j L_{ik} + m_k L_{ij}] - \frac{e^\tau}{L^2} [m_j m_k l_i + m_i m_k l_j + m_i m_j l_k - m_i m_j m_k], \quad (2.12)\]
where $\tau = \frac{\beta}{L}$, $m_i = b_i - \tau l_i$. The normalised suporting element, the metric tensor and angular metric tensor of $^*F^n$ are obtained as [5]

\[ ^*l_i = e^\tau (m_i + l_i), \quad (2.13) \]

\[ ^*g_{ij} = \nu e^{2\tau} g_{ij} + e^{2\tau} \left( 2\tau^2 - \tau - \rho \right) l_il_j + e^{2\tau} \left( 1 - 2\tau \right) (b_il_j + b_jl_i) + 2e^{2\tau} b_ib_j, \quad (2.14) \]

\[ ^*h_{ij} = \nu e^{2\tau} h_{ij} + e^{2\tau} m_im_j, \quad (2.15) \]

where $\nu = 1 + \rho - \tau$. Differentiating the angular metric tensor $h_{ij}$ with respect to $y^k$, we get

\[ \partial_k h_{ij} = 2C_{ijk} - \frac{1}{L} (l_i h_{jk} - l_j h_{ik}), \]

which gives

\[ L_{ijk} = \frac{2}{L} C_{ijk} - \frac{1}{L^2} (h_{ij} l_k + h_{jk} l_i + h_{ik} l_j). \]

Using this, the equation (2.12) may be re-written as

\[ ^*C_{ijk} = \nu e^{2\tau} C_{ijk} + \frac{2}{L} e^{2\tau} m_i m_j m_k + \frac{1}{2L} e^{2\tau} (2\nu - 1) (m_i h_{kj} + m_j h_{ki} + m_k h_{ij}). \quad (2.16) \]

The inverse metric tensor of $^*F^n$ is derived as follows [5]

\[ ^*g^{ij} = \frac{e^{-2\tau}}{\nu} \left[ g^{ij} - \frac{1}{m^2 + \nu} b^i b^j + \frac{\tau - \nu}{m^2 + \nu} \left( b^i l^j + b^j l^i \right) - l^i l^j \left\{ \frac{\tau - \nu}{m^2 + \nu} (m^2 + \tau) - \rho \right\} \right], \quad (2.17) \]

where $b$ is magnitude of the vector $b^i = g^{ij} b_j$.

### 3 Finsler Space $^*F^n = (M^n, ^*L)$

From the definition of $m_i$, we have

\[ (a) \ m_i l^i = 0, \quad (b) \ m_i b^i = b^2 - \frac{\beta^2}{L^2}, \quad (3.1) \]

\[ (c) \ g_{ij} m^j = h_{ij} m^j = m_i, \quad (d) \ C_{ijk} h^i = L^{-1} \rho h_{ij}. \]

From (2.3), (2.16), (2.17) and (3.1), we have

\[ ^*C_{ij} = C_{ij} + \frac{1}{m^2 + \nu} C_{ijk} b^k \left( -b^h + 2\tau t^h - \rho t^h - l^h \right) \]

\[ + \frac{2}{\nu L} \left[ m_i m_j m^h + \frac{1}{m^2 + \nu} m_i m_j m^2 \left( -b^h + 2\tau t^h - \rho t^h - l^h \right) \right] \]

\[ + \frac{1}{2\nu L} (2\nu - 1) \left[ m_i h^h_j + m_j h^h_i + m^h h_{ij} \! + \! \frac{1}{m^2 + \nu} \left( -b^h + 2\tau t^h - \rho t^h - l^h \right) (2m_i m_j + m^2 h_{ij}) \right]. \quad (3.2) \]

Contracting $h$ and $k$ in above equation, we have

\[ ^*C_i = C_i + \Omega m_i, \quad (3.3) \]

where

\[ \Omega = -\rho + 2m^2 + (1 - \frac{1}{2\nu}) \left( (n - 2)m^2 + (n + 1)\nu \right). \]
Theorem 3.3. Thus, we have
\[ \mathcal{C}_{ijk} = V_{ijk} + \Sigma \mathcal{C}_i H_{jk}, \] (3.4)
where
\[ V_{ijk} = \nu e^\tau C_{ijk} + \frac{2}{L} e^{2\tau} m_i m_j m_k - \frac{1}{2L} e^{2\tau} (2\nu - 1) \Sigma C_i h_{jk} / \Omega, \]
and
\[ H_{jk} = \frac{1}{2L} e^{2\tau} (2\nu - 1) h_{jk}. \]
and $\Sigma$ means cyclic interchange of indices $i, j, k$ and summation. Thus, taking account the definition of Quasi-$C$-reducible, we have

**Theorem 3.1.** The Finsler space $^*F^n$ given by $h$-exponential change of Finsler metric is Quasi-$C$-reducible if the tensor $V_{ijk}$ vanishes.

Now, $v$-curvature for the Finsler space $^*F^n$ is given by
\[ \mathcal{S}_{hijk} = ^*C_{ijr} * C_{rkh} - ^*C_{ikr} * C_{rjh}. \] (3.5)
Using (2.16) and (3.2), above equation becomes
\[ \mathcal{S}_{hijk} = \nu e^{2\tau} \mathcal{S}_{hijk} + I(h_{jk} m_i m_j - h_{ij} m_i m_i + h_{ij} m_i m_k - h_{ik} m_i m_j) + J(h_{ij} h_{jk} - h_{ik} h_{hj}), \] (3.6)
where
\[ I = \left[ m^2 (\nu^2 - \frac{1}{4}) + \nu (\nu^2 - \nu + \frac{1}{4} + \rho) m^2 \right] \frac{e^{2\nu}}{(m^2 + \nu)L^2}, \] (3.7)
and
\[ J = \left[ 2\nu (\nu - \frac{1}{2}) \rho - \nu \rho^2 + \nu (\nu - \frac{1}{2} - \frac{1}{4\nu}) m^2 \right] \frac{e^{2\nu}}{(m^2 + \nu)L^2}. \] (3.8)
Equation (3.6) may be re-written as
\[ \mathcal{S}_{hijk} = \nu e^{2\tau} \mathcal{S}_{hijk} + \left[ h_{kh} (I m_i m_j + \frac{J}{2} h_{ij}) + h_{ij} (I m_k m_i + \frac{J}{2} h_{kh}) - j/k \right]. \] (3.9)
Thus, we have

**Theorem 3.2.** The $v$-curvature tensor $\mathcal{S}_{hijk}$ of Finsler space $^*F^n$ characterised by $h$-exponential change of Finsler metric is given by (3.9).

Using (2.15), equation (3.9) gives us
\[ \nu L^2 \mathcal{S}_{hijk} = L^2 \nu e^{4\tau} \mathcal{S}_{hijk} + \nu h_{kh} M_{ij} + \nu h_{ij} M_{kh} - \nu h_{ik} M_{jkh}, \] (3.10)
where
\[ M_{ij} = \frac{L^2}{2\nu} m_i m_j + \frac{L^2}{4\nu} h_{ij}. \]
Thus, we have

**Theorem 3.3.** If the $v$-curvature tensor $\mathcal{S}_{hijk}$ of Finsler space $F^n$ vanishes, then the Finsler space $^*F^n$ is $S$-$4$ like Finsler space.

Further, if $F^n = (M^n, L)$ is $S$-$4$ like space, i.e.
\[ L^2 \mathcal{S}_{hijk} = (h_{hj} K_{ik} + h_{ik} K_{hj} - j/k), \]
Then equation (3.10) gives us
\[ \nu L^2 \mathcal{S}_{hijk} = [\nu h_{hj} H_{ik} + \nu h_{ik} H_{hj} - j/k] - A_{ijkh}, \] (3.11)
where
\[ H_{ij} = L^2 e^{2\tau} K_{ij} - M_{ij} \quad \text{and} \quad A_{ijkh} = L^2 e^{4\tau} [m_i m_j K_{ik} + m_i m_k K_{hj} - j/k]. \]
Thus, we have

**Theorem 3.4.** If $F^n$ is $S$-$4$ like Finsler space. Then $h$-exponential changed Finsler space $^*F^n$ is $S$-$4$ like Finsler space provided $A_{ijkh}$ vanishes.
References


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