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Some geometric properties on h-exponential change of Finsler metric

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Abstract. In this paper, we study the v-curvature of Finsler space characterised by h-exponential change of Finsler metric and derive some results on C-reducibility for the change.

1 Introduction

There are two important transformations in Finsler geometry: the conformal change and the β-change. In 1984, C. Shibata [10] has dealt with β -change of Finsler metric. For a β-change of Finsler metric, the differential 1-form β play very important role. The β-change of Riemannian metric gives (α, β)-metric which has many application in physics, mechanics, seismology, biology, informatics and control theory [1, 2, 3, 4]. The β-change has many classes such as Randers change, Kropina change etc. These changes are finte in nature *i.e.* numbers of terms are finite. An important of class of β-change is exponential change which is infinite in nature *i.e.* number of terms are infinite.

In 2006, YU Yao-yong and YOU Ying [12] studied a Finsler space with metric function given by exponential change of Riemannian metric. In 2012, H. S. Shukla et.al.[11] considered a Finsler space $\overline{F}^n = (M^n, \overline{L})$, whose Fundamental metric function is an exponential change of Finsler metric function given by

$$\overline{L} = L\, e^{\frac{\beta}{L}} \,, \tag{1.1}$$

where $\beta = b_i(x)y^i$ is 1-form on manifold M^n. Present authors have also discussed hypersurface of Finsler space characterised by h-exponential change of Finsler metric [6].

In the present paper, we consider a Finsler space ${}^*F^n = (M^n, {}^*L)$, whose metric function ${}^*L$, an h-exponential change of metric , is given by

$${}^*L = L\, e^{\frac{\beta}{L}} \,, \tag{1.2}$$

where $\beta = b_i(x, y)y^i$ and b_i is an h-vector. Authors obtain the v-curvature tensor for the Finsler space characterised by h-exponential change of metric and derive some results on C-reducibility.

2 Preliminaries

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with the Fundamental function $L(x, y)$. The metric tensor, angular metric tensor and Cartan tensor are defined by $g_{ij} = \frac{1}{2}\dot\partial_i\dot\partial_j L^2$, $h_{ij} = g_{ij} - l_i l_j$ and $C_{ijk} = \frac{1}{2}\dot\partial_i g_{jk}$ respectively, where $\dot\partial_k = \frac{\partial}{\partial y^k}$. The Cartan connection is given by $C\Gamma = (F^i_{jk}, N^i_k, C^i_{jk})$. The h- and v-covariant derivatives $X_{i|j}$ and $X_i|_j$ of a covarient vector field X_i are defined by [9]

$$X_{i|j} = \partial_j X_i - N^r_j\, \dot\partial_r X_i - X_r F^r_{ij} \,, \tag{2.1}$$

and

$$X_i|_j = \dot\partial_j X_i - X_r C^r_{ij} \,, \tag{2.2}$$

where $\partial_k = \frac{\partial}{\partial x^k}$.

H. Izumi [7] introduced the concept of an *h*-vector $b_i(x,y)$ which is *v*-covariant constant with respect to the Cartan connection and satisfies $L\,C^h_{ij}\,b_h = \rho\,h_{ij}$, where ρ is a non-zero scalar function and C^i_{jk} are components of Cartan tensor. Thus if b_i is an *h*-vector then

$$(i)\,b_i|_k = 0, \qquad (ii)\,L\,C^h_{ij}b_h = \rho h_{ij}\,. \tag{2.3}$$

From the above definition, we have

$$L\,\dot{\partial}_j b_i = \rho h_{ij}\,, \tag{2.4}$$

which shows that b_i is a function of directional argument also. H. Izumi [7] proved that the scalar ρ is independent of directional argument.

A Finsler space $F^n = (M^n, L)$ with $n \geq 3$ is said to be Quasi-C-reducible if Cartan tensor C_{ijk} satisfies [8]

$$C_{ijk} = Q_{ij}C_k + Q_{jk}C_i + Q_{ki}C_j\,, \tag{2.5}$$

where Q_{ij} is symmetric indicatory tensor. A Finsler space $F^n = (M^n, L)$ with $n \geq 3$ is said to be C-reducible if Cartan tensor C_{ijk} satisfies [8]

$$C_{ijk} = \frac{1}{(n+1)}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)\,, \tag{2.6}$$

where $C_i = g^{jk}C_{ijk}$.

The v-curvature tensor S_{hijk} of a Finsler space with respect to Cartan space $C\Gamma$ is defined by [8]

$$S_{hijk} = C_{ijr}C^r_{hk} - C_{ikr}C^r_{hj}\,. \tag{2.7}$$

A Finsler space $F^n = (M^n, L)$ is said to be S-3 like Finsler space if the v-curvature tensor has the form [8]

$$L^2 S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij})\,, \tag{2.8}$$

where scalar S is function of x alone. A Finsler space $F^n = (M^n, L)$ is said to be S-4 like Finsler space if there exists a symmetric and indicatory tensor K_{ij} such that the v-curvature tensor has the form [8]

$$L^2 S_{hijk} = S(h_{hj}K_{ik} + h_{ik}K_{hj} - j/k)\,, \tag{2.9}$$

where $-j/k$ means interchange of j and k and subtract the quantities within the bracket.

We use following notations $L_i = \dot{\partial}_i L = l_i\,,\;\; L_{ij} = \dot{\partial}_i\dot{\partial}_j L\,,\;\; L_{ijk} = \dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L$. The quantities corrosponding to $^*F^n$ is denoted by asterisk over that quantity.
From (1.2), we have

$$^*L_i = e^{\tau}\big(m_i + l_i\big)\,, \tag{2.10}$$

$$^*L_{ij} = e^{\tau}(1+\rho-\tau)L_{ij} + \frac{e^{\tau}}{L}m_i m_j\,, \tag{2.11}$$

$$\begin{aligned} ^*L_{ijk} =& e^{\tau}\big(1+\rho-\tau\big)L_{ijk} + \big(\rho-\tau\big)\frac{e^{\tau}}{L}\big[m_iL_{jk} + m_jL_{ik} + m_kL_{ij}\big] \\ &- \frac{e^{\tau}}{L^2}\big[m_jm_kl_i + m_im_kl_j + m_im_jl_k - m_im_jm_k\big], \end{aligned} \tag{2.12}$$

where $\tau = \frac{\beta}{L}$, $m_i = b_i - \tau l_i$. The normalised suporting element, the metric tensor and angular metric tensor of ${}^*F^n$ are obtained as [5]

$$ {}^*l_i = e^{\tau}\big(m_i + l_i\big), \tag{2.13} $$

$$ {}^*g_{ij} = \nu\, e^{2\tau} g_{ij} + e^{2\tau}\Big(2\tau^2 - \tau - \rho\Big) l_i l_j + e^{2\tau}\Big(1 - 2\tau\Big)\big(b_i l_j + b_j l_i\big) + 2e^{2\tau} b_i b_j , \tag{2.14} $$

$$ {}^*h_{ij} = \nu e^{2\tau} h_{ij} + e^{2\tau} m_i m_j \,, \tag{2.15} $$

where $\nu = 1 + \rho - \tau$.
Diffrentiating the angular metric tensor h_{ij} with respect to y^k, we get

$$ \dot\partial_k h_{ij} = 2C_{ijk} - \frac{1}{L}(l_i h_{jk} - l_j h_{ik}) \,, $$

which gives

$$ L_{ijk} = \frac{2}{L} C_{ijk} - \frac{1}{L^2}(h_{ij} l_k + h_{jk} l_i + h_{ki} l_j) \,. $$

Using this, the equation (2.12) may be re-written as

$$ {}^*C_{ijk} = \nu\, e^{2\tau} C_{ijk} + \frac{2}{L} e^{2\tau} m_i m_j m_k + \frac{1}{2L} e^{2\tau}(2\nu - 1)\big(m_i h_{kj} + m_j h_{ki} + m_k h_{ij}\big) \,. \tag{2.16} $$

The inverse metric tensor of ${}^*F^n$ is derived as follows[5]

$$ {}^*g^{ij} = \frac{e^{-2\tau}}{\nu}\Big[g^{ij} - \frac{1}{m^2+\nu} b^i b^j + \frac{\tau - \nu}{m^2+\nu}\Big(b^i l^j + b^j l^i\Big) - l^i l^j \Big\{\frac{\tau-\nu}{m^2+\nu}(m^2+\tau) - \rho\Big\}\Big], \tag{2.17} $$

where b is magnitude of the vector $b^i = g^{ij} b_j$.

3 Finsler Space ${}^*F^n = (M^n, {}^*L)$

From the definition of m_i, we have

$$ \begin{aligned} &(a)\ m_i l^i = 0 \,, && (b)\ m_i b^i = b^2 - \frac{\beta^2}{L^2} \,, \\ &(c)\ g_{ij} m^j = h_{ij} m^j = m_i \,, && (d)\ C_{ihj} m^h = L^{-1} \rho h_{ij} \,. \end{aligned} \tag{3.1} $$

From (2.3), (2.16), (2.17) and (3.1), we have

$$ \begin{aligned} {}^*C^h_{ij} =& C^h_{ij} + \frac{1}{m^2+\nu} C_{ijk} b^k(-b^h + 2\tau\, l^h - \rho l^h - l^h) \\ &+ \frac{2}{\nu L}\Big[m_i m_j m^h + \frac{1}{m^2+\nu} m_i m_j m^2(-b^h + 2\tau\, l^h - \rho l^h - l^h)\Big] \\ &+ \frac{1}{2\nu L}(2\nu - 1)\Big[m_i h^h_j + m_j h^h_i + m^h h_{ij} \\ &+ \frac{1}{m^2+\nu}(-b^h + 2\,\tau\, l^h - \rho l^h - l^h)\,(2m_i m_j + m^2 h_{ij})\Big]. \end{aligned} \tag{3.2} $$

Contracting h and k in above equation, we have

$$ {}^*C_i = C_i + \Omega\, m_i \,, \tag{3.3} $$

where

$$ \Omega = -\rho + 2m^2 + (1 - \tfrac{1}{2\nu})((n-2)m^2 + (n+1)\nu) \,. $$

Now, using (3.3), equation (2.16) becomes

$$^*C_{ijk} = V_{ijk} + \Sigma\ ^*C_i H_{jk}\,, \tag{3.4}$$

where

$$V_{ijk} = \nu e^{\tau} C_{ijk} + \frac{2}{L} e^{2\tau} m_i m_j m_k - \frac{1}{2L} e^{2\tau}(2\nu - 1)\Sigma \frac{C_i h_{jk}}{\Omega}$$

and

$$H_{jk} = \frac{1}{2L} e^{2\tau}(2\nu - 1)\, h_{jk}\,.$$

and Σ means cyclic interchange of indices $i, j,\ k$ and summation. Thus, taking account the definition of Quasi C-reducible, we have

Theorem 3.1. *The Finsler space $^*F^n$ given by h-exponential change of Finsler metric is Quasi C-reducible if the tensor V_{ijk} vanishes.*

Now, v-curvature for the Finsler space $^*F^n$ is given by

$$^*S_{hijk} =^* C_{ijr}{}^*C^r_{hk} -^* C_{ikr}{}^*C^r_{hj}\,. \tag{3.5}$$

Using (2.16) and (3.2), above equation becomes

$$^*S_{hijk} = \nu e^{2\tau} S_{hijk} + I(h_{hk} m_i m_j - h_{hj} m_i m_k + h_{ij} m_h m_k - h_{ik} m_h m_j) + J(h_{ij} h_{hk} - h_{ik} h_{hj})\,, \tag{3.6}$$

where

$$I = \left[m^2(\nu^2 - \frac{1}{4}) + \nu(\nu^2 - \nu + \frac{1}{4} + \rho)m^2\right] \frac{e^{2\nu}}{(m^2 + \nu)L^2}\,, \tag{3.7}$$

and

$$J = \left[2\nu(\nu - \frac{1}{2})\rho - \nu\rho^2 + \nu(\nu - \frac{1}{2} - \frac{1}{4\nu})m^2\right] \frac{e^{2\nu}}{(m^2 + \nu)L^2}\,, \tag{3.8}$$

Equation (3.6) may be re-written as

$$^*S_{hijk} = \nu e^{2\tau} S_{hijk} + \left[h_{kh}(I m_i m_j + \frac{J}{2} h_{ij}) + h_{ij}(I m_k m_h + \frac{J}{2} h_{kh}) - j/k\right]. \tag{3.9}$$

Thus, we have

Theorem 3.2. *The v-curvature tensor $^*S_{hijk}$ of Finsler space $^*F^n$ characterised by h-exponential change of Finsler metric is given by (3.9).*

Using (2.15), equation (3.9) gives us

$$^*L^2\ ^*S_{hijk} = L^2 \nu e^{4\tau} S_{hijk} + {}^*h_{kh} M_{ij} + {}^*h_{ij} M_{kh} - {}^*h_{jh} M_{ik} - {}^*h_{ik} M_{jh}\,, \tag{3.10}$$

where

$$M_{ij} = \tfrac{L^2 I}{\nu} m_i m_j + \tfrac{L^2 J}{2\nu} h_{ij}\,.$$

Thus, we have

Theorem 3.3. *If the v-curvature tensor S_{hijk} of Finsler space F^n vanishes, then the Finsler space $^*F^n$ is S-4 like Finsler space.*

Further, if $F^n = (M^n, L)$ is S-4 like space , *i.e.*

$$L^2 S_{hijk} = (h_{hj} K_{ik} + h_{ik} K_{hj} - j/k)\,,$$

Then equation (3.10) gives us

$$^*L^2\,^*S_{hijk} = [^*h_{hj} H_{ik} +^* h_{ik} H_{hj} - j/k] - A_{ijkh}, \tag{3.11}$$

where

$$H_{ij} = L^2 e^{2\tau} K_{ij} - M_{ij} \quad \text{and} \quad A_{ijkh} = L^2 e^{4\tau}[m_h m_j K_{ik} + m_i m_k K_{hj} - j/k]$$

Thus, we have

Theorem 3.4. *If F^n is S-4 like Finsler space. Then h-exponential changed Finsler space $^*F^n$ is S-4 like Finsler sace provided A_{ijkh} vanishes.*

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