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ON GENERALIZED CR-LIGHTLIKE SUBMANIFOLDS

Selcen Yüksel Perktaş and Feyza Esra Erdoğan

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Abstract. The aim of the article is to find the conditions for the integrability of the distributions which are defined on a generalized Cauchy-Riemann (GCR) lightlike submanifold of a para-Sasakian manifold. Also, we find a condition for the induced connection to be a metric connection on the GCR-lightlike submanifold.

1 Introduction

Submanifold theory, especially theory of lightlike (or null) submanifolds, is one of the important research area in semi-Riemannian geometry. A submanifold M of a semi-Riemannian manifold $\bar{M}$ is called a lightlike (null) submanifold if the induced metric tensor field on the submanifold is degenerate. If the induced metric tensor is degenerate then the classical theory of Riemannian submanifolds fails since the tangent bundle and the normal bundle of the submanifold have a non-zero intersection. To overcome this problem D. N. Küpeli [25] (intrinsic approach) and K. L. Duggal-A. Bejancu [15] (extrinsic approach) were introduced some new methods and studied lightlike submanifolds (see also [16]). On this topic, some applications of the theory to mathematical physics is inspired, especially general relativity [21], black hole horizons [16] and electromagnetism [15].

A CR-submanifold of Kaehler manifold was defined by A. Bejancu [10, 11], as a result of generalization of invariant and anti-invariant submanifolds. Contact CR-submanifolds of Sasakian manifolds were studied in [12].

In [17], K. L. Duggal-B. Şahin defined screen real, screen CR-lightlike, invariant and contact CR-lightlike submanifolds of indefinite Sasakian manifolds. Later on study of generalized CR-lightlike submanifolds of indefinite Sasakian manifolds were initiated in [18]. The authors in [22] gave some necessary and sufficient conditions on integrability of various distributions of GCR-lightlike submanifold of an indefinite Sasakian manifold. Moreover, CR-lightlike submanifolds of a Kaehlerian manifolds were studied in [19]. Also, a general notion of paracontact CR-lightlike submanifolds was introduced in [5, 6]. Recently, a huge number of research papers has appeared on lightlike submanifolds and its applications (for further read we refer [28, 26, 4, 23, 3] and references therein).

On a semi-Riemannian manifold $\bar{M}$, S. Kaneyuki-M. Konzai [24] introduced a structure which is known the almost paracontact structure and then they characterized the almost paracomplex structure on $\bar{M}^{2n+1} \times \mathbb{R}$. Recently, S. Zamkovoy [30] studied paracontact metric manifolds. The study of paracontact geometry has been continued by several papers ([8, 9, 2, 13, 14, 7, 20, 29]) which contain role of paracontact geometry about semi-Riemannian geometry, mathematical physics and relationships with the para-Kähler manifolds.

The purpose of this article is to investigate the conditions for the integrability of the distributions which are defined on GCR-lightlike submanifolds of para-Sasakian manifolds. Also, we obtain a condition for the induced connection to be a metric connection on the GCR-lightlike submanifolds of a para-Sasakian manifold.

2 Preliminaries

2.1 Lightlike Submanifolds

Let $(\bar{M}^{n+m}, \bar{g})$ be a semi-Riemannian manifold with index q, such that $m, n \geqslant 1, 1 \leq q \leq m+n-1$ and (M^m, g) be a submanifold of $\bar{M}$, where g is the induced metric from $\bar{g}$ on M. In this case, M is called a *lightlike* (*null*) *submanifold* of $\bar{M}$ if g is degenerate on M. Now let us consider a degenerate metric g on M. Thus $TM^{\perp}$ is a degenerate $n-$dimensional subspace of $T_x\bar{M}$ and orthogonal subspaces T_xM and $T_xM^{\perp}$ are degenerate but no longer complementary. So, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$ which is called radical space. If the mapping $RadTM : x \in M \to RadT_xM$ defines a distribution, namely *radical distribution*, on M of rank $r > 0$ then the submanifold M is called an $r-$*lightlike submanifold* [15].

Let $S(TM)$ be the screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM. So one can write

$$TM = S(TM) \perp RadTM, \tag{2.1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}$ and $RadTM$ in $S(TM^{\perp})^{\perp}$, respectively. In this case we arrive at

$$tr(TM) = S(TM^{\perp}) \perp ltr(TM), \tag{2.2}$$

$$T\bar{M} = TM \oplus tr(TM) = \{RadTM \oplus ltr(TM)\} \perp S(TM) \perp S(TM^{\perp}). \tag{2.3}$$

Theorem 2.1. *[15] Let $(M, g, S(TM))$ be a lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exist a complementary vector subbundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\,|_U$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\,|_U$, where U is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, E_i) = 1, \qquad \bar{g}(N_i, N_j) = 0, \tag{2.4}$$

where $\{E_1, E_2, ..., E_n\}$ is a lightlike basis of $\Gamma(RadTM)$.

For a lightlike submanifold $(M, g, S(TM))$ we have following four cases:
* If $r < min\{m, n\}$ then M is a $r-$*lightlike* submanifold,
* If $r = n < m$, $S(TM^{\perp}) = \{0\}$ then M is a *coisotropic lightlike* submanifold,
* If $r = m < n$, $S(TM) = \{0\}$ then M is a *isotropic lightlike* submanifold,
* If $r = m = n$, $S(TM) = \{0\} = S(TM^{\perp})$ then M is a *totally null* submanifold.

By use of (2.3), the Gauss and Weingarten formulas are defined by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \forall X, Y \in \Gamma(TM), \tag{2.5}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^t_X U, \quad \forall X \in \Gamma(TM), \ U \in \Gamma(tr(TM)), \tag{2.6}$$

respectively, where $\{\nabla_X Y, A_U X\}$ belongs to $\Gamma(TM)$ and $\{h(X, Y), \nabla^t_X U\}$ belongs to $\Gamma(tr(TM))$. $\bar{\nabla}$ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively.

In view of (2.2), we consider the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$. Therefore (2.5) and (2.6) become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad X, Y \in \Gamma(TM), \tag{2.7}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad X \in \Gamma(TM), N \in \Gamma(ltr(TM)), \tag{2.8}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \quad X \in \Gamma(TM), W \in \Gamma(S(TM^{\perp})), \tag{2.9}$$

where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $\nabla^l_X N$, $D^l(X, W) \in \Gamma(ltr(TM))$, $\nabla^s_X W$, $D^s(X, N) \in \Gamma(S(TM^{\perp}))$ and $\nabla_X Y$, $A_N X$, $A_W X \in \Gamma(TM)$.

Assume that P is a projection of TM on $S(TM)$. Then by use of (2.1), we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \quad X, Y \in \Gamma(TM), \tag{2.10}$$

$$\nabla_X E = -A^*_E X + \nabla^{*t}_X E, \quad X \in \Gamma(TM), \ E \in \Gamma(RadTM), \tag{2.11}$$

where $\{\nabla^*_X PY, A^*_E X\}$ belongs to $\Gamma(S(TM))$ and $\{h^*(X, PY), \nabla^{*t}_X E\}$ belongs to $\Gamma(RadTM)$.
Using (2.10) and (2.11), we get

$$\bar{g}(h^*(X, PY), N) = \bar{g}(A_N X, PY), \tag{2.12}$$

$$\bar{g}(h^l(X, PY), E) = \bar{g}(A^*_E X, PY), \tag{2.13}$$

$$\bar{g}(h^l(X, E), E) = 0, \quad A^*_E E = 0. \tag{2.14}$$

In genereal, the induced connection ∇ on M is not a metric connection. Since $\bar{\nabla}$ is a metric connection then from (2.7), we have

$$(\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y). \tag{2.15}$$

However, ∇^* is a metric connection on $S(TM)$.

2.2 Almost paracontact metric manifolds

A paracontact manifold $\bar{M}$ is a differentiable manifold equipped with a $1-$form η, a characteristic vector field ξ and a tensor field $\bar{\phi}$ of type $(1, 1)$ such that [24]:

$$\eta(\xi) = 1, \tag{2.16}$$

$$\bar{\phi}^2 = I - \eta \otimes \xi, \tag{2.17}$$

$$\bar{\phi}\xi = 0, \tag{2.18}$$

$$\eta \circ \bar{\phi} = 0. \tag{2.19}$$

Moreover, if the manifold $\bar{M}$ is equipped with a semi-Riemannian metric $\bar{g}$ of signature $(n + 1, n)$, which is called *compatible metric,* satisfying [30]

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y) = -\bar{g}(X, Y) + \eta(X)\eta(Y), \tag{2.20}$$

then we say that $\bar{M}$ is an *almost paracontact metric manifold* with an *almost paracontact metric structure* $(\bar{\phi}, \xi, \eta, \bar{g})$.
From the definition, one can see that [30]

$$\bar{g}(\bar{\phi}X, Y) = -\bar{g}(X, \bar{\phi}Y), \tag{2.21}$$

$$\bar{g}(X, \xi) = \eta(X). \tag{2.22}$$

If $\bar{g}(X, \bar{\phi}Y) = d\eta(X, Y)$, then the almost paracontact metric manifold is said to be a *paracontact metric manifold*.

For an almost paracontact metric manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$, one can always find a local orthonormal basis $(X_i, \bar{\phi}X_i, \xi)$, $i = 1, 2, ..., n$, which is called $\bar{\phi}-$*basis* [30].

An almost paracontact metric manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ is a *para-Sasakian manifold* if and only if [30]

$$(\bar{\nabla}_X \bar{\phi})Y = -\bar{g}(X, Y)\xi + \eta(Y)X, \tag{2.23}$$

where $\bar{\nabla}$ is a Levi-Civita connection on $\bar{M}$.
From (2.23), one arrive at

$$\bar{\nabla}_X \xi = -\bar{\phi}X. \tag{2.24}$$

Example 2.2. [1] Let $\bar{M} = \mathbb{R}^{2n+1}$ be the $(2n+1)-$dimensional real number space with standard coordinate system $(x_1, y_1, x_2, y_2, ..., x_n, y_n, z)$. Defining

$$\begin{aligned} \overline{\phi}\frac{\partial}{\partial x_\alpha} &= \frac{\partial}{\partial y_\alpha}, \qquad \overline{\phi}\frac{\partial}{\partial y_\alpha} = \frac{\partial}{\partial x_\alpha}, \qquad \overline{\phi}\frac{\partial}{\partial z} = 0, \\ \xi &= \frac{\partial}{\partial z}, \qquad \overline{\eta} = dz, \\ \overline{g} &= \eta \otimes \eta + \sum_{\alpha=1}^{n}(dx_\alpha \otimes dx_\alpha - dy_\alpha \otimes dy_\alpha), \end{aligned} \tag{2.25}$$

where $\alpha = 1, 2, ..., n$, then the set $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ is an almost paracontact metric manifold.

3 Main Theorems

Definition 3.1. Let $(M, g, S(TM))$ be a lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then M is called a GCR-lightlike submanifold if the following conditions are provided,

$i)$ There exist two subbundles μ_1 and μ_2 of $Rad\,TM$ such that

$$Rad\,TM = \mu_1 \oplus \mu_2, \quad \bar{\phi}(\mu_1) = \mu_1, \quad \bar{\phi}(\mu_2) \subset S(TM), \tag{3.1}$$

$ii)$ There exist two subbundles $\tilde{D}_1$ and $\tilde{D}_2$ of $S(TM)$ such that

$$S(TM) = \{\bar{\phi}(\mu_2) \oplus \tilde{D}_2\} \perp \tilde{D}_1 \perp \{\xi\}, \quad \bar{\phi}(\tilde{D}_2) = D_0 \perp \tilde{D}_0, \tag{3.2}$$

where $\tilde{D}_1$ is an invariant non-degenerate distribution on M, $\{\xi\}$ is a 1-dimensional distribution spanned by ξ and D_0, $\tilde{D}_0$ are vector subbundles of $ltr(TM)$ and $S(TM^{\perp})$, respectively.

So one has the following:

$$TM = D \oplus \tilde{D}_2 \oplus \{\xi\}, \qquad D = RadTM \oplus \tilde{D}_1 \oplus \bar{\phi}(\mu_2). \tag{3.3}$$

Let Q, S_1, S_2 be the projection morphisms on D, $\bar{\phi}\tilde{D}_0$, $\bar{\phi}D_0$, respectively. So we have

$$X = QX + \xi + S_1X + S_2X,$$

for $X \in \Gamma(TM)$. Applying $\bar{\phi}$ to the both sides of above equation, we get

$$\bar{\phi}X = \alpha X + \beta S_1 X + \beta S_2 X, \tag{3.4}$$

where $\alpha X \in \Gamma(D)$, $\beta S_1 X \in \Gamma(\tilde{D}_0)$ and $\beta S_1 X \in \Gamma(D_0)$. Thus, one can write

$$\bar{\phi}X = \alpha X + \beta X, \tag{3.5}$$

where αX is the tangential component and βX is the transversal component of X.

Also, for any $U \in \Gamma(tr(TM))$ we have

$$\bar{\phi}U = BU + CU, \tag{3.6}$$

where $BU \in \Gamma(TM)$ and $CU \in \Gamma(tr(TM))$.

Differantiating (3.4) and in view of (2.8), (2.9) and (3.6), we get

$$\begin{aligned} D^l(X, \beta S_1 Y) &= -\nabla^l_X \beta S_2 Y + \beta S_2 \nabla_X Y \\ &\quad -h^l(X, \alpha Y) + Ch^l(X, Y), \end{aligned} \tag{3.7}$$

$$\begin{aligned} D^s(X, \beta S_2 Y) &= -\nabla^s_X \beta S_1 Y + \beta S_1 \nabla_X Y \\ &\quad -h^s(X, \alpha Y) + Chs(X, Y). \end{aligned} \tag{3.8}$$

By using (2.23) with (2.7) and (2.8), we give the following.

Lemma 3.2. *Let M be a GCR-lightlike submanifold of a para-Sasakian manifold. Then, we have*

$$(\nabla_X \alpha)Y = -g(X,Y)\xi + \eta(X)Y + A_{\beta Y}X + Bh(X,Y), \tag{3.9}$$

and

$$(\nabla^t_X \beta)Y = -h(X,\alpha Y) + Ch(X,Y), \tag{3.10}$$

for all $X,Y \in \Gamma(TM)$, where $(\nabla_X \alpha)Y = \nabla_X \alpha Y - \alpha\nabla_X Y$ and $(\nabla^t_X \beta)Y = \nabla^t_X \beta Y - \beta\nabla_X Y$.

Lemma 3.3. *Let M be a GCR-lightlike submanifold of a para-Sasakian manifold. Then, we have*

$$(\nabla_X B)U = A_{CU}X - \alpha A_U X, \tag{3.11}$$

$$(\nabla^t_X C)U = -\beta A_U X - h(X,BU), \tag{3.12}$$

for $X \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $(\nabla_X B)U = \nabla_X BU - B\nabla^t_X U$ and $(\nabla^t_X C)U = \nabla^t_X CU - C\nabla^t_X U$.

Theorem 3.4. *Let $(\bar{M},\bar{\phi},\xi,\eta,\bar{g})$ be a para-Sasakian manifold and $(M,g,S(TM))$ be a GCR-lightlike submanifold of $\bar{M}$. Then, we have*

$$g(\nabla_X Y, Z) = -g(\alpha A_{\beta Y}X, Z), \tag{3.13}$$

for any $Y \in \Gamma(\tilde{D}_2)$ and $Z \in \Gamma(D)$.

Proof. By use of (3.9), we get

$$-\alpha\nabla_X Y = -g(X,Y)\xi + A_{\beta Y}X + Bh(X,Y), \tag{3.14}$$

for any $Y \in \Gamma(\tilde{D}_2)$.

If we take $Z \in \Gamma(D)$ then we get $\bar{\phi}Z \in \Gamma(D)$. Using this result in (3.14), we obtain

$$g(\alpha\nabla_X Y, \bar{\phi}Z) = -g(A_{\beta Y}X, \bar{\phi}Z).$$

In view of (2.17) in above equation, we get (3.13).

Also taking $Z \in \Gamma(D_0)$, we obtain $\nabla_X Y = -\alpha A_{BY}X$. □

Theorem 3.5. *Let $(\bar{M},\bar{\phi},\xi,\eta,\bar{g})$ be a para-Sasakian manifold and $(M,g,S(TM))$ be a GCR-lightlike submanifold of $\bar{M}$. Then, $D \oplus \{\xi\}$ is integrable if and only if*

$$h(X,\alpha Y) = h(\alpha X, Y). \tag{3.15}$$

Proof. From (3.7) and (3.8), for any $X,Y \in \Gamma(D \oplus \{\xi\})$, we get

$$\beta S\nabla_X Y = h^l(X,\alpha Y) - Ch^l(X,Y). \tag{3.16}$$

Replacing X by Y in (3.16), we get

$$\beta S\nabla_Y X = h^l(Y,\alpha X) - Ch^l(Y,X).$$

Finally, we arrive at

$$\beta[X,Y] = h(X,\alpha Y) - h(Y,\alpha X),$$

which completes the proof. □

Theorem 3.6. *Let $(\bar{M},\bar{\phi},\xi,\eta,\bar{g})$ be a para-Sasakian manifold and $(M,g,S(TM))$ be a GCR-lightlike submanifold of $\bar{M}$. Then, $\tilde{D}_2$ is integrable if and only if*

$$A_{\bar{\phi}U}V = A_{\bar{\phi}V}U. \tag{3.17}$$

Proof. From (3.9), we get

$$-f(\nabla_U V) = -g(U,V)\xi + A_{\beta V}U + Bh(U,V), \tag{3.18}$$

for all $U,V \in \Gamma(\tilde{D}_2)$. Replacing U and V in (3.18), we have

$$-f(\nabla_V U) = -g(V,U)\xi + A_{\beta U}V + Bh(V,U). \tag{3.19}$$

Substracting (3.19) from (3.18), we obtain

$$f[U,V] = A_{\beta V}U - A_{\beta U}V,$$

which gives the equation (3.17) and completes the proof. □

Theorem 3.7. *Let $(\bar{M},\bar{\phi},\xi,\eta,\bar{g})$ be a para-Sasakian manifold and $(M,g,S(TM))$ be a GCR-lightlike submanifold of $\bar{M}$. Then, $D\oplus\{\xi\}$ defines a totally geodesic foliation if and only if*

$$Bh(X,\bar{\phi}Y) = 0.$$

Proof. In view of definition of GCR-lightlike submanifolds and the decomposition (3.3), $D\oplus\{\xi\}$ defines a totally geodesic foliation if and only if

$$g(\nabla_X Y,\bar{\phi}E) = 0, \qquad \forall X,Y \in \Gamma(D\oplus\{\xi\}),\ E\in\Gamma(\mu_2),$$

and

$$g(\nabla_X Y,\bar{\phi}W) = 0, \qquad \forall X,Y \in \Gamma(D\oplus\{\xi\}),\ W\in\Gamma(\tilde{D}_0).$$

Using (2.7) and (2.23), we have

$$g(\nabla_X Y,\bar{\phi}E) = -g(\bar{\phi}\nabla_X Y,E) = -g(\nabla_X\bar{\phi}Y,E) = -g(h^l(X,\alpha Y),E).$$

Similarly, we get

$$g(\nabla_X Y,\bar{\phi}W) = -g(\bar{\phi}\nabla_X Y,W) = -g(\nabla_X\bar{\phi}Y,W) = -g(h^s(X,\alpha Y),W).$$

So, this completes the proof. □

Theorem 3.8. *Let $(\bar{M},\bar{\phi},\xi,\eta,\bar{g})$ be a para-Sasakian manifold and $(M,g,S(TM))$ be a GCR-lightlike submanifold of $\bar{M}$. The distribution $\tilde{D}_2$ defines a totally geodesic foliation on M if and only if, for all $X,Y\in\Gamma(\tilde{D}_2)$ and $N\in\Gamma(ltr(TM))$,*

$i)$ $A_N X$ has no component on $\bar{\phi}(\mu_2)\perp\bar{\phi}\tilde{D}_0$,

and

$ii)$ $A_{\beta Y}X$ has no component on $\mu_2\perp\tilde{D}_2$.

Proof. In view of the definition of GCR-lightlike submanifolds and the decomposition (3.3), $\tilde{D}_2$ defines a totally geodesic foliation if and only if

$$g(\nabla_X Y,N) = 0, \qquad N\in\Gamma(ltr(TM)),$$

$$g(\nabla_X Y,\bar{\phi}P_1) = 0, \qquad P_1\in\Gamma(D_0),$$

$$g(\nabla_X Y,\bar{\phi}Z) = 0, \qquad Z\in\Gamma(\tilde{D}_1).$$

$$g(\nabla_X Y,\xi) = 0,$$

for all $X,Y\in\Gamma(\tilde{D}_2)$. Using (2.7) and (2.8), we find

$$g(\nabla_X Y,N) = g(\bar{\nabla}_X Y,N) = -g(Y,\bar{\nabla}_X N) = g(Y,A_N X). \tag{3.20}$$

Similarly using (2.7) and (2.23), we get

$$\begin{aligned} g(\nabla_X Y,\bar{\phi}P_1) &= -g(\bar{\phi}\nabla_X Y,P_1) \\ &= -g(\nabla_X\bar{\phi}Y,P_1) \\ &= -g(\bar{\nabla}_X\beta Y,P_1) \\ &= g(A_{\beta Y}X,P_1), \end{aligned} \tag{3.21}$$

$$\begin{aligned} g(\nabla_X Y, \bar{\phi} Z) &= -g(\bar{\phi}\nabla_X Y, Z) \\ &= -g(\nabla_X \bar{\phi} Y, Z) \\ &= -g(\bar{\nabla}_X \beta Y, Z) \\ &= g(A_{\beta Y} X, Z), \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} g(\nabla_X Y, \xi) &= g(\bar{\nabla}_X Y, \xi) \\ &= -g(Y, \bar{\nabla}_X \xi) \\ &= g(Y, \bar{\phi} X) \\ &= 0. \end{aligned} \tag{3.23}$$

So, in view of (3.20)-(3.23), the proof is completed. □

Theorem 3.9. *Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be a para-Sasakian manifold and $(M, g, S(TM))$ be a GCR-lightlike submanifold of $\bar{M}$. Then, the induced connection ∇ is a metric connection if and only if*

$$-A^*_{\bar{\phi}E} X + \nabla^{*t}_X \bar{\phi} E \in \Gamma(\bar{\phi}(\mu_2) \bot \mu_1), \quad X \in \Gamma(TM),\ E \in \Gamma(\mu_1),$$

$$\nabla^*_X \bar{\phi} E + h^*(X, \bar{\phi} E) \in \Gamma(\bar{\phi}(\mu_2) \bot \mu_1), \quad X \in \Gamma(TM),\ E \in \Gamma(\mu_2),$$

$$h(X, \bar{\phi} E) \in \Gamma(D_0 \bot \tilde{D}_0)^{\bot} \text{ and } A^*_E X \in \Gamma(\tilde{D}_2 \bot \tilde{D}_1 \bot \bar{\phi}(\mu_2)).$$

Proof. By use of (2.23), for any $X \in \Gamma(TM), E \in \Gamma(RadTM)$, we get

$$\bar{\nabla}_X \bar{\phi} E = \bar{\phi} \bar{\nabla}_X E. \tag{3.24}$$

Now, by considering (2.11) and (2.17), we get

$$\nabla_X E + h(X, E) = \bar{\phi}(\nabla_X \bar{\phi} E + h(X, \bar{\phi} E)) + g(A^*_E X, \xi)\xi. \tag{3.25}$$

Suppose that $E \in \Gamma(\mu_1)$, so $\bar{\phi} E \in \Gamma(\mu_1)$. Again in view of (2.11) and (3.25), we find

$$\begin{aligned} \nabla_X E + h(X, E) \quad = \quad & \bar{\phi}(-A^*_{\bar{\phi}E} X + \nabla^{*t}_X \bar{\phi} E + h(X, \bar{\phi} E)) \\ & + g(A^*_E X, \xi)\xi, \end{aligned} \tag{3.26}$$

which yields

$$\nabla_X E = -\alpha A^*_{\bar{\phi}E} X + \alpha \nabla^{*t}_X \bar{\phi} E + B h(X, \bar{\phi} E) + g(A^*_E X, \xi)\xi. \tag{3.27}$$

Thus $\nabla_X E \in \Gamma(RadTM)$ if and only if

$$\begin{aligned} Bh(X, \bar{\phi} E) \quad &= \quad 0, \\ -\alpha A^*_{\bar{\phi}E} X + \alpha \nabla^{*t}_X \bar{\phi} E \quad &\in \quad \Gamma(RadTM), \end{aligned}$$

and $g(A^*_E X, \xi) = 0$ if and only if

$$h(X, \bar{\phi} E) \in \Gamma(D_0 \bot \tilde{D}_0)^{\bot}, \tag{3.28}$$

$$A^*_E X \in \Gamma(\tilde{D}_2 \bot \tilde{D}_1 \bot \bar{\phi}(\mu_2)). \tag{3.29}$$

Now, suppose that $E \in \Gamma(\mu_2)$. By use of (2.10) with (3.25), we arrive at

$$\begin{aligned} \nabla_X E \quad = \quad & \alpha \nabla^*_X \bar{\phi} E + \alpha h^*(X, \bar{\phi} E) + B h(X, \bar{\phi} E) \\ & + g(A^*_E X, \xi)\xi. \end{aligned} \tag{3.30}$$

Therefore $\nabla_X E \in \Gamma(RadTM)$ if and only if

$$\begin{aligned} Bh(X, \bar{\phi} E) \quad &= \quad 0, \\ -\alpha A^*_{\bar{\phi}E} X + \alpha \nabla^{*t}_X \bar{\phi} E \quad &\in \quad \Gamma(RadTM), \end{aligned}$$

and $g(A^*_E X, \xi) = 0$ if and only if

$$h(X, \bar{\phi}E) \in \Gamma(L \perp S)^{\perp},$$

$$\nabla^*_X \bar{\phi}E + h^*(X, \bar{\phi}E) \in \Gamma(\bar{\phi}(\mu_2) \perp \mu_1), \tag{3.31}$$

$$A^*_E X \in \Gamma(\tilde{D}_2 \perp \tilde{D}_1 \perp \bar{\phi}(\mu_2)). \tag{3.32}$$

Thus, in view of (3.28), (3.29), (3.31) and (3.32), the proof is completed. □

References

[1] B. E. Acet, S. Yüksel Perktaş and E. Kılıç, *On lightlike geometry of para-Sasakian manifolds*, Scientific World Journal, Article ID 696231, 2014 .

[2] B. E. Acet, S. Yüksel Perktaş and E. Kılıç, *On submanifolds of para-Sasakian manifolds*, JP Journal of Geometry and Topology, 19 (2016), 1-18.

[3] B. E. Acet, S. Yüksel Perktaş and E. Kılıç, *Symmetric lightlike hypersurfaces of a para-Sasakian space form*, Analele Stiintifice Ale Universitatii Al.I. Cuza Iasi (N.S.), 2 (2016), 915-926.

[4] B. E. Acet and S. Yüksel Perktaş, *Screen slant radical transversal lightlike submanifolds of para-Sasakian manifolds*, Facta Universitatis (NIS) Ser. Math. Inform., 31 (2016), 543-557.

[5] B. E. Acet, S. Yüksel Perktaş and E. Kılıç, *Lightlike submanifolds of a para-Sasakian manifold*, General Math. Notes, 22 (2014), 22-45.

[6] B. E. Acet and S. Yüksel Perktaş, *On geodesic paracontact CR-lightlike submanifolds*, British J. Math. and Comp. Sci., 14(6), 1-11, 2016.

[7] B. E. Acet and S. Yüksel Perktaş, *On para-Sasakian manifolds with a canonical paracontact connection*, New Trends in Math. Sci., 3 (2016), 162-173.

[8] D. V. Alekseevski, V. Cortés, A.S. Galaev and T. Leistner, *Cones over pseudo-Riemannian manifolds and their holonomy*, J. Reine Angew. Math., 635 (2009), 23-69.

[9] D. V. Alekseevski, C. Medori and A. Tomassini, *Maximally homogeneous para-CR manifolds*, Ann. Global Anal. Geom., 30 (2006), 1-27.

[10] A. Bejancu, *CR-submanifold of a Kaehler manifold I*, Proc. Amer. Math. Soc., 69 (1978), 135-142.

[11] A. Bejancu, *CR-submanifold of a Kaehler manifold II*, Trans. Amer. Math. Soc., 250 (1979), 333-345.

[12] A. Bejancu, M. Kon and K. Yano, *CR-submanifold of a complex space form*, J. Diff. Geom., 16 (1981), 333137-145.

[13] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressing, *Special geometry of Euclidean supersymmetry 1. Vector multiplets*, J. High Energy Phys., 0403 (2004),73 pp.

[14] V. Cortés, M.A. Lawn and L. Schäfer, *Affine hyperspheres associated to special para-Kähler manifolds*, Int. J. Geom. Methods Mod. Phys., 3 (2006), 995-1009.

[15] K. L. Duggal and A. Bejancu, *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Mathematics and Its Applications, Kluwer Publisher, 1996.

[16] K. L. Duggal and B. Şahin, *Differential geometry of lightlike submanifolds*, Frontiers in Mathematics, 2010.

[17] K. L. Duggal and B. Şahin, *Lightlike submanifolds of indefinite Sasakian manifolds*, International Journal of Mathematics and Mathematical Sciences, Article ID 57585, 2007.

[18] K. L. Duggal and B. Şahin, *Generalized Cauchy-Riemann lightlike submanifolds of indefinite Sasakian manifolds*, Acta Math. Hungarica, 122 (2009), 45-58.

[19] K. L. Duggal and B. Şahin, *Generalized Cauchy-Riemann lightlike submanifolds of Kaehler manifolds*, Acta Math. Hungarica, 112 (2006), 107-130.

[20] S. Erdem, *On almost (para)contact (hyperbolic) metric manifolds and harmonicity of (φ, φ')-holomorphic maps between them*, Houston J. Math., 28 (2002), 21-45.

[21] G. J. Galloway, *Lecture notes on Spacetime Geometry*, Beijing International Mathematics Research Center, (2007), 1-55.

[22] V. Jain, R. Kumar, R. K. Nagaich, *Integrability of Distributions in GCR-Lightlike Submanifolds of Indefinite Sasakian Manifolds*, Kyungpook Math. J., 53(2013), 207–218.

[23] D. H. Jin, *Special lightlike hypersurfaces of inmdefinite Kaehler manifolds*, Filomat, 30 (2016), 1919-1930.

[24] S. Kaneyuki and M. Konzai, *Paracomplex structure and affine symmetric spaces*, Tokyo J. Math., 8 (1985), 301-308.

[25] D. N. Küpeli, *Singuler semi-Riemann geometry*, Mathematics and Its Application, vol 366, 1996.

[26] F. Massamba and S. Ssekajja, *Quasi generalized CR-lightlike submanifolds of indefinite nearly Sasakian manifolds*, Arab J. Math., 5 (2016), 87-101.

[27] B. O'Neill, *Semi-Riemannian geometry with appl. to relativity*, Academic Press, 1983.

[28] B. Şahin and R. Güneş, *Geodesic CR-lightlike submanifold, Contributions to Algebra and Geometry*, 42 (2001), 583-594.

[29] S. Yüksel Perktaş, *On para-Sasakian manifolds satisfying certain curvature conditions with canonical paracontact connection*, Khayyam J. Math., 3 (2017), 34-44.

[30] S. Zamkovoy, *Canonical connection on paracontact manifolds*, Ann. Glob. Anal. Geo., 36 (2009),37-60.

Author information

Selcen Yüksel Perktaş, Faculty of Arts and Sciences, Department of Mathematics, Adıyaman University, Adıyaman, Turkey.
E-mail: `sperktas@adiyaman.edu.tr`

Feyza Esra Erdoğan, Faculty of Science, department of mathematics, Ege University, Izmir, Turkey.
E-mail: `eyza.esra.erdogan@ege.edu.tr`

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