Palestine Journal of Mathematics

Vol. 8(1)(2019) , 209–216

Deformations of Nearly C-manifolds

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Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53D10, 53C15; Secondary 53C25, 53C35.

Keywords and phrases: Framed manifolds, nearly C-manifolds, constant Φ-sectional curvature, D-conformal transformations.

Abstract. A study of nearly C-manifold has been made. Using some suitable D conformal transformations, the deformation of the characteristic structure of nearly C-manifold has been investigated and it is seen that the characteristic structures are invariant under D conformal transformations. Necessary and sufficient conditions for the main and deformed manifolds have also been obtained.

1 Introduction

D-conformal transformations are very useful tools in differential geometry, particularly in manifold theory. In 1968, Tanno defined D-homothetic transformations which are special classes of D-conformal transformations [15] He used them to get Betti numbers of certain contact Riemannian manifolds [14]. On the other hand, D-conformal transformations are used to define new classes of manifolds from existed manifolds. While Jamssens and Vanhecke introduced almost α-Kenmotsu manifolds in 1981 [8], Kim and Pak defined almost α-cosymplectic manifolds by defining suitable D-conformal transformations [9].

In the present paper, we consider a class of D-conformal transformations which is defined by [11] on nearly C-manifolds (which were introduced by Balkan and Aktan in 2016 [1]). Section 2 contains few known definitions and results that are required for further investigation. Nearly C-manifolds have been studied in section 3 and finally the effect of D-conformal transformation on nearly C-manifold has been considered. A nearly C-manifold is not a C-manifold and it is a class of globally framed f-manifolds analogue to nearly Kähler manifolds [6].

The notion of globally framed manifold or globally framed f-manifold, which is generalization of complex and contact manifolds, was introduced by Nakagawa in 1966 [10]. Then, Blair defined three classes of globally framed manifolds, called K-manifold, S-manifold and C-manifold [2]. Many researchers studied on these manifolds. Falcitelli and Pastore introduced almost Kenmotsu f-manifolds in 2007 [3]. In 2014, Öztürk et al. defined almost α-cosymplectic f-manifolds, which are generalization of almost C-manifolds and almost Kenmotsu f-manifolds [12].

2 Preliminaries

Let M be $(2n+s)$-dimensional manifold and φ is a non-null $(1,\ 1)$ tensor field on M. If φ satisfies

$$\varphi^3+\varphi=0, \tag{2.1}$$

then φ is called an f-structure and M is called f-manifold [16]. If $rank\varphi=2n$, namely $s=0$, φ is called almost complex structure and if $rank\varphi=2n+1$, namely $s=1$, then φ reduces an almost contact structure [5]. $rank\varphi$ is always constant [13].

On an f-manifold M, P_1 and P_2 operators are defined by

$$P_1=-\varphi^2,\quad P_2=\varphi^2+I, \tag{2.2}$$

which satisfy

$$\begin{gathered} P_1+P_2=I,\qquad P_1^2=P_1,\qquad P_2^2=P_2,\\ \varphi P_1=P_1\varphi=\varphi,\quad P_2\varphi=\varphi P_2=0. \end{gathered} \tag{2.3}$$

These properties show that P_1 and P_2 are complement projection operators. There are D and $D^{\perp}$ distributions with respect to P_1 and P_2 operators, respectively [17]. Also, dim $(D) = 2n$ and dim $\left(D^{\perp}\right) = s$.

Let M be $(2n+s)$-dimensional f-manifold and φ is a $(1,\ 1)$ tensor field, ξ_i is vector field and η^i is 1-form for each $1 \leq i \leq s$ on M, respectively. If $(\varphi,\ \xi_i,\ \eta^i)$ satisfy

$$\eta^j\left(\xi_i\right) = \delta_i^j, \tag{2.4}$$

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \tag{2.5}$$

then $(\varphi,\ \xi_i,\ \eta^i)$ is called globally framed f-structure or simply framed f-structure and M is called globally framed f-manifold or simply framed f-manifold [10]. For a framed f-manifold M, the following properties are satisfied [10]:

$$\varphi\xi_i = 0, \tag{2.6}$$

$$\eta^i \circ \varphi = 0. \tag{2.7}$$

If on a framed f-manifold M, there exists a Riemannian metric which satisfies

$$\eta^i\left(X\right) = g\left(X,\ \xi_i\right), \tag{2.8}$$

and

$$g\left(\varphi X,\ \varphi Y\right) = g\left(X,\ Y\right) - \sum_{i=1}^{s} \eta^i\left(X\right)\eta^i\left(Y\right), \tag{2.9}$$

for all vector fields $X,\ Y$ on M, then M is called framed metric f-manifold [4]. On a framed metric f-manifold, fundamental 2-form Φ defined by

$$\Phi\left(X,\ Y\right) = g\left(X,\ \varphi Y\right), \tag{2.10}$$

for all vector fields $X,\ Y \in \chi\left(M\right)$ [4]. For a framed metric f-manifold,

$$N_{\varphi} + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0, \tag{2.11}$$

is satisfied, M is called normal framed metric f-manifold, where N_{φ} denotes the Nijenhuis torsion tensor of φ [7].

3 Nearly C-manifolds

In this section, we recall some basic facts about nearly C-manifolds for further properties from [1]

Definition 3.1. Let M be globally framed f-manifold and ∇ is Levi-Civita connection. For each $X,\ Y \in \chi\left(M\right)$,

$$\left(\nabla_X\varphi\right)Y + \left(\nabla_Y\varphi\right)X = 0, \tag{3.1}$$

is satisfied, then M is called nearly C-manifold.

On a nearly C-manifold, we can define H_i and H^i tensor fields by

$$H_iX = \nabla_X\xi_i, \tag{3.2}$$

and

$$H^iX = \nabla_{\xi_i}X, \tag{3.3}$$

for all vector fields X on M and for each $1 \leq i \leq s$.

Corollary 3.2. *For H_i and H^i tensor fields, the following relation holds:*

$$H_iX = -\varphi H^i \varphi X + \varphi^2 H^i X. \tag{3.4}$$

Proposition 3.3. *For each $1 \leq i \leq s$, H_i and H^i tensor fields satisfy the following properties:*

(1) $H_i(\xi_i) = 0,\ H^i(\xi_i) = 0,$

(2) $H_i\varphi X + \varphi H_i X = -\sum\limits_{j=1}^{s} \varphi H^i\left(\eta^j(X)\xi_j\right),$

(3) $tr(H_i) = 0,$

(4) $(\nabla_X \varphi)\xi_i = -\varphi H_i X,$

for all vector fields $X \in \chi(M)$.

Lemma 3.4. *On a nearly C-manifold M, the following identity holds:*

$$g(H_iX,\ Y) + g(X,\ H_iY) = \sum_{k=1}^{s}\left\{\eta^k(Y)\, g(X,\ H_k\xi_i) + \eta^k(X)\, g(Y,\ H_k\xi_i)\right\}, \tag{3.5}$$

for all vector fields $X,\ Y \in \chi(M)$.

Corollary 3.5. *On a nearly C-manifold M, for each $1 \leq i \leq s$, the characteristic vector field ξ_i is not Killing.*

Lemma 3.6. *Let M be a nearly C-manifold and R is the Riemannian curvature tensor of M. Then the following identities hold [1]:*

$$\begin{aligned} g(R(\varphi X,\ \varphi Y)\varphi Z,\ \varphi W) &= g(R(X,\ Y)Z,\ W) \\ -\textstyle\sum_{i=1}^{s}\left\{\eta^i(X)\, g(R(\xi_i,\ Y)Z,\ W) - \eta^i(Y)\, g(R(X,\ \xi_i)Z,\ W)\right\}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} g(R(\xi_i,\ X)Y,\ Z) &= -g((\nabla_X H_i)Y,\ Z) \\ &= \sum_{k=1}^{s}\{\eta^k(Y)\, g((H_i\circ H_k)X,\ Z) \\ &\quad -\eta^k(Z)\, g((H_i\circ H_k)X,\ Y)\}, \end{aligned} \tag{3.7}$$

$$(R(\varphi W,\ \varphi X)Y,\ Z) = g(R(W,\ X)Y,\ Z) + g((\nabla_W\varphi)X,\ (\nabla_Y\varphi)Z) + \sum_{k=1}^{s} g(H_kW,\ X)\, g(Z,\ H_kY). \tag{3.8}$$

Definition 3.7. Let M be a nearly C-manifold. If at any point $p \in M$ the sectional curvature $K(X,\ \varphi X)$, which is denoted by K_p, is independent of the choice of the tangent vector $X \in T_pM$, such that $X \neq 0$ and for each $1 \leq i \leq s$ $X \perp \xi_i$, then it is said that M has the pointwise constant φ-sectional curvature K_p.

4 D-conformal Transformations on Nearly C-manifolds

Definition 4.1. Let M be a globally framed f-manifold and β be a non-zero smooth function which satisfies $d\beta \wedge \eta^i = 0$ for each $1 \leq i \leq s$. Then, for each $\lambda \in R$, the D-conformal transformation of $(\varphi,\ \xi_i,\ \eta^i,\ g)$ on M are defined by

$$\widetilde{\varphi} = \varphi, \quad \widetilde{\xi}_i = \frac{1}{\beta}\xi_i, \quad \widetilde{\eta}^i = \beta\eta^i, \quad \widetilde{g} = \lambda g + \left(\beta^2 - \lambda\right)\sum_{i=1}^{s}\eta^i \otimes \eta^i. \tag{4.1}$$

Furthermore, $\widetilde{M}$ is a globally framed f-manifold admitting a $\left(\widetilde{\varphi},\ \widetilde{\xi}_i,\ \widetilde{\eta}^i,\ \widetilde{g}\right)$ f-structure. Additionally, we can define a fundamental 2-form $\widetilde{\Phi}(X,\ Y) = \widetilde{g}(X,\ \widetilde{\varphi}Y)$.

In view of this definition, we can give following corollaries:

Corollary 4.2. *$\widetilde{M}$ is a globally framed f-manifold if and only if M is a globally framed f-manifold.*

Corollary 4.3. *There is relation between $\widetilde{\Phi}$ and Φ such that*

$$\widetilde{\Phi}\left(X,\ Y\right)=\widetilde{g}\left(X,\ \widetilde{\varphi}Y\right)=\lambda g\left(X,\ \varphi Y\right), \tag{4.2}$$

namely, $\widetilde{\Phi}=\lambda\Phi$ for all X and Y vector fields. Where $\lambda\in R$ and $\widetilde{\Phi}$ and Φ are fundamental 2-forms on M and $\widetilde{M}$, respectively.

Proposition 4.4. *Let M and $\widetilde{M}$ be globally framed f-manifolds such as stated in the above. The following identities hold*

$$\widetilde{\nabla}_X\widetilde{\xi}_i=X\left(\frac{1}{\beta}\right)\xi_i+\nabla_X\xi_i+\frac{\varepsilon_i}{\beta}\eta^i\left(X\right)\xi_i, \tag{4.3}$$

$$\widetilde{\nabla}_{\widetilde{\xi}_i}X=\nabla_{\xi_i}X+\frac{\varepsilon_i}{\beta}\eta^i\left(X\right)\xi_i, \tag{4.4}$$

$$\sum_{k=1}^{s}\eta^k\left(\widetilde{\nabla}_XY\right)=\sum_{k=1}^{s}\left\{\eta^k\left(\nabla_XY\right)+\varepsilon_k\eta^k\left(X\right)\eta^k\left(Y\right)\right\}, \tag{4.5}$$

$$\widetilde{\nabla}_XY=\nabla_XY+\sum_{k=1}^{s}\varepsilon_k\eta^k\left(X\right)\eta^k\left(Y\right)\xi_k, \tag{4.6}$$

for all vector fields $X,\ Y$. Where ∇ and $\widetilde{\nabla}$ denotes Levi-Civita connections on M and $\widetilde{M}$, respectively. We set $\varepsilon_k:=\dfrac{d\beta\left(\xi_k\right)}{\beta}$ for simplicity.

Proof. From Koszul formula, we have

$$\begin{aligned}2\widetilde{g}\left(\widetilde{\nabla}_XY,\ Z\right)&=2\lambda g\left(\nabla_XY,\ Z\right)\\&+2\left(\beta^2-\lambda\right)\sum_{k=1}^{s}\eta^k\left(\nabla_XY\right)\eta^k\left(Z\right)\\&+2\sum_{k=1}^{s}d\beta\left(\xi_k\right)\eta^k\left(X\right)\eta^k\left(Y\right)\eta^k\left(Z\right).\end{aligned} \tag{4.7}$$

On the other hand, using (4.1), we find

$$\begin{aligned}2\widetilde{g}\left(\widetilde{\nabla}_XY,\ Z\right)&=2\lambda g\left(\widetilde{\nabla}_XY,\ Z\right)\\&+2\left(\beta^2-\lambda\right)\sum_{k=1}^{s}\eta^k\left(\widetilde{\nabla}_XY\right)\eta^k\left(Z\right).\end{aligned} \tag{4.8}$$

In view of (4.7) and (4.8), we get (4.5) and (4.6). Putting $Y=\widetilde{\xi}_i$ in (4.6), we have (4.3). Again, choosing $X=\widetilde{\xi}_i$ and $Y=X$ we obtain (4.4). □

Theorem 4.5. *Let M be globally framed f-manifold and $\widetilde{M}$ be a globally framed f-manifold which is gotten from M by using D-conformal transformations. Then, $\widetilde{M}$ is a nearly C-manifold if and only if M is a nearly C-manifold.*

Proof. Replacing Y by $\widetilde{\varphi}Y$ in (4.6) and using (2.6) and $\widetilde{\varphi}=\varphi$, we have

$$\left(\widetilde{\nabla}_X\widetilde{\varphi}\right)Y=\left(\nabla_X\varphi\right)Y. \tag{4.9}$$

Similarly, we get

$$\left(\widetilde{\nabla}_Y\widetilde{\varphi}\right)X=\left(\nabla_Y\varphi\right)X. \tag{4.10}$$

Now, adding (4.9) and (4.10) side by side, we have

$$\left(\widetilde{\nabla}_X\widetilde{\varphi}\right)Y+\left(\widetilde{\nabla}_Y\widetilde{\varphi}\right)X=\left(\nabla_X\varphi\right)Y+\left(\nabla_Y\varphi\right)X. \tag{4.11}$$

Since M is a nearly C-manifold, we find

$$\left(\widetilde{\nabla}_X\widetilde{\varphi}\right)Y+\left(\widetilde{\nabla}_Y\widetilde{\varphi}\right)X=0, \tag{4.12}$$

which means $\widetilde{M}$ is a nearly C-manifold.

Conversely, let $\widetilde{M}$ be a nearly C-manifold. Then (4.10) is satisfied. Considering (4.6), we have $\left(\nabla_X\widetilde{\varphi}\right)Y+\left(\nabla_Y\widetilde{\varphi}\right)X=0$. Since $\widetilde{\varphi}=\varphi$, we obtain $\left(\nabla_X\varphi\right)Y+\left(\nabla_Y\varphi\right)X=0$ which means M is a nearly C-manifold. This completes our proof. □

Let M and $\widetilde{M}$ be globally framed f-manifolds such as stated in the above. On globally framed f-manifold $\widetilde{M}$, $\widetilde{H}_i$ and $\widetilde{H}^i$ tensor fields are defined by $\widetilde{H}_iX=\widetilde{\nabla}_X\widetilde{\xi}_i$ and $\widetilde{H}^iX=\widetilde{\nabla}_{\widetilde{\xi}_i}X$, respectively. Then, considering definitions of H_i and H^i tensor fields, we have

$$\widetilde{H}_iX=\frac{1}{\beta}H_iX+X\left(\frac{1}{\beta}\right)\xi_i+\frac{\varepsilon_i}{\beta}\eta^i\left(X\right)\xi_i \tag{4.13}$$

and

$$\widetilde{H}^iX=\frac{1}{\beta}H_iX+\frac{\varepsilon_i}{\beta}\eta^i\left(X\right)\xi_i. \tag{4.14}$$

Now, we can give a corollary similar to Corollary 3.2.

Corollary 4.6. *There is a relation between $\widetilde{H}_i$ and $\widetilde{H}^i$ tensor fields such as*

$$\widetilde{H}_iX=-\widetilde{\varphi}\widetilde{H}^i\widetilde{\varphi}X+\widetilde{\varphi}^2\widetilde{H}^iX+\frac{\varepsilon_i}{\beta}\eta^i\left(X\right)\xi_i. \tag{4.15}$$

Proof. From (4.1), we find

$$\widetilde{\varphi}\widetilde{\xi}_i=\frac{1}{\beta}\varphi\xi_i, \tag{4.16}$$

for each $1\leq i\leq s$. Differentiating (4.16) with respect to X, we get

$$\widetilde{H}_iX=-\widetilde{\varphi}\widetilde{H}^i\widetilde{\varphi}X+\widetilde{\varphi}^2\widetilde{H}^iX+\sum_{k=1}^{s}\eta^k\left(\widetilde{\nabla}_X\widetilde{\xi}_i\right)\xi_k. \tag{4.17}$$

Finally, using (4.3) and (4.5), we obtain desired result. □

Theorem 4.7. *Let M be a nearly C-manifold and $\widetilde{M}$ be a nearly C-manifold which is gotten from M by using D-conformal transformations. Denote R and $\widetilde{R}$ the Riemannian curvature tensor of M and $\widetilde{M}$,respectively, then the following identity holds*

$$\widetilde{R}\left(X,\ Y\right)Z=R\left(X,\ Y\right)Z+\sum_{k=1}^{s}\left\{A_k\left(X,\ Y,\ Z\right)-A_k\left(Y,\ X,\ Z\right)\right\}, \tag{4.18}$$

for all vector fields $X,\ Y,\ Z$ and for each $1\leq k\leq s$. Here, for each $1\leq k\leq s$, the A_k tensor fields are defined by

$$\begin{aligned}A_k\left(X,\ Y,\ Z\right)&=X\left(\varepsilon_k\right)\eta^k\left(Y\right)\eta^k\left(Z\right)\xi_k+\varepsilon_kX\left(\frac{1}{\beta}\right)\eta^i\left(Y\right)\eta^k\left(Z\right)\xi_k\\&+\varepsilon_kg\left(Y,\ H_kX\right)\eta^k\left(Z\right)\xi_k+\varepsilon_kX\left(\frac{1}{\beta}\right)\eta^i\left(Z\right)\eta^k\left(Y\right)\xi_k+\varepsilon_kg\left(Z,\ H_kX\right)\eta^k\left(Y\right)\xi_k\\&+\varepsilon_kX\left(\frac{1}{\beta}\right)\eta^k\left(Y\right)\eta^k\left(Z\right)\xi_k+\varepsilon_k\eta^k\left(Y\right)\eta^k\left(Z\right)H_kX.\end{aligned} \tag{4.19}$$

Proof. Let ∇ and $\widetilde{\nabla}$ be Levi-Civita connections on M and $\widetilde{M}$. All vector fields $X,\ Y,\ Z,$ it is well-known that the Riemannian curvature $\widetilde{R}$ of $\widetilde{M}$ is defined by

$$\widetilde{R}\left(X,\ Y\right)Z=\widetilde{\nabla}_X\widetilde{\nabla}_YZ-\widetilde{\nabla}_Y\widetilde{\nabla}_XZ-\widetilde{\nabla}_{[X,\ Y]}Z. \tag{4.20}$$

Taking (4.6) into account of (4.20) and making necessary calculations, we easily get (4.19). □

Lemma 4.8. *Let M and $\widetilde{M}$ be nearly C-manifolds which are stated in the above. For all vector fields $X,\ Y,\ Z$ and for each $1 \leq k \leq s,$ the A_k tensor fields satisfy*

$$\sum_{k=1}^{s} A_k\left(X,\ \widetilde{\varphi}Y,\ Z\right) = \sum_{k=1}^{s} \varepsilon_k g\left(\widetilde{\varphi}Y,\ H_k X\right)\eta^k\left(Z\right)\xi_k \tag{4.21}$$

and

$$\sum_{k=1}^{s} A_k\left(X,\ Y,\ \widetilde{\varphi}Z\right) = \sum_{k=1}^{s} \varepsilon_k g\left(\widetilde{\varphi}Z,\ H_k X\right)\eta^k\left(Y\right)\xi_k. \tag{4.22}$$

Proof. Putting $Y = \widetilde{\varphi}Y$ in (4.19) and using (2.7), we obtain (4.21). Similarly, setting $Z = \widetilde{\varphi}Z$ in (4.19), then we have (4.22). □

Theorem 4.9. *Let M and $\widetilde{M}$ be nearly C-manifolds which are stated in the above and let R and $\widetilde{R}$ denote the Riemannian curvature tensors of M and $\widetilde{M}$,respectively. Then the following identities are satisfied:*

$$\begin{aligned}\widetilde{g}\left(\widetilde{R}\left(X,\ Y\right)Z,\ W\right) = \lambda\Big[g\left(R\left(X,\ Y\right)Z,\ W\right) + \sum_{k=1}^{s}\{g\left(A_k\left(X,\ Y,\ Z\right),\ W\right) \\ -g\left(A_k\left(Y,\ X,\ Z\right),\ W\right)\}\Big] + \left(\beta^2-\lambda\right)\sum_{k=1}^{s}\big\{\eta^k\left(R\left(X,\ Y\right)Z\right) \\ +\eta^k\left(A_k\left(X,\ Y,\ Z\right)\right) - \eta^k\left(A_k\left(Y,\ X,\ Z\right)\right)\big\}\eta^k\left(W\right),\end{aligned} \tag{4.23}$$

$$\begin{aligned}\widetilde{g}\left(\widetilde{R}\left(\widetilde{\varphi}X,\ Y\right)Z,\ W\right) + \widetilde{g}\left(\widetilde{R}\left(X,\ \widetilde{\varphi}Y\right)Z,\ W\right) + \widetilde{g}\left(\widetilde{R}\left(X,\ Y\right)\widetilde{\varphi}Z,\ W\right) \\ +\widetilde{g}\left(\widetilde{R}\left(X,\ Y\right)Z,\ \widetilde{\varphi}W\right) = \sum_{k=1}^{s}\{B_k\left(X,\ Y,\ Z,\ W\right) - B_k\left(Y,\ X,\ Z,\ W\right)\},\end{aligned} \tag{4.24}$$

$$\begin{aligned}\widetilde{g}\left(\widetilde{R}\left(\widetilde{\varphi}X,\ \widetilde{\varphi}Y\right)\widetilde{\varphi}Z,\ \widetilde{\varphi}W\right) = \lambda\{g\left(R\left(X,\ Y\right)Z,\ W\right) \\ -\sum_{k=1}^{s}\left\{\eta^k\left(X\right)g\left(R\left(\xi_k,\ Y\right)Z,\ W\right) - \eta^k\left(Y\right)g\left(R\left(X,\ \xi_k\right)Z,\ W\right)\right\}\Big\},\end{aligned} \tag{4.25}$$

$$\begin{aligned}\widetilde{g}\left(\widetilde{R}\left(\widetilde{\varphi}X,\ \widetilde{\varphi}Y\right)Z,\ W\right) = \lambda g\left(R\left(X,\ Y\right)\widetilde{\varphi}Z,\ \widetilde{\varphi}W\right) \\ +\left[\lambda + s\left(\beta^2-\lambda\right)\right]\sum_{k=1}^{s}\varepsilon_k\left\{g\left(\widetilde{\varphi}Y,\ H_k\widetilde{\varphi}X\right)\eta^k\left(Z\right)\eta^k\left(W\right)\right. \\ \left.-g\left(\widetilde{\varphi}X,\ H_k\widetilde{\varphi}Y\right)\eta^k\left(Z\right)\eta^k\left(W\right)\right\},\end{aligned} \tag{4.26}$$

$$\begin{aligned}\widetilde{g}\left(\widetilde{R}\left(\xi_i,\ X\right)Y,\ Z\right) = \sum_{k=1}^{s}\{C_k\left(\xi_i,\ X,\ Y,\ Z\right) - C_k\left(X,\ \xi_i,\ Y,\ Z\right)\} \\ +\sum_{k=1}^{s}\left\{\lambda\eta^k\left(Y\right)g\left(\left(H_i\circ H_k\right)X,\ Z\right) - \left[\lambda + s\left(\beta^2-\lambda\right)\right]\eta^k\left(Z\right)g\left(\left(H_i\circ H_k\right)X,\ Y\right)\right\}\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}\widetilde{g}\left(\widetilde{R}\left(\widetilde{\varphi}X,\ \widetilde{\varphi}Y\right)Z,\ W\right) = \lambda\{g\left(R\left(X,\ Y\right)Z,\ W\right) \\ +g\left(\left(\nabla_X\widetilde{\varphi}\right)Y,\ \left(\nabla_Z\widetilde{\varphi}\right)W\right) + \sum_{k=1}^{s}g\left(H_kX,\ Y\right)g\left(H_kZ,\ W\right)\Big\} \\ +\left[\lambda + s\left(\beta^2-\lambda\right)\right]\sum_{k=1}^{s}\left\{g\left(\widetilde{\varphi}Y,\ H_k\widetilde{\varphi}X\right)\eta^k\left(Z\right)\right. \\ \left.-g\left(\widetilde{\varphi}X,\ H_k\widetilde{\varphi}Y\right)\eta^k\left(Z\right)\right\}\eta^k\left(W\right) \\ +\left(\beta^2-\lambda\right)\sum_{k=1}^{s}\left\{g\left(\left(\nabla_X\widetilde{\varphi}\right)Y,\ H_k\widetilde{\varphi}X\right)\eta^k\left(W\right) - \eta^k\left(R\left(X,\ Y\right)Z\right)\eta^k\left(W\right)\right\},\end{aligned} \tag{4.28}$$

for all vector fields $X,\ Y,\ Z,\ W$, where B_k and C_k tensor fields are defined by

$$\begin{aligned}B_k\left(X,\ Y,\ Z,\ W\right) = \lambda g\left(A_k\left(\widetilde{\varphi}X,\ Y,\ Z\right),\ W\right) \\ \left(\beta^2-\lambda\right)\eta^k\left(A_k\left(\widetilde{\varphi}X,\ Y,\ Z\right)\right)\eta^k\left(W\right) \\ \left[\lambda + s\left(\beta^2-\lambda\right)\right]\varepsilon_k\left\{g\left(\widetilde{\varphi}Y,\ H_kX\right)\eta^k\left(W\right)\eta^k\left(Z\right)\right. \\ \left.g\left(\widetilde{\varphi}Z,\ H_kX\right)\eta^k\left(W\right)\eta^k\left(Y\right)\right\} + \varepsilon_k\lambda\eta^k\left(Z\right)\eta^k\left(Y\right)g\left(\widetilde{\varphi}W,\ H_kX\right)\end{aligned} \tag{4.29}$$

and

$$C_k(X, \xi_i, Y, Z) = \lambda g(A_k(X, \xi_i, Y), Z) + (\beta^2 - \lambda)\eta^k(A_k(X, \xi_i, Y))\eta^k(Z), \quad (4.30)$$

for each $1 \le k \le s$.

Proof. **(4.23):** It is clear from (4.18).

(4.24): Replacing X by $\widetilde{\varphi}X$ in (4.23) and using (4.19) and (4.21), we get

$$\begin{aligned} \widetilde{g}\left(\widetilde{R}(\widetilde{\varphi}X, Y)Z, W\right) &= \lambda\{g(R(\widetilde{\varphi}X, Y)Z, W) \\ &+ \sum_{k=1}^{s}\left[g(A_k(\widetilde{\varphi}X, Y, Z), W) - \varepsilon_k g(\widetilde{\varphi}X, H_kY)\eta^k(W)\eta^k(Z)\right]\Big\} \\ &+ (\beta^2 - \lambda)\sum_{k=1}^{s}\{\eta^k(R(\widetilde{\varphi}X, Y)Z) + \eta^k(A_k(\widetilde{\varphi}X, Y, Z)) \\ &- s\varepsilon_k g(\widetilde{\varphi}X, H_kY)\eta^k(Z)\}\eta^k(W). \end{aligned} \quad (4.31)$$

We can obtain the other terms of (4.23), similarly. Then adding the gotten equations side by side, we find (4.24).

(4.25): Replacing X, Y, Z and W by $\widetilde{\varphi}X$, $\widetilde{\varphi}Y$, $\widetilde{\varphi}Z$ and $\widetilde{\varphi}W$, respectively in (4.23) and considering (4.21) and (4.22), then we get

$$\widetilde{g}\left(\widetilde{R}(\widetilde{\varphi}X, \widetilde{\varphi}Y)\widetilde{\varphi}Z, \widetilde{\varphi}W\right) = \lambda g(R(\widetilde{\varphi}X, \widetilde{\varphi}Y)\widetilde{\varphi}Z, \widetilde{\varphi}W). \quad (4.32)$$

Putting (3.6) into (4.32), we have desired result.

(4.26): Similar to the last calculations, taking $X = \widetilde{\varphi}X$ and $Y = \widetilde{\varphi}Y$ in (4.23), we get (4.26) by using (4.21) and (4.22).

(4.27): We can get easily from (3.7) and (4.23).

(4.28): Finally, in (4.23), using (3.8), we have (4.28). □

Theorem 4.10. *Let M and $\widetilde{M}$ be nearly C-manifolds which are stated in the above. $\widetilde{M}$ has the pointwise constant φ-sectional curvature K if and only if M has the pointwise constant φ-sectional curvature K.*

Proof. Let M be a nearly C-manifold with pointwise constant φ-sectional curvature K. At any point $p \in M$, for each $1 \le k \le s$ and for all vector fields X such that $X \perp \xi_k$, we have

$$g(R(\varphi X, X)\varphi X, X) + K_p g(X, X)g(X, X) = 0. \quad (4.33)$$

Using (4.1) in the last equation and takin (4.23) into account of gotten equation, we obtain

$$\begin{aligned} &\widetilde{g}\left(\widetilde{R}(\widetilde{\varphi}X, X)\widetilde{\varphi}X, X\right) + K_p\widetilde{g}(X, X)\widetilde{g}(X, X) \\ &= \lambda^2\{g(R(\widetilde{\varphi}X, X)\widetilde{\varphi}X, X) + K_p g(X, X)g(X, X)\}. \end{aligned} \quad (4.34)$$

In view of (4.33), we have

$$\widetilde{g}\left(\widetilde{R}(\widetilde{\varphi}X, X)\widetilde{\varphi}X, X\right) + K_p\widetilde{g}(X, X)\widetilde{g}(X, X) = 0, \quad (4.35)$$

which means $\widetilde{M}$ has the pointwise constant φ-sectional curvature K_p at any point $p \in M$.

Conversely, let $\widetilde{M}$ has pointwise constant φ-sectional curvature K_p. Then (4.35) is satisfied. Here, we can see easily that (4.33) holds from (4.1) and (4.23). This completes the proof. □

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