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A New Subclass Of Bi-Univalent Functions Defined Using Convolution

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Abstract. In this paper , we introduce a new subclass of bi-univalent function by making use of convolution(or Hadamard product) of analytic functions. We obtain the coefficient bounds and initial coefficient inequalities of this class using Faber polynomial approach. Connections to earlier known results are briefly indicated.

1 Introduction, Definitions and Preliminaries

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : z \in \mathbb{C} \quad and \quad |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) = 1$.

For functions $f(z)$ defined by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ we define the convolution of $f(z)$ and $g(z)$ by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, z \in U$

Let $f(z)$ and $g(z)$ be analytic functions in U , we say that $f(z)$ is subordinate to $g(z)$, written as

$$f(z) \prec g(z)$$

if there exist a Schwarz functions $w(z)$ in U, such that $f(z) = g(w(z))$ with $w(0) = 0$ and $|w(z)| < 1 \qquad (z \in U)$ [10]
In particular, if the function $g(z)$ is univalent in U, then the above subordination is equivalent to

$$f(0) = g(0)$$

and

$$f(U) \subseteq g(U).$$

Let S be the class of A consisting of the functions of the form (1.1) which are also univalent in U. According to Koebe one quarter theorem [10], it is well known fact that every function $f \in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z(z \in U)$$

and

$$f(f^{-1}(w)) = w \quad for \quad |w| < 1/4.$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of all bi-univalent functions in U given by the Taylor-Maclaurin series

expansion (1.1). Recently Hamidi et.al [13] and S. Bulut [7] have used the Faber polynomial approach to find coefficient estimates for subclasses of bi-univalent functions. Motivated by their work, in the present paper we introduce a new subclass of bi-univalent functions using convolution and obtain the upper bounds for the same.
We now introduce the subclass $B_{\Sigma}^{\lambda,\beta}(f,g,h;\phi)$ of bi-univalent functions as follows:

Definition 1.1. A function $f(z) \in \Sigma$ is said to be in the class $B_{\Sigma}^{\lambda,\beta}(f,g,h;\phi)$, for $\lambda \geq 0$ if the following conditions are satisfied:

$$e^{i\beta}\frac{(1-\lambda)(f*g)(z)+\lambda(f*h)(z)}{z} \prec \phi(z)cos\beta + isin\beta \qquad (z\in U)$$

and

$$e^{i\beta}\frac{(1-\lambda)(F*g)(w)+\lambda(F*h)(w)}{w} \prec \phi(w)cos\beta + isin\beta, \qquad (w\in U)$$

where $\beta \in (-\pi/2, \pi/2)$ and $F = f^{-1}$.

Remark 1.2. By giving special values for $g, h, \phi, \lambda, \beta$, we obtain several subclasses of bi-univalent functions that were studied earlier in the literature ([6],[9],[11],[12],[14],[20],[15],[5], [16], [17], [4], [21]) out of which few are listed below.

(i) $B_{\Sigma}^{\lambda,0}\left[f, z+\sum_{n=2}^{\infty} n^p z^n, z+\sum_{n=2}^{\infty} n^{p+1} z^n; \phi\right] = B_{\Sigma}(p,\lambda,\phi)$, this class was introduced and studied by Altinkaya and Yalcin [6].

(ii) $B_{\Sigma}^{1,\beta}\left[f, g, z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1)-\lambda}z^n; h\right] = B_{\Sigma}^{\lambda,\beta}(h)$, this class was introduced and studied by Goyal and Goswami [12].

(iii) $B_{\Sigma}^{1,0}\left[f, g, z+\sum_{n=2}^{\infty} n z^n; \frac{1+(1-2\beta)z}{1-z}\right] = H_{\Sigma}(\beta)$, this class was introduced and studied by Srivastava et.al [20].

(iv) $B_{\Sigma}^{\lambda,0}\left[f, \frac{z}{1-z}, \frac{z}{(1-z)^2}; \frac{1+(1-2\alpha)z}{1-z}\right] = Q_{\lambda}(\alpha)$, this class was introduced by Ding et.al [9] and studied by Frasin and Aouf [11] and improved by Jhangiri and Hamidi [14]

2 Coefficient bounds for the class $B_{\Sigma}^{\lambda,\beta}(f,g,h;\phi)$

Using the Faber Polynomial expansion of functions $f(z) \in A$ of the form (1.1), the coefficients of its inverse map $F = f^{-1}$ may be expressed as [2], $F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2, a_3, ...)w^n$ where

$$K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3$$
$$+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 + \frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}[a_5+(-n+2)a_3^2]$$
$$+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6+(-2n+5)a_3a_4] + \sum_{i\geq 7}a_2^{n-i}V_i,$$

such that V_i with $7 \leq i \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$ [3].

In particular, the first three terms of K_{n-1}^{-n} are

$$\frac{1}{2}K_1^{-2} = -a_2$$
$$\frac{1}{3}K_2^{-3} = 2a_2^2 - a_3$$

$$\frac{1}{4}K_3^{-4} = -(5a_2^3 - 5a_2a_3 + a_4)$$

In general, for any $p \in N$, an expansion of K_n^p is as, [[2],p.183]

$$K_n^p = pa_n + \frac{p(p-1)}{2}D_n^2 + \frac{(p!)}{(p-3)!3!}D_n^3 + ... + \frac{p!}{(p-n)!n!}D_n^n$$

where $D_n^p = D_n^p(a_2, a_3, ...)$ and by [18]

$$D_n^m(a_1, a_2, ..., a_n) = \sum_{m=1}^{\infty} \frac{m!(a_1)^{\mu_1}...(a_n)^{\mu_n}}{\mu_1!...\mu_n!}$$

while $a_1 = 1$, and the sum is taken over all non negative integers $\mu_1, ..., \mu_n$ satisfying

$$\mu_1 + \mu_2 + ... + \mu_n = m,$$

$$\mu_1 + 2\mu_2 + ... + n\mu_n = n,$$

It is clear that $D_n^n(a_1, a_2, ..., a_n) = a_1^n$[1].

Theorem 2.1. *For $\lambda \geq 1$,if $f(z) \in \Sigma$ satisfies (1.1),is in the class $B_\Sigma^{\lambda,\beta}(f, g, h; \phi)$ If $a_k = 0; (2 \leq k \leq n-1)$, then*

$$|a_n| \leq \frac{2cos\beta}{(1-\lambda)g_n + \lambda h_n}, n \geq 4$$

where $\beta \in (-\pi/2, \pi/2)$.

Proof. Let $f(z) \in \Sigma$ be as given in (1.1). Therefore,

$$e^{i\beta}\frac{(1-\lambda)(f*g)(z) + \lambda(f*h)(z)}{z} = e^{i\beta}\left[1 + \sum_{n=2}^{\infty}[(1-\lambda)g_n + \lambda h_n]a_n z^{n-1}\right] \tag{2.1}$$

and for its inverse map, $F = f^{-1}$,

$$\begin{aligned} e^{i\beta}\frac{(1-\lambda)(F*g)(w) + \lambda(F*h)(w)}{w} &= e^{i\beta}\left[1 + \sum_{n=2}^{\infty}[(1-\lambda)g_n + \lambda h_n]b_n w^{n-1}\right] \\ &= e^{i\beta}\left[1 + \sum_{n=2}^{\infty}[(1-\lambda)g_n + \lambda h_n] \times \frac{1}{n}K_{n-1}^{-n}(a_2, a_3, ...)w^{n-1}\right] \end{aligned} \tag{2.2}$$

On the other hand, since $f \in B_\Sigma^{\lambda,\beta}(f, g, h; \phi)$ and $F = f^{-1} \in B_\Sigma^{\lambda,\beta}(f, g, h; \phi)$, by definition, there exist two Schwarz functions

$$p(z) = c_1 z + c_2 z^2 + c_3 z^3 + ...$$

and

$$q(w) = d_1 w + d_2 w^2 + d_3 w^3 + ...$$

such that

$$e^{i\beta}\left[1 + \sum_{n=2}^{\infty}[(1-\lambda)g_n + \lambda h_n]a_n z^{n-1}\right] = \phi(p(z))$$

$$e^{i\beta}\left[1 + \sum_{n=2}^{\infty}[(1-\lambda)g_n + \lambda h_n]b_n w^{n-1}\right] = \phi(q(w))$$

where

$$\phi(p(z)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \phi_k D_n^k(c_1, c_2, ..., c_n)z^n \tag{2.3}$$

and

$$\phi(q(w)) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \phi_k D_n^k(d_1, d_2, ..., d_n) w^n \tag{2.4}$$

Comparing the corresponding coefficients of (2.1) and (2.3) we have

$$e^{i\beta}\left[\sum_{n=2}^{\infty}[(1-\lambda)g_n + \lambda h_n]a_n z^{n-1}\right] = \sum_{k=1}^{n-1} \phi_k D_{n-1}^k(c_1, c_2, ..., c_{n-1}), n \geq 2 \tag{2.5}$$

Similarly from (2.2) and (2.4) we have

$$e^{i\beta}\left[\sum_{n=2}^{\infty}[(1-\lambda)g_n + \lambda h_n]b_n w^{n-1}\right] = \sum_{k=1}^{n-1} \phi_k D_{n-1}^k(d_1, d_2, ..., d_{n-1}), n \geq 2 \tag{2.6}$$

Now, (2.5) and (2.6) for $a_k = 0(2 \leq k \leq n-1)$,respectively, yield

$$[(1-\lambda)g_n + \lambda h_n]a_n e^{i\beta} = \phi_1 c_{n-1} cos\beta \tag{2.7}$$

and

$$-[(1-\lambda)g_n + \lambda h_n]a_n e^{i\beta} = \phi_1 d_{n-1} cos\beta \tag{2.8}$$

Taking the absolute values of a_n in (2.7) or (2.8) and using the facts $|\phi_1| \leq 2$, $|c_{n-1}| \leq 1$ and $|d_{n-1}| \leq 1$ we obtain

$$|a_n| \leq \frac{2cos\beta}{(1-\lambda)g_n + \lambda h_n} \tag{2.9}$$

Theorem 2.2. *For $\lambda \geq 1$,if $f(z) \in B_{\Sigma}^{\lambda,\beta}(f, g, h; \phi)$. Then*

$$(i)|a_2| \leq min\left\{\frac{2cos\beta}{[(1-\lambda)g_2 + \lambda h_2]}, \sqrt{\frac{4cos\beta}{[(1-\lambda)g_3 + \lambda h_3]}}\right\}$$

$$(ii)|a_3| \leq min\left\{\frac{4cos\beta}{[(1-\lambda)g_3 + \lambda h_3]}, \frac{2cos\beta}{(1-\lambda)g_3 + \lambda h_3]} + \frac{4cos^2(\beta)}{[(1-\lambda)g_2 + \lambda h_2]^2}\right\}$$

$$(iii)|2a_2^2 - a_3| \leq \frac{4cos\beta}{[(1-\lambda)g_3 + \lambda h_3]} where \quad \beta \in (-\pi/2, \pi/2).$$

Proof. Letting n=2 and n=3 in (2.5) and (2.6) respectively, imply

$$a_2[(1-\lambda)g_2 + \lambda h_2]e^{i\beta} = \phi_1 c_1 cos\beta \tag{2.10}$$

$$a_3[(1-\lambda)g_3 + \lambda h_3]e^{i\beta} = (\phi_1 c_2 + \phi_2 c_1^2)cos\beta \tag{2.11}$$

$$-a_2[(1-\lambda)g_2 + \lambda h_2]e^{i\beta} = \phi_1 d_1 cos\beta \tag{2.12}$$

$$(2a_2^2 - a_3)[(1-\lambda)g_3 + \lambda h_3]e^{i\beta} = (\phi_1 d_2 + \phi_2 d_1^2)cos\beta \tag{2.13}$$

From (2.10) or (2.12) we have

$$|a_2| = \left|\frac{\phi_1 c_1 cos\beta}{e^{i\beta}[(1-\lambda)g_2 + \lambda h_2]}\right| \leq \frac{2cos\beta}{[(1-\lambda)g_2 + \lambda h_2]} \tag{2.14}$$

From (2.10) and (2.12) we have

$$(2a_2^2)[(1-\lambda)g_2 + \lambda h_2]e^{2i\beta} = \phi_1^2(c_1^2 + d_1^2)cos^2\beta$$

$$a_2^2 = \frac{\phi_1^2(c_1^2 + d_1^2)}{2e^{2i\beta}[(1-\lambda)g_2 + \lambda h_2]^2} cos^2\beta \tag{2.15}$$

From (2.11) and (2.13) we have

$$(2a_2^2)[(1-\lambda)g_3+\lambda h_3]e^{i\beta}=[\phi_1(c_2+d_2)+\phi_2(c_1^2+d_1^2)]cos\beta$$

$$a_2^2=\frac{[\phi_1(c_2+d_2)+\phi_2(c_1^2+d_1^2)]}{2[(1-\lambda)g_3+\lambda h_3]e^{i\beta}}cos\beta$$

$$a_2=\sqrt{\frac{[\phi_1(c_2+d_2)+\phi_2(c_1^2+d_1^2)]}{2[(1-\lambda)g_3+\lambda h_3]e^{i\beta}}cos\beta}$$

$$|a_2|\leq\sqrt{\frac{4cos\beta}{[(1-\lambda)g_3+\lambda h_3]}}$$

and combining this with the inequality (2.14) we obtain the desired estimate on the coefficient $|a_2|$ as asserted in the theorem.
Again from (2.11) we have

$$|a_3|=\left|\frac{(\phi_1c_2+\phi_2c_1^2)cos\beta}{e^{i\beta}[(1-\lambda)g_3+\lambda h_3]}\right|\leq\frac{4cos\beta}{[(1-\lambda)g_3+\lambda h_3]} \tag{2.16}$$

On the other hand, from (2.11) and (2.13) we have

$$2(a_3-a_2^2)[(1-\lambda)g_3+\lambda h_3]e^{i\beta}=[\phi_1(c_2-d_2)+\phi_2(c_1^2-d_1^2)]cos\beta$$

Since $c_1=-d_1$, therefore we find that

$$(a_3-a_2^2)=\frac{\phi_1(c_2-d_2)}{2e^{i\beta}[(1-\lambda)g_3+\lambda h_3]}cos\beta$$

and using (2.15)

$$a_3=\frac{\phi_1(c_2-d_2)}{2e^{i\beta}[(1-\lambda)g_3+\lambda h_3]}cos\beta+\frac{\phi_1^2(c_1^2+d_1^2)}{2e^{2i\beta}[(1-\lambda)g_2+\lambda h_2]^2}cos^2\beta$$

which implies

$$|a_3|\leq\frac{2cos\beta}{[(1-\lambda)g_3+\lambda h_3]}+\frac{4cos^2\beta}{[(1-\lambda)g_2+\lambda h_2]^2}$$

and combining this with the inequality (2.15) we obtain the desired estimate on the coefficient$|a_3|$ as asserted in the theorem 2.2.
Also from (2.13) we have

$$|2a_2^2-a_3|\leq\frac{4cos\beta}{[(1-\lambda)g_3+\lambda h_3]}$$

Remark 2.3. By specializing on the parameters g,h,ϕ,λ,β as in Remark 1.2, we obtain the bounds on $|a_2|$ and $|a_3|$ which are improvement of the estimates given in ([12], [19], [5], [20], [9], [11], [17], [4]) and corresponding results due to ([6], [15], [16], [14]).

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